Guaranteed, locally space-time efficient, and polynomial-degree robust a posteriori error estimates for high-order discretizations of parabolic problems

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Outline

Model parabolic problem

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

$$f \in L^2(0,T;L^2(\Omega)), \quad u_0 \in L^2(\Omega)$$

Spaces

$$X := L^2(0,T;H^1_0(\Omega)),$$

$$\|v\|_X^2 := \int_0^T \|\nabla v\|^2 \,\mathrm{d}t,$$

$$Y := L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),$$

$$\|v\|_Y^2 := \int_0^T \|\partial_t v\|^2_{H^{-1}(\Omega)} + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v\rangle + (\nabla u, \nabla v)\,\mathrm{d}t = \int_0^T (f, v)\,\mathrm{d}t \qquad \forall\, v \in X.$$

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An optimal a posteriori estimate for evolutive problems

Guaranteed upper bound

- $\|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\mathrm{eff}} \|u - u_{h\tau}\|_{?,\text{neighbors of } K\times(t^{n-1},t^n)}$
- optimal space-time mesh refinement
- **local** in **time** and in **space** error lower bound

Robustness

- C_{eff} independent of data, domain Ω, **final time** T, meshes, solution u, **polynomial degrees** of $u_{h\tau}$ in space and in time

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \searrow 1$
- overestimation factor goes to one with increasing effort

Small evaluation cost

- estimators can be evaluated cheaply (locally)

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Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the misfit of u_h in the weak formulation:

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v) \qquad v \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega);\, \|\nabla v\|=1} \langle \mathcal{R}(u_h), v \rangle$$

Energy error is the dual norm of the residual

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$$\|\nabla(u - u_h)\| = \sup_{v \in H_0^1(\Omega);\, \|\nabla v\|=1} \{(f, v) - (\nabla u_h, \nabla v)\} = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

Equivalence error–residual (steady case)

Laplace equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the misfit of u_h in the weak formulation:

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v) \qquad v \in H_0^1(\Omega)$$

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Equivalence error–residual (unsteady case)

Theorem (Parabolic inf–sup identity)

For every $\varphi \in Y$, we have

$$\|\varphi\|_Y^2 = \left[\sup_{v\in X,\, \|v\|_X=1} \int_0^T \langle \partial_t \varphi, v\rangle + (\nabla\varphi, \nabla v)\,\mathrm{d}t\right]^2 + \|\varphi(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v\rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v\rangle - (\nabla u_{h\tau}, \nabla v)\,\mathrm{d}t$$

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$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v\in X,\, \|v\|_X=1} \langle \mathcal{R}(u_{h\tau}), v\rangle$$

Y norm error is the dual X norm of the residual + IC error

$$\|u - u_{h\tau}\|_Y^2 = \|\mathcal{R}(u_{h\tau})\|_{X'}^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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Proof of the parabolic inf–sup identity: $\varphi \in Y$

Proof.

- let $w_* \in X$ be defined by, a.e. in $(0, T)$,

$$(\nabla w_*, \nabla v) = \langle \partial_t \varphi, v\rangle \quad \forall v \in H_0^1(\Omega) \Rightarrow \|\nabla w_*\|^2 = \|\partial_t \varphi\|^2_{H^{-1}(\Omega)}$$

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Outline

High-order space-time discretization

CG in space & DG in time

- ***p***-degree **continuous** piecewise polynomials in **space**

$$V_h^n := \left\{ v_h \in H_0^1(\Omega),\ v_h|_K \in \mathcal{P}_{p_K}(K) \quad \forall K \in \mathcal{T}^n \right\}$$

- ***q***-degree **discontinuous** piecewise polynomials in **time**

$$\mathcal{Q}_{q_n}(I_n; V) := \left\{ V\text{-valued pols of degree at most } q_n \text{ over } I_n \right\}$$

High-order discretization

Find $u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n}(I_n; V_h^n)$ with $u_{h\tau}(0) = \Pi_h u_0$ such that

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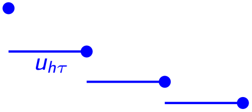
Approximate solution and Radau reconstruction



Approximate solution

- ✗ $u_{h\tau}$ is a piecewise discontinuous polynomial in time
- ✗ $u_{h\tau} \notin Y \Rightarrow$ impossible to estimate $\|u - u_{h\tau}\|_Y$

Radau reconstruction

- $\mathcal{I}u_{h\tau} \in Y$, $\mathcal{I}u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n+1}(I_n; \widetilde{V_h^n})$ (Makridakis–Nochetto)

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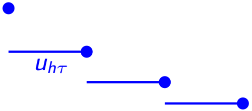

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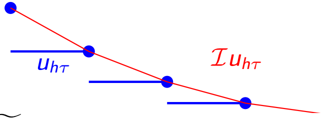

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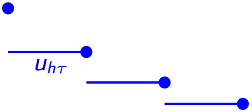

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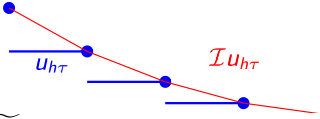

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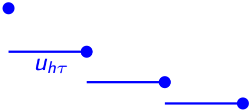

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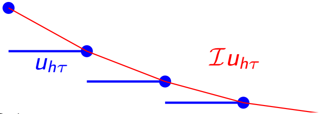

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Outline

Results in the Y norm

Theorem (Reliability in the Y norm)

Suppose no data oscillation for simplicity. Then, for any $\boldsymbol{\sigma}_{h\tau} \in L^2(0, T; \mathbf{H}(\mathrm{div}, \Omega))$ *with* $\nabla \cdot \boldsymbol{\sigma}_{h\tau} = f - \partial_t \mathcal{I} u_{h\tau}$*, there holds*

$$\|u - \mathcal{I} u_{h\tau}\|_Y^2 \leq \int_0^T \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I} u_{h\tau}\|^2 \, \mathrm{d}t.$$

Proof of the upper bound

Proof.

- equivalence error-residual (no error in the initial condition):

$$\|u - \mathcal{I}u_{h\tau}\|_Y = \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle$$

- Green theorem

$$\int_0^T (\boldsymbol{\sigma}_{h\tau}, \nabla \mathcal{I}u_{h\tau}) + (\nabla \cdot \boldsymbol{\sigma}_{h\tau}, \mathcal{I}u_{h\tau})\, \mathrm{d}t = 0$$

- residual definition, Cauchy–Schwarz inequality:

$$\langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle = \int_0^T (f, v) - (\partial_t \mathcal{I}u_{h\tau}, v) - (\nabla \mathcal{I}u_{h\tau}, \nabla v)\, \mathrm{d}t$$

$$= \int_0^T (\underbrace{f - \partial_t \mathcal{I}u_{h\tau} - \nabla \cdot \boldsymbol{\sigma}_{h\tau}}_{=0}, v) - (\nabla \mathcal{I}u_{h\tau} + \boldsymbol{\sigma}_{h\tau}, \nabla v)\, \mathrm{d}t$$

$$\leq \left\{ \int_0^T \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2\, \mathrm{d}t \right\}^{\frac{1}{2}} \|v\|_X$$

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$$\int_0^T (\boldsymbol{\sigma}_{h\tau}, \nabla \mathcal{I}u_{h\tau}) + (\nabla \cdot \boldsymbol{\sigma}_{h\tau}, \mathcal{I}u_{h\tau})\,\mathrm{d}t = 0$$

- residual definition, Cauchy–Schwarz inequality:

$$\langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle = \int_0^T (f, v) - (\partial_t \mathcal{I}u_{h\tau}, v) - (\nabla \mathcal{I}u_{h\tau}, \nabla v)\,\mathrm{d}t$$

$$= \int_0^T (\underbrace{f - \partial_t \mathcal{I}u_{h\tau} - \nabla \cdot \boldsymbol{\sigma}_{h\tau}}_{=0}, v) - (\nabla \mathcal{I}u_{h\tau} + \boldsymbol{\sigma}_{h\tau}, \nabla v)\,\mathrm{d}t$$

$$\leq \left\{ \int_0^T \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2\,\mathrm{d}t \right\}^{\frac{1}{2}} \|v\|_X$$

Proof of the upper bound

Proof.

- equivalence error-residual (no error in the initial condition):

$$\|u - \mathcal{I}u_{h\tau}\|_Y = \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle$$

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Global efficiency ∼ missing Galerkin orthogonality

Efficiency

For suitable $\boldsymbol{\sigma}_{h\tau}$, there holds

$$\int_{I_n} \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I} u_{h\tau}\|_{\Omega}^2 \,\mathrm{d}t \leq C_{\mathrm{eff}}^2 \|u - \mathcal{I} u_{h\tau}\|_{Y(I_n)} \qquad \forall 1 \leq n \leq N.$$

- ✗ local-in-time but **global-in-space** only (as in Verfürth & Bergam–Bernardi–Mghazli)

Reason

- ✗ $\mathcal{I} u_{h\tau}$ misses the Galerkin orthogonality:

$$\int_{I_n} (f, v_{h\tau}) - (\partial_t \mathcal{I} u_{h\tau}, v_{h\tau}) - (\nabla \mathcal{I} u_{h\tau}, \nabla v_{h\tau}) \mathrm{d}t$$

- ✓ the misfit is known: $u_{h\tau} - \mathcal{I} u_{h\tau}$

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✓ the misfit is known: $u_{h\tau} - \mathcal{I} u_{h\tau}$

Remedy

Augmented norm

- augment the norm: $\|v\|^2_{\mathcal{E}_Y} := \|\mathcal{I}v\|^2_Y + \|v - \mathcal{I}v\|^2_X$, $v \in Y + V_{h\tau}$
- $\mathcal{I}u = u \Rightarrow$

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} = \|u - \mathcal{I}u_{h\tau}\|^2_Y + \underbrace{\|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_X}_{\text{known, computable}}$$

- we are adding to Y norm the time jumps in X norm (Schötzau–Wihler):

$$\|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_{X(I_n)} = \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 \,\mathrm{d}t$$
$$= \frac{\tau_n(q_n+1)}{(2q_n+1)(2q_n+3)} \|\nabla (\!| u_{h\tau} |\!)_{n-1}\|^2$$

Equivalence between the Y and $\mathcal{E}_Y$ norms

Theorem (Global equivalence)

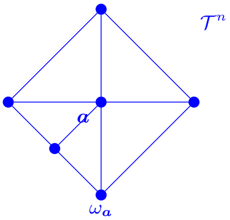
Suppose no source term oscillation or no coarsening. Then there holds

$$\|u - \mathcal{I}u_{h\tau}\|_Y \leq \|u - u_{h\tau}\|_{\mathcal{E}_Y} \leq 3\|u - \mathcal{I}u_{h\tau}\|_Y$$

- the two norms $\|\cdot\|_Y$ and $\|\cdot\|_{\mathcal{E}_Y}$ still may differ locally
- in general, an additional source term oscillation or coarsening term appears

Outline

Handling mesh adaptivity



- refinement & coarsening can also involve changing polynomial degrees

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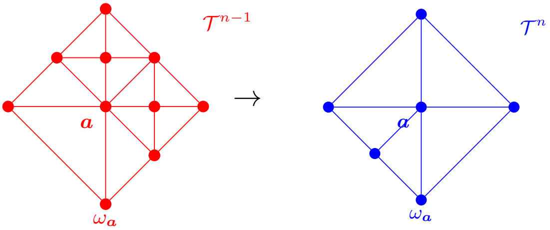

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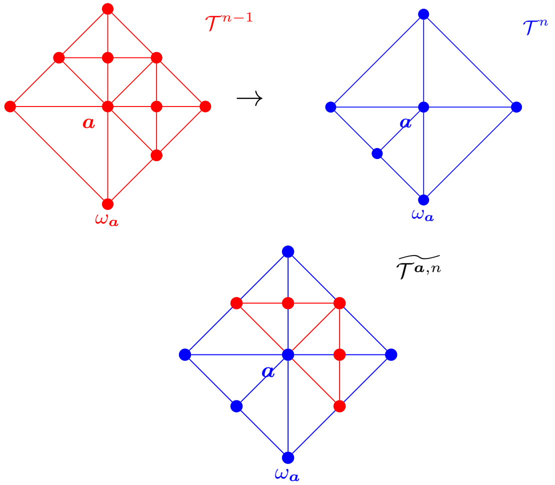

- refinement & coarsening can also involve changing polynomial degrees

Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\boldsymbol{v}_h = \psi_{\mathbf{a}}(f - \partial_t \mathcal{I} u_{h\tau}) - \nabla\psi_{\mathbf{a}}\cdot\nabla u_{h\tau}}} \int_{I_n} \|\boldsymbol{v}_h + \psi_{\mathbf{a}} \nabla u_{h\tau}\|_{\omega_{\mathbf{a}}}^2 \, \mathrm{d}t.$$

Then set

$$\boldsymbol{\sigma}_{h\tau} := \sum_{n=1}^{N} \sum_{\mathbf{a}\in\mathcal{V}^n} \boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n}.$$

Comments

- ✓ satisfies $\boldsymbol{\sigma}_{h\tau} \in L^2(0,T;\mathbf{H}(\mathrm{div},\Omega))$ with $\nabla\cdot\boldsymbol{\sigma}_{h\tau} = f - \partial_t \mathcal{I} u_{h\tau}$
- works on the common refinement $\widetilde{\mathcal{T}}^{\mathbf{a},n}$ of the patch $\omega_{\mathbf{a}}$
- ✗ a priori a local space-time problem, $\boldsymbol{V}_{h\tau}^{\mathbf{a},n} := \mathcal{Q}_{q_n}(I_n; \boldsymbol{V}_h^{\mathbf{a},n})$
- ✓ actually uncouples to q_n elliptic problems posed in $\boldsymbol{V}_h^{\mathbf{a},n}$

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Guaranteed upper bound

Theorem (Guaranteed upper bound)

In the absence of data oscillation, there holds

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \int_{I_n} \|\boldsymbol{\sigma}_{h\tau} + \nabla\mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 \,\mathrm{d}t.$$

Data oscillation

- initial condition

$$\eta_{\mathrm{osc,init}} := \|u_0 - \Pi_h u_0\|$$

- temporal oscillation of the source term

$$\eta_{\mathrm{osc},\tau}(t) := \|f(t) - f_\tau(t)\|_{H^{-1}(\Omega)}$$

- spatial oscillation of the source term

$$\eta_{\mathrm{osc},h}^n(t) := \left[\sum_{\widetilde{K}\in\widetilde{\mathcal{T}}^n} \frac{h_{\widetilde{K}}^2}{\pi^2}\|f_\tau(t) - f_{h\tau}(t)\|_{\widetilde{K}}^2\right]^{\frac{1}{2}}$$

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Outline

Local space-time efficiency and robustness

Local error contributions

$$|u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}} = \int_{I_n} \|\partial_t(u - \mathcal{I}u_{h\tau})\|^2_{H^{-1}(\omega_{\mathbf{a}})} + \|\nabla(u - \mathcal{I}u_{h\tau})\|^2_{\omega_{\mathbf{a}}}\,\mathrm{d}t + \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2_{\omega_{\mathbf{a}}}\,\mathrm{d}t$$

Theorem (Local space-time efficiency and robustness)

For each time-step interval I_n and for each element $K \in \mathcal{T}^n$, there holds, in the absence of data oscillation,

$$\int_{I_n} \|\sigma_{h\tau} + \nabla\mathcal{I}u_{h\tau}\|^2_K + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2_K \mathrm{d}t \leq C^2_{\mathrm{eff}} \sum_{\mathbf{a}\in\mathcal{V}_K} |u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}}.$$

Comments

- ✓ **local** in **space** and **time**
- ✓ C_{eff} only depends on shape regularity ⇒ **robustness** w.r.t the final time T and the **polynomial degrees** p and q
- ✓ **no restriction on coarsening** between $\mathcal{T}^{n-1}$ and $\mathcal{T}^n$

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recall

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} = \int_0^T \|\partial_t(u - \mathcal{I}u_{h\tau})\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t + \int_0^T \|\nabla(u - \mathcal{I}u_{h\tau})\|^2\,\mathrm{d}t + \int_0^T \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2\,\mathrm{d}t + \|(u - \mathcal{I}u_{h\tau})(T)\|^2$$

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Fundamental results on a reference tetrahedron

Bounded right inverse of the divergence operator

- polynomial volume data
- Costabel & McIntosh (2010):

Let $K \in \mathcal{T}$ and $r \in \mathcal{P}_p(K)$. Then there exists $\xi_h \in \mathbf{RTN}_p(K)$ s.t. $\nabla{\cdot}\xi_h = r$ and

$$\|\xi_h\|_K \leq C\|r\|_{H^{-1}(K)} = \sup_{v \in H^1_0(K),\, \|\nabla v\|_K = 1} (r, v)_K.$$

Polynomial extensions in $\mathbf{H}(\mathrm{div})$

- polynomial boundary data
- Demkowicz, Gopalakrishnan, Schöberl (2012):

Let $K \in \mathcal{T}$ and $r \in \mathcal{P}_p(\mathcal{F}_K)$ satisfying $(r, 1)_{\partial K} = 0$. Then there exists $\xi_h \in \mathbf{RTN}_p(K)$ s.t. $\xi_h{\cdot}\mathbf{n}_K = r$ on ∂K, $\nabla{\cdot}\xi_h = 0$ in K, and

$$\|\xi_h\|_K \leq C\|r\|_{H^{-1/2}(\partial K)} = \sup_{v \in H^1(K),\, \|\nabla v\|_K = 1} (r, v)_{\partial K}.$$

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General result on a physical tetrahedron

Lemma ($\mathbf{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K .$$

General result on a physical tetrahedron

Lemma (H(div) polynomial extension on a tetrahedron)

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Context

$$-\Delta \zeta_K = r_K \qquad \text{in } K,$$
$$-\nabla \zeta_K \cdot \mathbf{n}_K = r_F \qquad \text{on all } F \in \mathcal{F}_K^{\mathrm{N}},$$
$$\zeta_K = 0 \qquad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}.$$

Set $\xi_K := -\nabla \zeta_K$.

General result on a physical tetrahedron

Lemma (H(div) polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K = C \|\xi_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\xi_K := -\nabla \zeta_K$.

General result on a physical tetrahedron

Lemma ($\mathbf{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F\in\mathcal{F}_K}(r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

$$\|\xi_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\mathbf{v}_h\in\mathbf{RTN}_p(K)\\ \mathbf{v}_h\cdot\mathbf{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\mathbf{v}_h=r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v}\in\mathbf{H}(\mathrm{div},K)\\ \mathbf{v}\cdot\mathbf{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\mathbf{v}=r_K}} \|\mathbf{v}\|_K = C\|\xi_K\|_K.$$

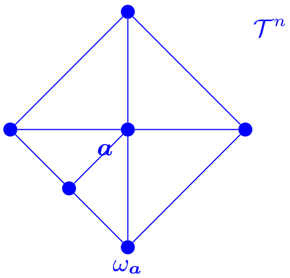
Context

$$-\Delta\zeta_K = r_K \quad \text{in } K,$$
$$-\nabla\zeta_K\cdot\mathbf{n}_K = r_F \quad \text{on all } F \in \mathcal{F}_K^{\mathrm{N}},$$
$$\zeta_K = 0 \quad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}.$$

Set $\xi_K := -\nabla\zeta_K$.

Stable broken $\mathbf{H}(\mathrm{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
- Ern & V. (2016), 3D
- Ern, Smears, & V. (2017), 2-3D, patches with subrefinement



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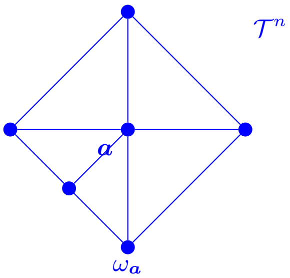

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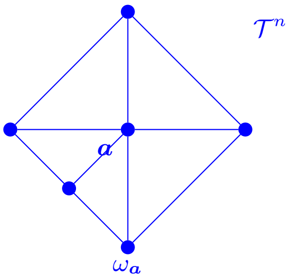

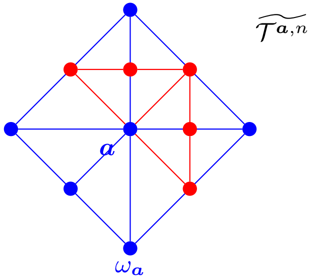

Outline

Conclusions and future directions

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- ✓ local **space-time** efficiency is possible (adding the time jumps to the Y-norm error)
- ✓ **robustness** with respect to both spatial and temporal polynomial degree
- ✓ arbitrarily large **coarsening** allowed

Future directions

- estimates in the X norm
- nonlinear problems

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Bibliography

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Thank you for your attention!

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