Measure-valued solutions to compressible Euler system

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joint work with Eduard Feireisl, Ondřej Kreml, Agnieszka Świerczewska-Gwiazda and Emil Wiedemann

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and
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Polish Academy of Sciences

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Plan of the talk

- Measure-valued strong uniqueness to compressible Euler and Navier-Stokes like systems
  - P. G, A. Świerczewska.-Gwiazda., E. Wiedemann, *Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models*, Nonlinearity, 2015
  - E. Feireisl, P. G, A. Świerczewska.-Gwiazda., E. Wiedemann, *Dissipative measure-valued solutions to the compressible Navier-Stokes system*, Calculus of Variations and Partial Differential Equations, 2016
- Measure-valued strong uniqueness for general hyperbolic conservation laws
  - P. G, O. Kreml, A. Świerczewska.-Gwiazda, *Dissipative measure-valued solutions for general hyperbolic conservation laws*, preprint

Incompressible Euler equations

This system models the flow of an inviscid, incompressible fluid with constant density in the absence of external forces. The vector-valued function $v(t,x)$ is the velocity of the fluid and the scalar-valued function $p(t,x)$ is the pressure

$$\begin{aligned} v_t + \operatorname{div}(v \otimes v) + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{aligned}$$

Leonhard Euler, 1757

Compressible Euler equations

For isentropic Euler equations, in contrast to the incompressible system, the pressure $p = p(\rho)$ is no longer a Lagrange multiplier, but a constitutively given function of the density

$$
\begin{aligned}
\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) &= 0, \\
\partial_t \rho + \operatorname{div}(\rho v) &= 0.
\end{aligned}
$$

Polyconvex elasticity

Consider the equations of nonlinear elasticity

$$\partial_t F = \nabla_x \mathbf{u}$$
$$\partial_t \mathbf{u} = \mathrm{div}_x\,(D_F W(F))$$

for an unknown matrix field F and an unknown vector field $\mathbf{u}$. Function $W(F)$ is given. The associated entropy is given by

$$\eta(\mathbf{u}, F) = \frac{1}{2}|\mathbf{u}|^2 + W(F).$$

Notation

$C_0(\mathbb{R}^d)$ – closure of the space of continuous functions on $\mathbb{R}^d$ with compact support w.r.t. the $\|\cdot\|_\infty$-norm.
$(C_0(\mathbb{R}^d))^* \cong \mathcal{M}(\mathbb{R}^d)$ – the space of signed Radon measures with finite mass. The duality pairing is given by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(\lambda)\, d\mu(\lambda).$$

Definition

A map $\mu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is called weakly* measurable if the functions $x \mapsto \langle \mu(x), f \rangle$ are measurable for all $f \in C_0(\mathbb{R}^d)$.

Young measures

- The main feature of Young measure theory is that it allows us to pass to a limit in the expression $f(v^j)$ with nonlinear f and only weakly-star convergent v^j.
- What's the strategy?
  Instead of considering $f(v^j)$ we embed the problem in a larger space, but gain linearity, i.e. $\langle f, \delta_{v^j(x)} \rangle$.
- If $f \in C_0(\mathbb{R}))$ using the duality

$$(L^1(\Omega; C_0(\mathbb{R}^d)))^* \cong L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^d)),$$

  Banach-Alaoglu theorem and weak-star continuity of linear operators allows for limit passage to get $\langle f, \nu_x \rangle$.
- What's the cost?
  We end up with a weaker notion of solution: *measure-valued solution*

Can the Young measures describe a concentration effect?

Definition

A bounded sequence $\{z^j\}$ in $L^1(\Omega)$ converges in biting sense to a function $z \in L^1(\Omega)$, written $z^j \xrightarrow{b} z$ in Ω, provided there exists a sequence $\{E_k\}$ of measurable subsets of Ω, satisfying $\lim_{k\to\infty} |E_k| = 0$, such that for each k

$$z^j \rightharpoonup z \quad \text{in} \quad L^1(\Omega \setminus E_k).$$

Remarks

Biting limit can be also express as $\lim_{n\to\infty} \lim_{j\to\infty} T^n(z^j)$, where by $T^n(\cdot)$ we denote standard truncation operator.

Lemma

Let u^j be a sequence of measurable functions and ν_x a Young measure associated to a subsequence u_{j_k}. Then $f(\cdot, u^{j_k}) \stackrel{b}{\rightarrow} \langle \nu_x, f \rangle$ for every Carathéodory function $f(\cdot,\cdot)$ s.t. $f(\cdot, u^{j_k})$ is a bounded sequence in $L^1(\Omega)$. Here $\langle \nu_x, f \rangle = \int_{\mathbb{R}^d} f \, d\nu_x$.

Remarks

In view of the above facts the classical Young measures prescribe only the oscillation effect, not the concentration one. The attempt to prescribe also concentration effect by some generalizations of the Young measures was initiated by DiPerna and Majda

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Weak-strong uniqueness for mvs to concrete systems

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,
  Incompressible Euler -oscillation and concentration measure,
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012
  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

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- P. G, A. Świerczewska-Gwiazda, E. Wiedemann, Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models, Nonlinearity, 2015
  Oscillatory and vector-valued concentration measure both in weak formulation and entropy inequality
- E. Feireisl, P. G, A. Świerczewska-Gwiazda, E. Wiedemann, Dissipative measure-valued solutions to the compressible Navier–Stokes system, Calculus of Variations and Partial Differential Equations, 2016
  Instead of vector-valued concentration measure the dissipation defect is introduced
- J. Brezina, E. Feireisl, Measure-valued solutions to the complete Euler system, arXiv:1702.04870

Generalized Young measures

A (generalized) Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times \mathbb{R}^+$ is a triple $(\nu_{x,t}, m, \nu^\infty_{x,t})$, where

- $\nu_{x,t} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (oscillation measure)
- $m \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+)$ (concentration measure)
- $\nu^\infty_{x,t} \in \mathcal{P}(\mathcal{S}^{d-1})$ for m-a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (concentration-angle measure)

R. J. DiPerna and A. J. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 1987.

J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 1997.

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Measure-valued solutions to incompressible Euler system

We say that (ν, m, ν^∞) is a measure-valued solution of IE with initial data u_0 if for every $\phi \in C^1_{c,\mathrm{div}}([0,T) \times \mathbb{T}^n; \mathbb{R}^n)$ it holds that

$$\int_0^T \int_{\mathbb{T}^n} \partial_t \phi \cdot \overline{u} + \nabla \phi : \overline{u \otimes u} dxdt + \int_{\mathbb{T}^n} \phi(\cdot, 0) \cdot u_0 dx \quad = 0.$$

Where

$$\overline{u} = \langle \lambda, \nu \rangle$$
$$\overline{u \otimes u} = \langle \lambda \otimes \lambda, \nu \rangle + \langle \beta \otimes \beta, \nu^\infty \rangle m$$

If the solution is generated by some approximation sequences, then the black terms on right-hand side correspond to the biting limit of sequences whereas the blue ones corespond to concentration measure

Admissibility

Let us set

$$E_{mvs}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{|u|^2}(t,x)dx$$

for almost every t, where

$$\overline{|u|^2} = \langle |\lambda|^2, \nu \rangle + \langle |\beta|^2, \nu^\infty \rangle m$$

and

$$E_0 := \int_{\mathbb{T}^n} \frac{1}{2}|u_0|^2(x)dx.$$

We then say that a measure-valued solution is admissible if

$$E_{mvs}(t) \leq E_0$$

in the sense of distributions.

Weak-Strong Uniqueness

Theorem (Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., 2011)

Let $U \in C^1([0,T]\times\mathbb{T}^n)$ *be a solution of IE . If* (ν, m, ν^∞) *is an admissible measure-valued solution with the same initial data, then*

$$\nu_{t,x} = \delta_{U(t,x))} \text{ for a.e. } t, x, \text{ and } m = 0.$$

Remark:

Some generalization of this result: Emil Wiedemann, *Weak-strong uniqueness in fluid dynamics*, arXiv:1705.04220

Skeach of the proof:

Let's define relative energy (entropy):

$$E_{rel}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{|u-U|^2}(t,x)dx$$

where

$$\overline{|u-U|^2} = \langle |\lambda - U|^2, \nu\rangle + \langle |\beta|^2, \nu^\infty\rangle m$$

then

$$\frac{d}{dt}E_{rel}(t) \leq \|U\|_{C^1} \cdot E_{rel}(t).$$

Measure-valued solutions to compressible Euler system

We say that (ν, m, ν^∞) is a measure-valued solution of CE with initial data (ρ_0, u_0) if for every $\tau \in [0, T]$, $\psi \in C^1([0, T] \times \mathbb{T}^n; \mathbb{R})$, $\phi \in C^1([0, T] \times \mathbb{T}^n; \mathbb{R}^n)$ it holds that

$$\int_0^\tau \int_{\mathbb{T}^n} \partial_t \psi \bar{\rho} + \nabla \psi \cdot \overline{\rho u} dxdt + \int_{\mathbb{T}^n} \psi(x, 0)\rho_0 - \psi(x, \tau)\bar{\rho}(x, \tau) dx = 0,$$

$$\int_0^\tau \int_{\mathbb{T}^n} \partial_t \phi \cdot \overline{\rho u} + \nabla \phi : \overline{\rho u \otimes u} + \mathrm{div}\, \phi \overline{\rho^\gamma} dxdt$$
$$+ \int_{\mathbb{T}^n} \phi(x, 0) \cdot \rho_0 u_0 - \phi(x, \tau) \cdot \overline{\rho u}(x, \tau) dx = 0.$$

Where

$$\bar{\rho} = \langle \lambda_1, \nu \rangle$$
$$\overline{\rho^\gamma} = \langle \lambda_1^\gamma, \nu \rangle + \langle \beta_1^\gamma, \nu^\infty \rangle m$$
$$\overline{\rho u} = \langle \sqrt{\lambda_1} \lambda', \nu \rangle$$
$$\overline{\rho u \otimes u} = \langle \lambda' \otimes \lambda', \nu \rangle + \langle \beta' \otimes \beta', \nu^\infty \rangle m$$
$$\overline{\rho |u|^2} = \langle |\lambda'|^2, \nu \rangle + \langle |\beta'|^2, \nu^\infty \rangle m$$

If the solution is generated by some approximation sequences, then the black terms on right-hand side correspond to the biting limit of sequences whereas the blue ones corespond to concentration measure

Note that:

$$\begin{aligned} \|\langle \beta' \otimes \beta', \nu^\infty \rangle m\|_{\mathrm{TV}} &\leq C \|\langle \mathrm{tr}(\beta' \otimes \beta'), \nu^\infty \rangle m\|_{\mathrm{TV}} \\ &= C \|\langle |\beta'|^2, \nu^\infty \rangle m\|_{\mathrm{TV}} \end{aligned}$$

Admissibility

Let us set

$$E_{mvs}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{\rho|u|^2}(t,x) + \frac{1}{\gamma-1}\overline{\rho^\gamma}(t,x)dx$$

for almost every t, and

$$E_0 := \int_{\mathbb{T}^n} \frac{1}{2}\rho_0|u_0|^2(x) + \frac{1}{\gamma-1}\rho_0^\gamma(x)dx.$$

We then say that a measure-valued solution is admissible if

$$E_{mvs}(t) \leq E_0$$

in the sense of distributions.

Weak-Strong Uniqueness

Theorem (G., Świerczewska-Gwiazda, Wiedemann, 2015)

Let $R \in W^{1,\infty}([0,T] \times \mathbb{T}^n), U \in C^1([0,T] \times \mathbb{T}^n)$ *is a solution of CE with initial data* $\varrho_0 \geq c > 0$, $\varrho_0 \in L^\gamma(\mathbb{T}^n)$, $\varrho_0 |u_0|^2 \in L^1(\mathbb{T}^n)$, *and* $R(x,t) \geq c > 0$ *for some constant* c *and all* $(t,x) \in [0,T] \times \mathbb{T}^n$. *If* (ν, m, ν^∞) *is an admissible measure-valued solution with the same initial data, then*

$$\nu_{t,x} = \delta_{(R(t,x),\sqrt{R(t,x)}U(t,x))} \text{ for a.e. } t,x, \text{ and } m = 0.$$

Remark

An analogue result holds for slightly weaker notion of solutions

General hyperbolic conservation law

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,
  General hyperbolic systems - only oscillation measure, both in weak formulation and entropy inequality
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012
  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure

General hyperbolic conservation law

- C. Christoforou, A. Tzavaras, Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity, arXiv:1603.08176
  An analogue result for more general form of a system, hyperbolic-parabolic case, also only with a non-negative concentration measure in entropy inequality
- P. G., O. Kreml, A. Świerczewska-Gwiazda. Dissipative measure valued solutions for general hyperbolic conservation laws *Concentration measure both in the weak formulation and the entropy inequality*

General hyperbolic conservation law

We consider the hyperbolic system of conservation laws in the form

$$\partial_t A(u) + \partial_\alpha F_\alpha(u) = 0 \tag{1}$$

with the initial condition $u(0) = u_0$. Here $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^n$. There exists an open convex set $X \subset \mathbb{R}^n$ such that the mappings $A : \overline{X} \to \mathbb{R}^n$, $\mathrm{F}_\alpha : X \to \mathbb{R}^n$ are C^2 maps on X, A is continuous on $\bar{X}$ and $\nabla A(u)$ is nonsingular for all $u \in X$.

Hypotheses

(H1) There exists an entropy-entropy flux pair (η, q_α), $\eta(u) \geq 0$ and $\lim_{|u|\to\infty} \eta(u) = \infty$

This yields the existence of a C^1 function $G : \overline{X} \to \mathbb{R}^n$ such that

$$\nabla\eta = G \cdot \nabla A, \quad \nabla q_\alpha = G \cdot \nabla F_\alpha, \qquad \alpha = 1, ..., d.$$

(H2) The symmetric matrix

$$\nabla^2\eta(u) - G(u) \cdot \nabla^2 A(u)$$

is positive definite for all $u \in X$.

(H3) The vector $A(u)$ and the fluxes $F_\alpha(u)$ are bounded by the entropy, i.e.

$$|A(u)| \leq C(\eta(u) + 1), \quad |F_\alpha(u)| \leq C(\eta(u) + 1), \qquad \alpha = 1, ..., d.$$

Define for a strong solution U taking values in a compact set $D \subset X$ the relative entropy

$$\begin{aligned} \eta(u|U) &:= \eta(u) - \eta(U) - \nabla\eta(U) \cdot \nabla A(U)^{-1}(A(u) - A(U)) \\ &= \eta(u) - \eta(U) - G(U) \cdot (A(u) - A(U)) \end{aligned}$$

and the relative flux as

$$F_\alpha(u|U) := F_\alpha(u) - F_\alpha(U) - \nabla F_\alpha(U)\nabla A(U)^{-1}(A(u) - A(U))$$

for $\alpha = 1, ..., d$.

If we assume (H1) – (H3) hold and $\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = 0$ then

$$|F_\alpha(u|U)| \leq C_D \eta(u|U).$$

An analogue lemma under more restrictive assumptions

$$\lim_{|u|\to\infty} \frac{A(u)}{\eta(u)} = \lim_{|u|\to\infty} \frac{F_\alpha(u)}{\eta(u)} = 0, \tag{2}$$

was proved in Christoforou & Tzavaras 2017. Note however that this condition is satisfied for polyconvex elastodynamics but is not satisfied e.g. for compressible Euler equations.

G. Kreml, Świerczewska-Gwiazda 2017

Theorem

Let $(\nu, m_A, m_{F_\alpha}, m_\eta)$, $\alpha = 1, ..., d$, *be a dissipative measure-valued solution to* (1) *generated by a sequence of approximate solutions. Let* $U \in W^{1,\infty}(Q)$ *be a strong solution to* (1) *with the same initial data* $\eta(u_0) \in L^1(\mathbb{T}^d)$, *thus* $\nu_{0,x} = \delta_{u_0(x)}$, $m_A^0 = m_{F_\alpha}^0 = m_\eta^0 = 0$. *Then* $\nu_{t,x} = \delta_{U(x)}$, $m_A = m_{F_\alpha} = m_\eta = 0$ *and* $u = U$ *a.e. in* Q.

Overview of the topic

E. Wiedemann.
Weak-strong uniqueness in fluid dynamics,
arXiv:1705.04220.

T. Debiec, P. Gwiazda, K. Łyczek, A. Świerczewska-Gwiazda,
Relative entropy method for measure-valued solutions in natural sciences
preprint

Thank you for your attention