On the Dirichlet vs. Neumann problem related to convex, variational integrals of linear growth

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MORE Workshop – Implicitly constituted materials: Modeling, Analysis and Computing

Setup – Variational integrals with linear growth

Functionals: we consider variational integrals

$$F[w] := \int_\Omega f(|\nabla w|)\,\mathrm{d}x$$

for functions $w\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$, a bounded Lipschitz domain Ω, and a C^1-integrand f with $f'(0) = 0$ which is strictly convex and of linear growth, i. e.

$$t \le f(t) \le L(1+t) \quad \text{for } t \in \mathbb{R}\,.$$

Notice:

- These functionals are well-defined in the Sobolev space $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$.
- Simple model integrands are

$$f_p(t) = (1+t^p)^{1/p} \qquad \text{or} \qquad f_p(t) = \int_0^t (1+s^p)^{-1/p} s\,\mathrm{d}s$$

for $t \in \mathbb{R}_0^+$ and $p \in (1,\infty)$, which both for $p = 2$ give the area integrand.

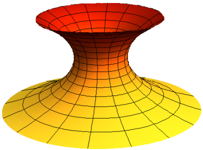
(All derivatives exhibit the same growth in $t \ge 1$, but the second integrand is of class C^2 also for $p < 2$, which avoids the need of regularization later on.)

Setup – Dirichlet and Neumann problem

Dirichlet problem: for fixed boundary values $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ we study the minimization problem

$$\inf \left\{ \int_\Omega f(|\nabla w|)\, \mathrm{d}x \colon w \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N) \right\}.$$

For $f_2(t) = (1+t^2)^{1/2}$ and $N = 1$ this is the classical non-parametric least area problem.



(The parametric least area problem instead refers to minimizing the area among surfaces which are not necessarily given by graphs of functions)

Setup – Dirichlet and Neumann problem

Dirichlet problem: for fixed boundary values $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ we study the minimization problem

$$\inf \left\{ \int_\Omega f(|\nabla w|) \, \mathrm{d}x \colon w \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N) \right\}.$$

Notice:

- If a minimizer exists in $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$, then it is unique.
- For $u_0 \equiv 0$ we have only the trivial minimizer $u \equiv 0$.
- Finding this minimizer is equivalent to finding the unique weak solution $u \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$ to the Euler–Lagrange system

$$\int_\Omega \frac{f'(|\nabla u|) \nabla u}{|\nabla u|} \cdot \nabla \varphi \, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N),$$

i.e.

$$\operatorname{div} \left(\frac{f'(|\nabla u|) \nabla u}{|\nabla u|} \right) = 0 \qquad \text{in } \Omega.$$

Setup – Dirichlet and Neumann problem

Neumann problem: for fixed $T_0 \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^{Nn})$ we study the minimization problem

$$\inf\left\{ \int_\Omega f(|\nabla w|) - T_0 \cdot \nabla w \, \mathrm{d}x \colon w \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N) \right\}.$$

Notice:

- If a minimizer in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ exists, it is unique up to additive constants.
- For $T_0 \equiv 0$ we have the trivial minimizer $u \equiv 0$.
- Finding a minimizer is equivalent to finding a function $u \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ satisfying

  $$\int_\Omega \frac{f'(|\nabla u|)\nabla u}{|\nabla u|} \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega T_0 \cdot \nabla \varphi \, \mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)\,,$$

  which implies that the validity of the Euler–Lagrange system

  $$\operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\right) = \operatorname{div} T_0 \qquad \text{in } \Omega\,.$$

  and, under sufficient regularity, a Neumann-type constraint

  $$\frac{f'(|\nabla u|)\nabla u}{|\nabla u|} \cdot \nu_{\partial\Omega} = T_0 \cdot \nu_{\partial\Omega} \qquad \text{on } \partial\Omega\,.$$

Part I:
Existence of BV-minimizers

Direct method in $\mathrm{W}^{1,1}(\Omega)$?

The **direct method of the calculus of variations** is a tool to find a minimizer of a functional F over a vector space X, which traces back to Hilbert, Lebesgue, Tonelli, ... and consists in

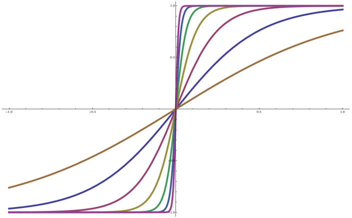
- choosing a topology such that we have compactness (allows to select a convergent subsequence from a minimizing sequence) and lower semicontinuity of F (which guarantees that the limit is a minimizer),

First attempt: with the topology of weak convergence in $\mathrm{W}^{1,1}(\Omega)$.

- largest space on which the functionals are well-defined;
- lower semicontinuity via convexity of the integrands;
- lack of compactness: bounded sequences $(w_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,1}(\Omega)$ may have no weakly convergent subsequence.

Example: Take $w(x) := \mathrm{sgn}(x)$, $w_k(x) := \tanh(kx)$ for $k \in \mathbb{N}$:

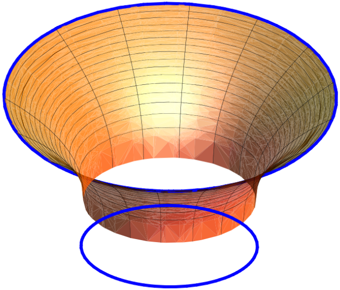
- $w_k \to w$ in $L^1((-1,1))$;
- $\|w_k\|_{\mathrm{W}^{1,1}} \leq 4$ for $k \in \mathbb{N}$;
- $w_k \not\rightharpoonup w$ in $W^{1,1}((-1,1))$.



Example of Finn (1965)

Let $\Omega = B_2 \setminus B_1$, take $f \in C^2$ strictly convex of linear growth, and look for minimizers of F with prescribed boundary values

$$u_0 := \begin{cases} -M & \text{on } \partial B_1\,, \\ M & \text{on } \partial B_2\,. \end{cases}$$



- If a solution exists, it is rotationally symmetric, i.e. $u(x) = \bar{u}(|x|)$, and $\bar{u}$ is increasing;
- The Euler–Lagrange equation reduces to the ODE

$$\big(f'(\bar{u}'(r))r^{n-1}\big)' = 0 \qquad \text{for } r \in (1,2)$$

- Thus, after integration of $\bar{u}'(r) = (f')^{-1}(cr^{1-n})$ for some integration constant c, one obtains that the solution $\bar{u}$ is bounded independently of the choice of M – a contradiction to $\bar{u}(2) = M$ – under the assumption

$$\int_0^\infty t f''(t)\,\mathrm{d}t < \infty\,.$$

Remarks:

- The solvability of the Dirichlet problem depends on Ω and u_0 (which is ultimately linked to the lack of weak compactness properties of $\mathrm{W}^{1,1}(\Omega)$).
- Necessary and sufficient criterion for its solvability (for all regular Ω and u_0) for the scalar, rotational symmetric case is essentially

$$\int_0^\infty t f''(t)\,\mathrm{d}t = \infty$$

[see Bernstein (1912), Leray (1939), Serrin (1969) ... B.–Bulíček–Maringová (2016)].

- For the model integrand with $f_p'(t) = (1+t^p)^{-1/p} t$ we have $f_p''(t) = (1+t^p)^{-1/p-1}$ (cancellation effect!), thus

$$\int_0^\infty t f_p''(t)\,\mathrm{d}t < \infty \qquad \text{for all } p \geq 1$$

and no minimizer in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ exists for this specific example (for suitable choices of $M > 0$).

Extension of the functionals to $\mathrm{BV}(\Omega)$

Many minimization problems of linear growth are in general not solvable in the original formulation, in particular not the Dirichlet problem for the area functional and the vectorial case!

The **direct method of the calculus of variations (II)** to find a minimizer of a functional F over a vector space X consists in

- choosing a space $Y \supset X$ and a topology such that we have compactness and lower semicontinuity of F, which guarantees the existence of a minimizer of F in Y,
- showing that the minimizer belongs to X.

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- showing that the minimizer belongs to X.

F is not defined on Y?

Extension by lower semicontinuity (Lebesgue–Serrin):
We define the relaxed functional (in $y \in Y$) as

$$\mathcal{F}[y] := \inf \Big\{ \liminf_{k\to\infty} F[x_k] : x_k \in X \text{ and } x_k \text{ “} \rightharpoonup \text{” } y \text{ in } Y \Big\}$$

Extension of the functionals to $\mathrm{BV}(\Omega)$

Extension by lower semicontinuity (Lebesgue–Serrin) to the space $\mathrm{BV}(\Omega)$ leads in the model cases to the following relaxed functionals (evaluated in $w \in \mathrm{BV}(\Omega)$):

- for the **Dirichlet problem**

$$\mathcal{F}_p^{u_0}[w] = \int_\Omega f_p(|\nabla w|)\,\mathrm{d}x + |\mathrm{D}^s w|(\Omega) + \int_{\partial\Omega} |w - u_0|\mathrm{d}\mathcal{H}^{n-1}$$

- for the **Neumann problem**

$$\mathcal{F}_p^{T_0}[w] = \int_\Omega f_p(|\nabla w|)\,\mathrm{d}x + |\mathrm{D}^s w|(\Omega) - \int_\Omega T_0 \cdot \mathrm{d}\mathrm{D}w$$

(integral representation by Goffman–Serrin 1964, Giaquinta–Modica–Souček 1979).

Recall the space of functions of bounded variation

$$\mathrm{BV}(\Omega) := \{w \in \mathrm{L}^1(\Omega) : \text{the weak derivative } \mathrm{D}w = \mathrm{D}^s w + \nabla w \mathcal{L}^n \text{ exists as finite Radon measure}\} \supsetneq \mathrm{W}^{1,1}(\Omega)\,.$$

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- for the **Neumann problem**

$$\mathcal{F}_p^{T_0}[w] = \int_\Omega f_p(|\nabla w|)\,\mathrm{d}x + |\mathrm{D}^s w|(\Omega) - \int_\Omega T_0 \cdot \mathrm{d}\mathrm{D}w$$

Note: These functionals are lower semicontinuous in $(\mathrm{BV}(\Omega), \text{weak-}*)$ and have the same infimum as for the original problems (Reshetnyak 1968).

- This leads to the concept of BV- or generalized minimizers as minimizers of the relaxed functionals in $\mathrm{BV}(\Omega)$ (or equivalently, as weak-$*$-limit of a minimizing sequence of the original minimization problem);
- Existence of such BV-minimizers follows from the direct method applied in $\mathrm{BV}(\Omega)$ ...

Necessary condition on T_0 for the Neumann problem

Existence of such BV-minimizers for the Neumann problem follows only under the assumption $T_0 \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^{Nn})$ such that

$$\|T_0\|_{\mathrm{L}^\infty(\Omega)} < \lim_{t\to\infty} f_p(t)/t = 1\,.$$

- This condition guarantees coerciveness, i.e. we have

$$\|\nabla w\|_{\mathrm{L}^1(\Omega)} \leq C(1 + F_p^{T_0}[w]) \qquad \text{for all } w \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)\,.$$

- Non-existence of (BV-)minimizers for $T_0 \equiv 1$: for $\Omega = (-1,1)$ we have

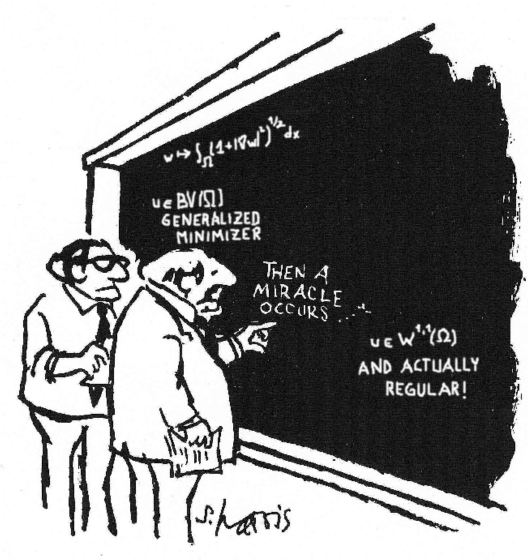
$$0 = \inf_{\mathrm{W}^{1,1}(\Omega)} F_p^{T_0} \leq F_p^{T_0}[kx] \quad \overset{k\to\infty}{\longrightarrow} \quad 0$$

  but the integrand is strictly positive!
- Unboundedness from below already for $F_p^{T_0}$ in $\mathrm{W}^{1,1}(\Omega)$ for $T_0 \equiv c > 1$.

Part II:

Regularity of BV-minimizers
or
Existence of $\mathrm{W}^{1,1}$-minimizers

$\mathrm{W}^{1,1}$-regularity of BV-minimizers?



"I think you should be more explicit here in step two."

$\mathrm{W}^{1,1}$-regularity of BV-minimizers?

The **typical strategy** is via the proof of higher integrability of ∇u, by using suitable test functions $\varphi \in \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$ in the Euler–Lagrange system

$$\int_\Omega \frac{f'(|\nabla u|)\nabla u}{|\nabla u|} \cdot \nabla\varphi \, \mathrm{d}x = 0 \,.$$

The analytic difficulty for proving estimates is the non-uniform ellipticity condition of the system, arising from

$$t^{-1-p} \lesssim f''(t) \lesssim t^{-1}$$

(cp. functionals under p-q-growth condition)

which limits the following strategy to $p \in (1, 2]$:

1. local boundedness of u;
2. weighted $\mathrm{W}^{2,2}_{\mathrm{loc}}$-estimates (with weights $(1 + |\nabla u|)^{-1-p}$);
3. local superlinear integrability of ∇u.

Regularity results up to the critical case $p=2$

Theorem (Bildhauer 2002, B.–Schmidt 2013-2015)

If u is a BV-minimizer of the Dirichlet problem for F_p with $p \in (1,2]$, then

(i) $u \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ *with* $|\nabla u|\log(1+|\nabla u|) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *if* $p=2$;

(ii) $u \in \mathrm{C}^1(\Omega,\mathbb{R}^N)$ *if* $p \in (1,2)$.

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(ii) $u \in \mathrm{C}^1(\Omega, \mathbb{R}^N)$ *if* $p \in (1,2)$*.*

Remarks:

- Uniform regularity estimates are established for a suitable minimizing sequence $(u_k)_{k \in \mathbb{N}}$ (via a vanishing viscosity approach), the result on minimizers then "follows" from compactness

a priori:	$u_k \overset{*}{\rightharpoonup} u$	in $\mathrm{BV}(\Omega, \mathbb{R}^N)$
now:	$u_k \rightharpoonup u$	in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$;

- $p{=}2,\ N{=}1$: (minimal graphs) all BV-minimizers are of class $\mathrm{C}^\infty(\Omega)$ (GMT-arguments, by Bombieri, De Giorgi, Giusti, Miranda ... $\sim$ 1969);
- $p{=}2,\ N{>}1$: higher regularity (such as $\mathrm{W}^{1,q}$ for some $q > 1$) is open;
- The result transfers directly to the Neumann problem (local estimates!) for regular T_0.

Autonomous case:

- Every BV-minimizer u is partially C^∞-regular, i.e. outside of a set of $\mathcal{L}^n$-measure zero.

  (Anzellotti–Giaquinta 1988, Schmidt 2014)

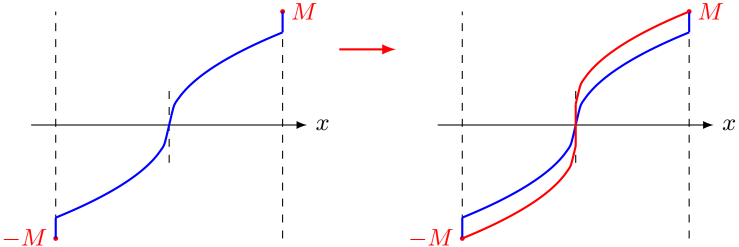
Notice: We always have $\mathcal{L}^n(\mathrm{supp}(\mathrm{D}^s u)) = 0$, so we

cannot exclude $u \in \mathrm{BV} \setminus \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$!

Non-autonomous case: We study the minimization of

$$\widetilde{F}_p[w] := \int_\Omega (1 + a(x)|\nabla w|^p)^{1/p}\, \mathrm{d}x \qquad \text{in } \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)\,.$$

- Smooth x-dependence does not chance the positive theory for $p \in (1, 2]$. (Bildhauer 2003)
- Let $n = 1$, $p > 2$, $\Omega = (-1, 1)$ and $a(x) = 1 + |x|^2$. For $u_0(\pm 1) = \pm M$ for M sufficiently large, the BV-minimizer
  - attains the boundary values,



... and what about $p > 2$ for the Dirichlet problem?

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  - attains the boundary values,
  - has a jump singularity at $\{0\}$, i.e. is in $\mathrm{BV} \setminus \mathrm{W}^{1,1}(\Omega)$!

  Jumps are cheaper in the interior of Ω, where $a(x)$ is small!
  (Giaquinta–Modica–Souček 1979)

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  Jumps are cheaper in the interior of Ω, where $a(x)$ is small!
  (Giaquinta–Modica–Souček 1979)
- Extension to $n > 1$ in rotational symmetric setting: for $\Omega = B_2 \setminus B_1$, $a(x) = 1 + (|x| - \frac{3}{2})^2$ and $u_0(\partial B_2) = M$, $u_0(\partial B_1) = -M$, one can show that the BV-minimizer
  - is rotationally symmetric
  - attains the boundary values,
  - is in $\mathrm{BV} \setminus \mathrm{W}^{1,1}(\Omega)$!

  (Bildhauer 2003)

A general existence result for $\mathrm{W}^{1,1}$-minimizers for the Neumann problem

Surprisingly, we find $\mathrm{W}^{1,1}$-minimizers for the Neumann problem for the whole range $p \in (1, \infty)$ in the following situation:

Theorem (B.–Bulíček–Gmeineder 2017)

Consider a bounded, simply connected Lipschitz domain Ω, a strictly convex C^2-integrand f of linear growth with $f'(0) = 0$ and a function $T_0 \in \mathrm{W}^{2,\infty}(\Omega, \mathbb{R}^{Nn})$ satisfying

$$\|T_0\|_{\mathrm{L}^\infty(\Omega)} < \lim_{t\to\infty} f(t)/t\,.$$

Then there exists a (unique) solution $u \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ (with fixed average) of the Neumann problem

$$\inf\left\{ \int_\Omega f(|\nabla w|) - T_0 \cdot \nabla w \,\mathrm{d}x \colon w \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N) \right\}.$$

Open: Is the construction of a BV-minimizer $u \notin \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ for a non-simply connected domain (like the annulus) possible?

Strategy of proof

We work with solutions $u_k \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ to the approximate Neumann problems

$$\text{to minimize } w \mapsto \mathcal{F}_p^{T_0}[w] + (2k)^{-1} \|\nabla w\|_{\mathrm{L}^2(\Omega)}^2 \qquad \text{in } \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$$

and show:

1. Boundedness of $(u_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, and (up to subsequences)

$$u_k \overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{BV}(\Omega, \mathbb{R}^N) \quad \text{and} \quad \nabla u_k \overset{b}{\rightharpoonup} E \quad \text{in } \mathrm{L}^1(\Omega, \mathbb{R}^{Nn});$$

2. $(u_k)_{k\in\mathbb{N}}$ is a minimizing sequence;
3. Uniform weighted $\mathrm{W}^{2,2}_{\mathrm{loc}}$ estimates;

— so far as for the Dirichlet problem —

4. Pointwise convergence $\nabla u_k \to E$ a.e. in Ω (relying on arguments used before in [B.–Bulíček–Málek–Süli, 2017]);
5. Gradient structure $E = \nabla v$ for some function $v \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ (via $\operatorname{curl} E = 0$ and a Sobolev-type version of Poincaré's Lemma);
6. Minimization property of v by (by pointwise convergence $\nabla u_k \to \nabla v$, Step 2 and Fatou).

Thank you for your attention!

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