On the states of stress and strain adjacent to a crack in a strain limiting viscoelastic body

Victor A. Kovtunenko

Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, NAWI Graz, AUSTRIA;
Lavrent'ev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, RUSSIA

⋈ joint works with Hiromichi Itou, Kumbakonam R. Rajagopal

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https://homepage.uni-graz.at/de/victor.kovtunenko/

Motivation

• In contrast to the linearized model, even when the strains are "small", e.g. metallic alloys response nonlinearly [Rajagopal (2014)]

• The boundedness, respectively smallness, of strains is required a-priori, ensured by a so-called limiting strain model

• Since strains are constrained, they are complained by singular stresses within measure spaces [Beck, Bulicek, Malek, Süli (2017)]

• While boundary tractions are problematic, contact conditions are suitable for limiting strain within nonlinear elastic model [Itou, Kovtunenko, Rajagopal (2017a, 2017b)]

• Here we extend the limiting strain to nonlinear viscoelastic model [Itou, Kovtunenko, Rajagopal (2017c)]

Outline

Strain limiting viscoelastic model

Modeling issues

For linearized strain ε, time rate of the linearized strain $\dot{\varepsilon}$, and stress σ

Linearized Kelvin–Voigt viscoelastic model:

$$\varepsilon + \alpha \, \dot{\varepsilon} = \beta\sigma \qquad \text{(Kelvin–Voigt)}$$

material parameters $\alpha, \beta > 0$ ($1/\beta$ is the shear modulus, $\alpha/(2\beta)$ the viscosity)

Nonlinear strain limiting viscoelastic model:

$$\varepsilon + \alpha \, \dot{\varepsilon} = \mathcal{F}(\sigma), \quad \|\mathcal{F}(\sigma)\| \leq M_1 \qquad \text{(Strain limiting)}$$

constant $M_1 > 0$, $\mathcal{F}$ is a response function

Dynamic stability

Component-wisely, the dynamic relations in isolation:

$$\varepsilon_{ij} + \alpha\,\dot{\varepsilon}_{ij} = \mathcal{F}_{ij}(\sigma), \quad -M_1 \le \mathcal{F}_{ij}(\sigma) \le M_1$$

imply two differential inequalities

$$\frac{d}{dt}(\varepsilon_{ij} - M_1) \le -\frac{1}{\alpha}(\varepsilon_{ij} - M_1), \quad \frac{d}{dt}(\varepsilon_{ij} + M_1) \ge -\frac{1}{\alpha}(\varepsilon_{ij} + M_1)$$

which are solved analytically such that

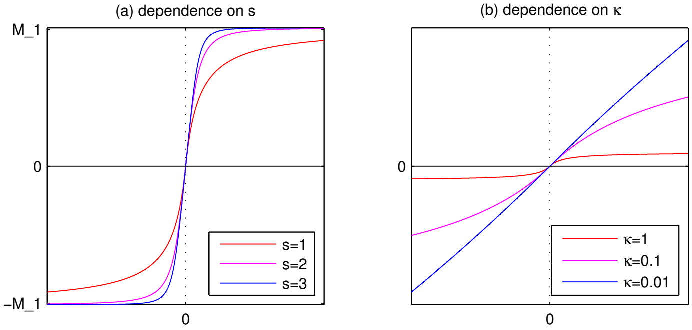
$$-M_1 + (\varepsilon_{ij}(0) + M_1)e^{-t/\alpha} \le \varepsilon_{ij}(t) \le M_1 + (\varepsilon_{ij}(0) - M_1)e^{-t/\alpha}$$

Therefore, the uniform boundedness $|\varepsilon_{ij}(t)| \le M_1$ is provided by $|\varepsilon_{ij}(0)| \le M_1$ and the strain limiting model is dynamically stable in the sense of Lyapunov

Generic response function

$$\mathcal{F}(\sigma) = \frac{\beta\sigma}{(1+\kappa\|\sigma\|^s)^{1/s}} \qquad \text{(Generic response function)}$$

in dependence of parameters $\kappa, s > 0$



If $\kappa \searrow 0$ in (Generic response function), it turns into (Kelvin–Voigt) model

Properties of generic response function

For (Generic response function), the following principal properties hold in $\mathbb{R}^d$:

uniform boundedness: $$\|\mathcal{F}(\sigma)\| \leq \frac{\beta}{\kappa^{1/s}} \quad (1)$$

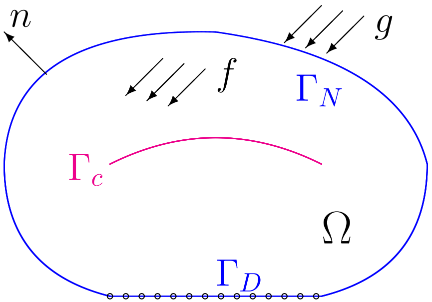
monotony: $$\big(\mathcal{F}(\sigma) - \mathcal{F}(\bar{\sigma})\big) : (\sigma - \bar{\sigma}) \geq 0 \quad (2)$$

Lipschitz continuity: $$(\mathcal{F}(\sigma) - \mathcal{F}(\bar{\sigma})) : (\sigma - \bar{\sigma}) \leq 2\beta\|\sigma - \bar{\sigma}\|^2 \quad (3)$$

semi-coercivity: $$-\frac{\beta}{\kappa^{2/s}c_s} + \frac{\beta}{d\kappa^{1/s}c_s} \sum_{i,j=1}^{d} |\sigma_{ij}| \leq \mathcal{F}(\sigma) : \sigma \quad (4)$$

where $c_s = 2^{1/s-1}$ for $s \in (0,1)$ and $c_s = 1$ for $s \geq 1$

Governing equations



The reference domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, consisted of the Dirichlet Γ_D and the Neumann Γ_N parts with the normal vector n, which contains crack $\Gamma_c \subset \Omega$

The domain with crack $\Omega_c := \Omega \setminus \overline{\Gamma}_c$ founds the time-cylinder $Q_c^T := (0,T) \times \Omega_c$

Response function

For an abstract response function given by a map:

$$\mathcal{F}: \ \mathrm{Sym}(\mathbb{R}^{d\times d}) \mapsto \mathrm{Sym}(\mathbb{R}^{d\times d}), \quad \mathcal{F}(0) = 0 \tag{5}$$

let constant $M_1, M_3 > 0$ and $M_2 \geq 0$ exist such that $\mathcal{F}$ is

uniform bounded:
$$\|\mathcal{F}(\sigma)\| \leq M_1 \tag{6}$$

monotone:
$$\big(\mathcal{F}(\sigma) - \mathcal{F}(\bar\sigma)\big) : (\sigma - \bar\sigma) \geq 0 \tag{7}$$

hemi-continuous:
$$r \mapsto \mathcal{F}\big(\sigma + r\bar\sigma\big) : \bar\sigma \quad \text{is continuous at } r = 0 \tag{8}$$

semi-coercive:
$$-M_2 + M_3 \sum_{i,j=1}^{d} |\sigma_{ij}| \leq \mathcal{F}(\sigma) : \sigma \tag{9}$$

Find:

vector of the displacement	$u(t,x) = (u_1, \ldots, u_d)$
vector of the velocity	$\dot{u}(t,x) = (\dot{u}_1, \ldots, \dot{u}_d)$
tensor of the Cauchy–Green strain	$\varepsilon(t,x) \in \mathrm{Sym}(\mathbb{R}^{d\times d})$
tensor of the Cauchy stress	$\sigma(t,x) \in \mathrm{Sym}(\mathbb{R}^{d\times d})$

satisfying component-wise for $i, j = 1, \ldots, d$ the quasi-static equations in Q_c^T:

equilibrium equation: $$-\sum_{j=1}^d \frac{\partial}{\partial x_j}\sigma_{ij} = f_i \tag{10}$$

constitutive equation: $$\varepsilon_{ij}(u) + \alpha\varepsilon_{ij}(\dot{u}) = \mathcal{F}_{ij}(\sigma) \tag{11}$$

linearized strain: $$\varepsilon_{ij}(u) := \frac{1}{2}\big(\frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\big) \tag{12}$$

under the initial and boundary conditions:

$$\text{initial condition:} \quad u_i(0,\,\cdot\,) = u_i^0 \quad \text{in } \Omega_c \tag{13}$$

$$\text{Dirichlet boundary condition:} \quad u_i = 0 \quad \text{on } (0,T)\times\Gamma_D \tag{14}$$

$$\text{Neumann boundary condition:} \quad \sum_{j=1}^{d} \sigma_{ij} n_j = g_i \quad \text{on } (0,T)\times\Gamma_N \tag{15}$$

$$\text{stress-free crack:} \quad \sum_{j=1}^{d} \sigma_{ij} n_j = 0 \quad \text{on } (0,T)\times\Gamma_c \tag{16}$$

for the given body force $f(t,x) = (f_1,\dots,f_d) \in C([0,T]; L^2(\Omega_c;\mathbb{R}^d))$
the boundary traction $g(t,x) = (g_1,\dots,g_d) \in C([0,T]; L^2(\Gamma_N;\mathbb{R}^d))$
and the initial state $u^0 \in W^{1,\infty}(\Omega_c;\mathbb{R}^d)$

Concept of generalized solution

Weak formulation

$$\int_{\Omega_c} \sigma : \varepsilon(\overline{u})\, dx = \int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx := \int_{\Omega_c} f \cdot \overline{u}\, dx + \int_{\Gamma_N} g \cdot \overline{u}\, dS_x$$

$$\varepsilon(u + \alpha \dot{u}) = \mathcal{F}(\sigma)$$

Generalized formulation

$$\int_{\Omega_c} \sigma : \varepsilon(\overline{u})\, dx = \int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx$$

$$\int_{\Omega_c} \varepsilon(u + \alpha \dot{u}) : \overline{\sigma}\, dx \le \int_{\Omega_c} \mathcal{F}(\overline{\sigma}) : (\overline{\sigma} - \sigma)\, dx + \int_{\Omega_c} \sigma^E : \varepsilon(u + \alpha \dot{u})\, dx$$

Elliptic regularization

$$\int_{\Omega_c} \sigma^\delta : \varepsilon(\overline{u})\, dx = \int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx$$

$$\varepsilon(u^\delta + \alpha \dot{u}^\delta) = \mathcal{F}(\sigma^\delta) + \delta \sigma^\delta$$

Well-posedness theorems

For a small parameter $\delta > 0$, the regularized problem:

Find $u^\delta \in C([0,T]; H^1(\Omega_c;\mathbb{R}^d))$, $\varepsilon(u^\delta) \in C^1([0,T]; L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$, and $\sigma^\delta \in C([0,T]; L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$u^\delta(0,\cdot) = u^0 \quad \text{in } \Omega_c \tag{17a}$$

$$u^\delta = 0 \quad \text{on } (0,T)\times\Gamma_D \tag{17b}$$

$$\int_{\Omega_c} \sigma^\delta : \varepsilon(\overline{u})\,dx = \int_{\Omega_c} f\cdot\overline{u}\,dx + \int_{\Gamma_N} g\cdot\overline{u}\,dS_x, \quad t\in(0,T) \tag{17c}$$

$$\int_{\Omega_c} \big(\varepsilon(u^\delta) + \alpha\varepsilon(\dot{u}^\delta) - \mathcal{F}(\sigma^\delta) - \delta\sigma^\delta\big) : \overline{\sigma}\,dx = 0, \quad t\in(0,T) \tag{17d}$$

for all test functions $\overline{u} \in H^1(\Omega_c;\mathbb{R}^d)$ such that $\overline{u} = 0$ at Γ_D and $\overline{\sigma} \in L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$

Theorem (well-posedness of regularized problem)

For δ fixed, there exists solution $(u^\delta, \varepsilon(u^\delta), \sigma^\delta)$ to the regularized problem (17). The solution satisfies a-priori estimates:

$$\int_{\Omega_c} \Big(\frac{\delta}{2}\|\sigma^\delta\|^2 + M_3 \sum_{i,j=1}^{d} |\sigma^\delta_{ij}|\Big)\, dx \le \int_{\Omega_c} \Big(M_2 + \frac{\delta}{2}\|\sigma^E\|^2 + M_1\|\sigma^E\|\Big)\, dx =: K_1 \tag{18a}$$

$$\frac{1}{2}\int_{Q_c^T} \|\varepsilon(u^\delta)\|^2\, dxdt + \frac{\alpha}{2}\max_{t\in[0,T]}\int_{\Omega_c} \|\varepsilon(u^\delta)\|^2\, dx \le \frac{\alpha}{2}\int_{\Omega_c} \|\varepsilon(u^0)\|^2\, dx + M_1^2|Q_c^T| + 2\delta T K_1 =: K_2 \tag{18b}$$

$$\frac{\alpha^2}{4}\int_{\Omega_c} \|\varepsilon(\dot{u}^\delta)\|^2\, dx \le M_1^2|\Omega_c| + 2\delta K_1 + \frac{2K_2}{\alpha} =: K_3 \tag{18c}$$

Generalized formulation of the problem

For the space of bounded measures $\mathcal{M}^1(\Omega_c)$ which is dual to the space $C_c(\Omega_c)$ of continuous functions with compact support in Ω_c, the generalized problem:

Find $u \in C([0,T]; H^1(\Omega_c;\mathbb{R}^d))$, $\varepsilon(u) \in C^1([0,T]; L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$, and $\sigma \in C([0,T]; \mathcal{M}^1(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$u(0,\,\cdot\,) = u^0 \quad \text{in } \Omega_c \tag{19a}$$

$$u = 0 \quad \text{on } (0,T)\times\Gamma_D \tag{19b}$$

$$\langle \sigma : \varepsilon(\overline{u})\rangle_{\Omega_c} = \int_{\Omega_c} f\cdot\overline{u}\,dx + \int_{\Gamma_N} g\cdot\overline{u}\,dS_x, \quad t\in(0,T) \tag{19c}$$

$$\int_{\Omega_c} \big(\varepsilon(u) + \alpha\varepsilon(\dot{u})\big) : \overline{\sigma}\,dx \le \langle(\sigma-\overline{\sigma}) : \mathcal{F}(\overline{\sigma})\rangle_{\Omega_c} + \int_{\Omega_c} f\cdot(u+\alpha\dot{u})\,dx + \int_{\Gamma_N} g\cdot(u+\alpha\dot{u})\,dS_x, \quad t\in(0,T) \tag{19d}$$

for all test functions $\overline{u}\in H^1(\Omega_c;\mathbb{R}^d)$ such that $\overline{u}=0$ at Γ_D and $\varepsilon(\overline{u}), \overline{\sigma} \in C_c(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$

Theorem (well-posedness of generalized problem)

(i) *As* $\delta \to 0$, *there exists an accumulation point* $(u, \varepsilon(u), \sigma)$ *of the solutions* $(u^\delta, \varepsilon(u^\delta), \sigma^\delta)$ *of the regularized problem* (17).
It solves the generalized problem (19).

(ii) *If the stress is regular such that* $\sigma \in C([0,T]; L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, *then the triple* $(u, \varepsilon(u), \sigma)$ *satisfies the weak formulation* (21) *and a-priori estimates:*

$$\|\varepsilon(u)\|^2 \leq \frac{1}{\alpha} M_1^2 T + \|\varepsilon(u^0)\|^2 =: K_7 \tag{20a}$$

$$\alpha \|\varepsilon(\dot{u})\| \leq \sqrt{K_7} + M_1 \tag{20b}$$

$$M_3 \int_{\Omega_c} \sum_{i,j=1}^{d} |\sigma_{ij}|\, dx \leq M_2 |\Omega_c| + M_1 \int_{\Omega_c} \|\sigma^E\|\, dx \tag{20c}$$

If the monotone property (7) *of* $\mathcal{F}$ *is strict, then the stress* σ *is unique.*

The weak formulation of the problem:

Find $u \in C([0,T]; H^1(\Omega_c; \mathbb{R}^d))$, $\varepsilon(u) \in C^1([0,T]; L^\infty(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, and $\sigma \in C([0,T]; L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$u(0,\,\cdot\,) = u^0 \quad \text{in } \Omega_c \tag{21a}$$

$$u = 0 \quad \text{on } (0,T)\times\Gamma_D \tag{21b}$$

$$\int_{\Omega_c} \sigma : \varepsilon(\overline{u})\,dx = \int_{\Omega_c} f\cdot\overline{u}\,dx + \int_{\Gamma_N} g\cdot\overline{u}\,dS_x, \quad t\in(0,T) \tag{21c}$$

$$\int_{\Omega_c} \big(\varepsilon(u) + \alpha\varepsilon(\dot{u}) - \mathcal{F}(\sigma)\big) : \overline{\sigma}\,dx = 0, \quad t\in(0,T) \tag{21d}$$

for all test functions $\overline{u} \in H^1(\Omega_c; \mathbb{R}^d)$ such that $\overline{u} = 0$ at Γ_D and $\overline{\sigma} \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$

Conclusion

- Limiting strain models provide regularity, boundedness, smallness of strain
- Stress is defined by bounded measures
- Cracks are admissible within the modeling
- Elastic and viscoelastic responses are suitable
- Elliptic regularization provides generalized solution as an accumulation point
- If stress is smooth, then the generalized solution turns into weak solution

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