Non-standard damped oscillators

D. Pražák[1], K.R. Rajagopal[2], J. Slavík[1]

[1]Department of Mathematical Analysis
Charles University, Prague

[2]Department of Mechanical Engineering
Texas A&M University, USA

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OUTLINE:

- history of infinitesimal calculus
- non-standard analysis
- non-standard oscillators

history of infinitesimal calculus

- Eudoxos, Archimedes (408-355 BC, 287-212 BC)
- Zeno of Elea (490-430 BC): paradoxes
- Descartes, Fermat, Pascal, Barrow (17th century)
- Leibniz, Newton (1675, 1666): Calculus Infinitesimalis
- indivisibles banned by jesuits (1632)
- Berkeley (1734): famous critique
- golden (non-rigorous!) era: Euler, Gauss, Riemann, ...
- Bolzano, Cauchy, ... quest for foundations (vs. necessity to teach)
- great triumvirate: Cantor, Dedekind, Weierstrass (end of 19th century): $\forall\epsilon\ \exists\delta$

end of story? (a premature one)

20th century: towards a true infinitesimal calculus

- development of logic: syntax vs. semantics
- Th. Skolem (1934): non-standard models of arithmetic
- Lowenheim-Skolem theorem (1915-1920), Gödel's theorem (1931)
- Abraham Robinson: NSA (1960)
- Elias Zakon: superstructure construction
- Edward Nelson (ca 1970): internal set theory
- Petr Vopěnka (1970): alternative set theory
- still many (famous) critiques: Alain Connes, Paul Halmos, Errett Bishop
- modern advocate of NSA: Mikhail Katz

non-standard analysis

Q: What is NSA ?

A: Some obscure theory to give you infinitesimals rigorously.
(Not worth the labour; we have ε and δ.)

Q: What is NSA, truly?

A: In fact, you not only have NS-A; you can have NS-{measure theory, functional analysis, combinatorics, PDEs, ODEs . . . }

It works for any mathematical theory: the more advanced, the better.

Language vs. universum

Two aspects of mathematical theory:

① universum $\mathcal{U}$: numbers, sets, functions, relations, functionals, . . .

② language $\mathcal{L}$: constants ($\sim$ objects of $\mathcal{U}$), logical symbols, quantifiers, variables, . . .

NSA . . . works with **two universa**: standard $\mathcal{U}$ and enlarged $\mathcal{U}'$ and so-called elementary embedding

$$\begin{aligned} * : \mathcal{U} &\to \mathcal{U}' \\ \alpha &\mapsto {}^*\alpha \end{aligned}$$

*-transform of a formula

On the side of language, one has ***-transform of formula** φ:
${}^*\varphi$ is obtained by replacing any *constant c* by ${}^*c$,
leaving everything else (variables, logical symbols) in place

$$\phi : \qquad (\forall x)\,(\exists y)\ldots P(x, y, \ldots, c_1, \ldots, c_n)$$

$$^*\phi : \qquad (\forall x)\,(\exists y)\ldots P(x, y, \ldots, {}^*c_1, \ldots, {}^*c_n)$$

Note. Examples of constants:
0, 1, π,
$\mathbb{R}$, $\mathbb{N}$, ...
sin, $+$, $\leq$, ...
$C(\mathbb{R}, \mathbb{R})$, $L^2(\Omega)$

Axioms of NSA

Axiom 1. [Transfer.]
A sentence φ holds true in $\mathcal{U}$ iff ${}^*\varphi$ holds in $\mathcal{U}'$

Axiom 2. [Enlargement.] For any *infinite* set A in $\mathcal{U}$, one has

$$\{{}^*a;\ a \in A\} \subsetneq {}^*A$$

Transfer – examples

①

$$(\forall x, y \in \mathbb{R}) \quad x + y = y + x$$

$$(\forall x, y \in {}^*\mathbb{R}) \quad x \,{}^*\!\!+ y = y \,{}^*\!\!+ x$$

②

$$(\forall x \in \mathbb{R}) \quad |\sin(x)| \leq 1$$

$$(\forall x \in {}^*\mathbb{R}) \quad {}^*|{}^*\sin(x)|{}^*\!\!\leq {}^*1$$

Convention. We replace ^{*1} by 1, ${}^*\sin$ by $\sin$, ${}^*\!\leq$ by $\leq$ etc.
Hence in particular $\mathbb{R} \subset {}^*\mathbb{R}$, $\mathbb{N} \subset {}^*\mathbb{N}$
So more suggestively though less rigorously, the transfered formulas read:

$$(\forall x, y \in {}^*\mathbb{R}) \quad x + y = y + x$$

$$(\forall x \in {}^*\mathbb{R}) \quad |\sin(x)| \leq 1$$

Terminology

Pronounciation. $*$ reads as *hyper*, i.e.,
${}^*\mathbb{R}$ … hyperreal numbers, ${}^*\mathbb{N}$ … hypernatural numbers,
${}^*C(\mathbb{R},\mathbb{R})$ … hypercontinuous functions, etc.

Definition. An object $\beta \in \mathcal{U}'$ is called:

- *standard*, if $\beta = {}^*\alpha$ for some $\alpha \in \mathcal{U}$
- *internal*, if $\beta \in {}^*\alpha$ for some $\alpha \in \mathcal{U}$
- *external*, if it is not internal

Enlargement – consequences

① there exist $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, necessarily infinitely large, i.e. $\forall n \in \mathbb{N} : n < N$

② there exist $\alpha \in {}^*(0, +\infty) \setminus (0, +\infty)$, infinitely small, i.e. $\alpha > 0$ such that $\forall x \in \mathbb{R}, x > 0 : \alpha < x$

③ there exist $f \in {}^*C(\mathbb{R}, \mathbb{R}) \setminus C(\mathbb{R}, \mathbb{R})$ ... (hyper)-continuous functions with singular properties

④ Stone-Weierstrass theorem: $f \in C([0, 1]; \mathbb{R})$ continuous can be arbitrarily close approximated by $p \in P[x]$ ... it can be infinitely close approximated by $p \in *P[x]$ ("hyper-polynomial", presumably of infinite degree)

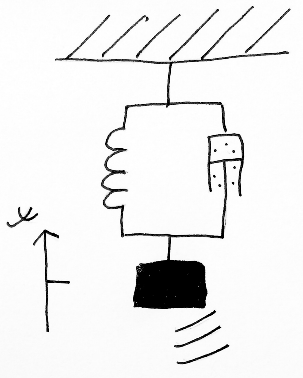
Some concluding observations on NSA

- NSA is conservative extension (no new truths about $\mathcal{U}$)
- makes no sense for *finite* universa
- dark side: external objects (old paradoxes arise), e.g. $\mathbb{N} \subsetneq {}^*\mathbb{N}$ „violates“ the induction axiom
- NSA = to visit a foreign country
- NSA results are hard to publish

oscillators

Standard oscillator

$$x'' + F_d + F_s = f(t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$



Standard = explicit constitutive relations:

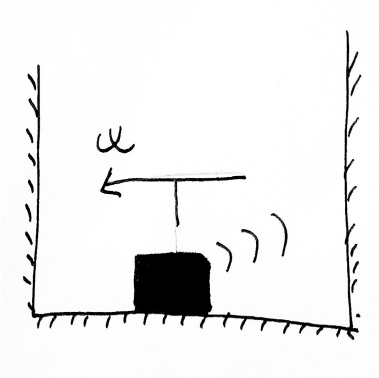
$$F_d = F_d(x, x') \qquad F_s = F_s(x, x')$$

$\implies$ 2nd order ODE, well-understood theory (existence, uniqueness, qualitative analysis, explicit integration for the linear case)

Q: What if F_d, F_s and x, x' cannot be related explicitely?
This is a more common situation than one would expect ...

Non-standard (non-smooth) oscillator

$$\begin{aligned} x'' + F_d + F_s &= f(t) \\ x(0) &= x_0 \\ x'(0) &= v_0 \end{aligned}$$



Assume: F_d and/or F_s cannot be written *explicitely* in terms of x, x'.

In particular, we will consider two situations:

1. Coulomb friction
2. obstacle (bouncing problem, inextensible string)

Coulomb friction

$$
\begin{aligned}
x'' + F_d + s(x) &= f(t) \\
x(0) &= x_0 \\
x'(0) &= v_0
\end{aligned}
\qquad
F_d \begin{cases} = \phi_0, & x' > 0 \\ = -\phi_0, & x' < 0 \\ \in [-\phi_0, \phi_0], & x' = 0 \end{cases}
$$

Can be treated classically, using:

- differential inclusions
- weak convergence
- monotone graph theory

This is a lot of advanced functional analysis ...

Coulomb via NSA

IDEA: replace vertical segments of the graph by an infinitely steep (linear) growth.

Hence: $F_d = \gamma(x')$, where γ is *-Lipschitz function
$\implies$ well-posedness by the usual ODE theory (modulo transfer)
a number of other tools are at our disposal ...

PROBLEM: macroscopic stability, in other words:

Does infinitely small change of initial condition imply infinitely small change of solution ??

In fact, easily solved here due to monotonicity of $\gamma(\cdot)$.

obstacle bounce

$$x'' = f(t) \qquad \text{if } x(t_1) = 0 \text{ with } x'(t_1) = v_1 > 0$$

$$\text{if } x(t) < 0 \qquad \text{then set } x'(t_1) = -\theta v_1, \text{ where } \theta \in [0, 1]$$

Can be solved classically (with some limitations) by „patching“ the solutions together.

BUT: there are some issues, namely close to the wall.

obstacle bounce via NSA

How can we model the bounce, using NSA?

IDEA: for $x > 0$, use classical linear oscillator

$$x'' + Dx' + Lx = 0$$

with D and/or $L > 0$ infinitely large (hyper-oscillator)

obstacle bounce via NSA - equation

More precisely, we need to combine the equations

$$\begin{aligned} x'' &= f(t) \qquad x \leq 0 \\ x'' + Dx' + Lx &= f(t) \qquad x \geq 0 \end{aligned}$$

use a simple trick: set $z = x' + \Delta(x)$, where

$$\Delta(x) = \begin{cases} Dx, & x \geq 0 \\ 0, & x \leq 0 \end{cases} \qquad \sigma(x) = \begin{cases} Lx, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$$

hence we obtain a system

$$\begin{aligned} x' &= z - \Delta(x) \\ z' &= -\sigma(x) + f(t) \end{aligned}$$

This is a *-classical ODE for (x, z).

Discussion: good news

① global existence of (unique) solutions (transfer Picard theorem)

② various types of „bounce“ can be modeled, namely:

- $D = 0$, $L \approx \infty$ . . . an elastic bounce
- $L \approx D^2 \approx \infty$ . . . bounce with partial/complete loss of energy
- $D \approx \infty$, $L = 0$ . . . stick to the wall

Problem

Is the system „stable“, in the sense that infinitely small change of initial condition only results in infinitely small change of solution?

This is not guaranteed by *-Picard theorem (infinitely large Lipschitz constant!)

Some mathematical work has to be done (essentially an estimate independent of infinitely large L, D).

Partial result.
D. Pražák, K. R. Rajagopal, J. Slavík:
A non-standard analysis approach to a constrained forced oscillator.
J. Log. Anal. 9 (2017), 1–22.

thank you