

Boundary Values of Functions with Bounded Deviatoric Variation

Lars Diening



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Problem

Minimize $\mathfrak{F}(u) = \int_{\Omega} f(x, \nabla u(x)) dx$ subject to Dirichlet data u_0 .

linear growth: $c_1 |\nabla u| \leq f(\cdot, \nabla u) \leq c_2 |\nabla u| + c_3$

continuity: $|f(x, Q) - f(y, Q)| \leq \omega(|x - y|)(1 + |Q|)$.

quasi-convexity: linear functions are local minimizers.

Singularities (line discontinuities) will develop!

Questions

Functions spaces: Compactness

Boundary values: Existence of traces

Now, replace ∇u by a first order differential operator Δ .

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Now, replace ∇u by a first order differential operator \mathbb{A} .

Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^N$ and $N(\mathbb{A}) = \{u : \mathbb{A}u = 0 \text{ as distribution}\}$.

Gradient

$$\mathbb{A}u = \nabla u, \quad N(\nabla) = \{x \mapsto b\}.$$

Symmetric Gradient ($n = N \geq 2$)

$$\mathbb{A} = \mathcal{E}u = \frac{1}{2}((\nabla u) + (\nabla u)^T),$$

$$N(\mathcal{E}) = \{x \mapsto Ax + b : A + A^T = 0\}.$$

Trace Free Gradient ($n = N \geq 2$)

$$\mathbb{A} = \mathcal{E}^D u = \mathcal{E}u - \frac{1}{n} \text{Id} \text{div} u.$$

$$n = 2: \quad N(\mathcal{E}^D) = \text{holomorphic functions}.$$

$$n \geq 3: \quad N(\mathcal{E}^D) = \mathcal{N}(\mathcal{E}) \oplus \{2(a \cdot x)x - |x|^2 a\}.$$

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Sobolev Space

$$W^{\mathbb{A},1}(\Omega) := \{u \in L^1_{\text{loc}} : \mathbb{A}u \in L^1(\Omega)\}.$$

$$\|u\|_{W^{\mathbb{A},1}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\mathbb{A}u\|_{L^1(\Omega)}.$$

To obtain some weak compactness we need:

Space of Bounded Variation

$$BV^{\mathbb{A}}(\Omega) := \{u \in L^1_{\text{loc}} : \mathbb{A}u \text{ is a Radon measure}\},$$

$$\|u\|_{BV^{\mathbb{A}}(\Omega)} := \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega).$$

Functions in $BV^{\mathbb{A}}$ may jump:

Take indicator function χ_B of ball B , then $\chi_B \in BV^{\nabla}(\Omega)$ and

$$\nabla \chi_B = -\nu \mathcal{H}^{n-1} \llcorner_{\partial B}.$$

with ν outer normal and Hausdorff measure \mathcal{H}^{n-1} .

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Korn's Inequality

If $1 < p < \infty$, then $W^{1,p}(\Omega) \approx W^{\mathcal{E},p}(\Omega)$.

Ornstein's Non-Inequality

$$W^{1,1}(\Omega) \subsetneq W^{\mathcal{E},1}(\Omega).$$

Additional properties:

Poincaré: $\|u - \Pi u\|_{L^1(B_r)} \lesssim r \|\mathcal{E}u\|_{L^1(B_r)}$.

Embedding: $\|u - \Pi u\|_{L^{\frac{n}{n-1}}(B_r)} \lesssim \|\mathcal{E}u\|_{L^1(B_r)}$.

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Classical Trace

There exists a trace operator $\text{tr} : W^{1,1}(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1})$.

By density it suffices to show estimates for smooth u .

Estimate is based on fundamental theorem of calculus:

$$u(y) - u(x) = \int_0^1 (\nabla u)(x + t(y - x)) \cdot (y - x) dt.$$

Choose $y \in \partial\Omega$ and $x \rightarrow \infty$.

Trace of Functions with Bounded Variation

There exists a trace operator $\text{tr} : BV^\nabla(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1})$.

Same argument. But we only get **interior traces!**

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BV-Trace [Strang-Teman '81, Babadjian '13]

There exists a trace operator

$$\begin{aligned} \text{tr} : W^{\mathcal{E},1}(\Omega) &\rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1}), \\ \text{BV}^{\mathcal{E}}(\Omega) &\rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1}). \end{aligned}$$

Idea: For every direction $a \in \mathbb{R}^n$

$$\begin{aligned} (u(x+a) - u(x)) \cdot a &= \int_0^1 (\nabla u)(x+ta) : (a \otimes a) dt \\ &= \int_0^1 (\mathcal{E}u)(x+a) : (a \otimes a) dt. \end{aligned}$$

Enough to recover u_n with $a = e_n$ and then the other u_j .

A Counterexample

Consider $\mathbb{A} = \mathcal{E}^D$ for $n = N = 2$. Then with $\mathbb{R}^2 = \mathbb{C}$

$$\mathcal{N}(\mathcal{E}^D) = \{u : u \text{ holomorphic}\}.$$

Let $u(z) := \frac{1}{z}$, then $\mathcal{E}^D u = 0 \in \mathcal{D}'(\mathbb{R}^2 \setminus \{0\})$.

Hence, $u \in W^{1,\mathcal{E}^D}(B_1(1))$, but $u \notin L^1(\partial B_1(1))$.

\Rightarrow No trace operator!



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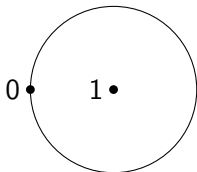
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For $\mathbb{A} = \sum_{\alpha=1}^N \mathbb{A}_\alpha \partial_\alpha$ with $\mathbb{A}_\alpha \in L(\mathbb{R}^N; \mathbb{R}^K)$

define the *symbol mapping* $\mathbb{A}[\xi] : \mathbb{R}^N \rightarrow \mathbb{R}^K$ by

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^n \xi_\alpha \mathbb{A}_\alpha v.$$

Example: $v \otimes_{\nabla} \xi = v \otimes \xi$.

Definition

\mathbb{A} is **\mathbb{R} -elliptic** if $\mathbb{A}[\xi] : \mathbb{R}^N \rightarrow \mathbb{R}^K$ is injective for all $\xi \neq 0$.

\mathbb{A} is **\mathbb{C} -elliptic** if $\mathbb{A}[\xi] : \mathbb{C}^N \rightarrow \mathbb{C}^K$ is injective for all $\xi \neq 0$.

∇ , \mathcal{E} and \mathcal{E}^D are \mathbb{R} -elliptic.

They are also \mathbb{C} -elliptic with the exception: \mathcal{E}^D for $n = N = 2$.

Recall $N(\mathbb{A}) = \{u : \mathbb{A}u = 0 \text{ as distribution}\}$.

Characterization

The following are equivalent

- \mathbb{A} is \mathbb{C} -elliptic.
- $N(\mathbb{A})$ is finite dimensional.
- $N(\mathbb{A})$ is a finite dimensional set of polynomials.

For the variational part we also need that \mathbb{A} is *cancelling*, i.e.

$$\bigcap_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^N) = \{0\}.$$

Lemma

If \mathbb{A} is \mathbb{C} -elliptic, then it is cancelling.

The reverse implication fails.

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Trace Theorem [Breit, Diening, Gmeineder '17]

Let Ω be bounded with $\partial\Omega$ Lipschitz and let \mathbb{A} be \mathbb{C} -elliptic.

Then there exists a trace operator $BV^{\mathbb{A}}(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1})$.

It is continuous with respect to *strict convergence*, i.e. for

$$u_n \rightarrow u \text{ in } L^1 \text{ and } |\mathbb{A}u_n|(\Omega) \rightarrow |\mathbb{A}u|(\Omega).$$

Necessity

If \mathbb{A} is not \mathbb{C} -elliptic, then there is no trace for $BV^{\mathbb{A}}(B_1(0))$.

Idea: Use the $\frac{1}{z}$ example.

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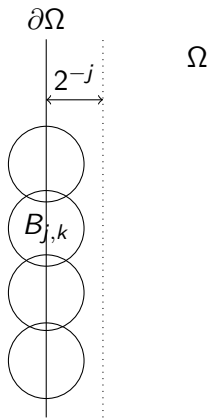
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 $\partial\Omega$ Ω

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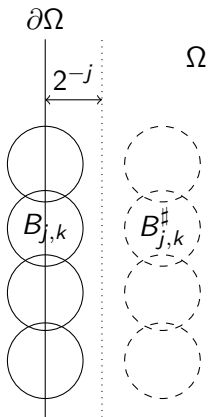


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For every $B_{j,k}$ choose reflected ball $B_{j,k}^{\#}$ in Ω .



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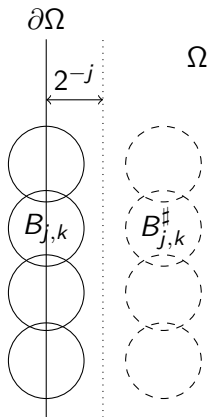
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Project u on $B_{j,k}^{\#}$ to $\Pi_{j,k}u \in N(\mathbb{A})$ and

replace u on $B_{j,k}$ by $\Pi_{j,k}u$ i.e.

$$T_j u := (1 - \rho_j)u + \rho_j \sum_k \eta_{j,k} \Pi_{j,k} u.$$



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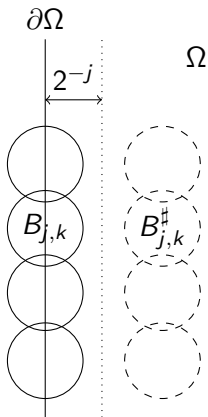
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Then $T_j u \rightarrow u$ in $BV^{\mathbb{A}}(\Omega)$

and $\text{tr}(T_j u) \rightarrow \text{tr}(u)$ in $L^1(\partial\Omega)$.

Based on inverse estimates for polyomials!



Gauß-Green Formula

For $u \in BV^{\mathbb{A}}(\Omega)$ and $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^K)$ we have

$$\int_{\Omega} \mathbb{A}u \cdot \varphi \, dx = - \int_{\Omega} u \cdot \mathbb{A}^* \varphi \, dx + \int_{\partial\Omega} (\operatorname{tr}(u) \otimes_{\mathbb{A}} \nu) \cdot \varphi \, d\mathcal{H}^{n-1}.$$

This allows gluing of $u \in BV^{\mathbb{A}}(\Omega)$ and $v \in BV^{\mathbb{A}}(U \setminus \Omega)$.

Functional with Linear Growth [Breit, Diening, Gmeineder '17]

The functional $\mathfrak{F}(u) = \int_{\Omega} f(x, (\mathbb{A}u)(x)) \, dx$ on $W_{u_0}^{1,\mathbb{A}}(\Omega)$

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Application

The \mathbb{A} -variational problem with linear growth and L^1 -boundary data has a solution in $BV^{\mathbb{A}}(\Omega)$.

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Thank you for your attention.

Problem: $\mathfrak{F}(u) = \int_{\Omega} f(x, (\mathbb{A}u)(x)) dx$ subject to Dirichlet data u_0 .

Recession function: $f_{\infty}(x, A) := \lim_{t \rightarrow \infty, x' \rightarrow x, A' \rightarrow A} \frac{f(x', tA')}{t}$.

Existence [Breit, Diening, Gmeineder '17]

Define $\bar{\mathfrak{F}}_{u_0} : BV^{\mathbb{A}}(\Omega) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \bar{\mathfrak{F}}_{u_0}[\mathbf{u}] &:= \int_{\Omega} f\left(x, \frac{d\mathbb{A}\mathbf{u}}{d\mathcal{L}^n}\right) d\mathcal{L}^n + \int_{\Omega} f^{\infty}\left(x, \frac{d\mathbb{A}\mathbf{u}}{d|\mathbb{A}^s\mathbf{u}|}\right) d|\mathbb{A}^s\mathbf{u}| \\ &\quad + \int_{\partial\Omega} f^{\infty}\left(x, \nu_{\partial\Omega} \otimes_{\mathbb{A}} \text{tr}(\mathbf{u} - \mathbf{u}_0)\right) d\mathcal{H}^{n-1}. \end{aligned}$$

Then $\bar{\mathfrak{F}}_{u_0}$ has a minimizer and $\min_{BV^{\mathbb{A}}(\Omega)} \bar{\mathfrak{F}}_{u_0} = \inf_{u_0 + W_0^{1,\mathbb{A}}(\Omega)} \mathfrak{F}$.