On compressible fluids interacting with an elastic shell

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in collaboration with **Dominic Breit**

Implicitly constituted materials: Modeling, Analysis and Computing

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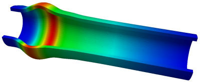
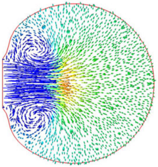
Happy birthday to the chairmen, Eduard and Zdeněk!

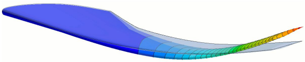
Fluid structure interaction

In this talk we will consider a compressible fluid which is floating in a body that is flexible.

- The fluid forces are interacting with a membrane that is assumed to be a part of the boundary.
- The geometry changes in time.

Examples:

Blood vessels:  Gas balloon: 



Airplane wing:

The Setting

- $\Omega \in \mathbb{R}^3$ is the **initial geometry** and the **reference geometry**
- $\partial\Omega = \Gamma \cup M$, Γ is the fixed part of the boundary
- M is the flexible part of the boundary–hence the **domain of definition** for the time-changing domain
- The displacement of the boundary is prescribed via a two dimensional surface representing a **Kirchhoff-Love** plate.
- It is a model reduction assuming small strains and plane stresses parallel to the middle surface.
- $\eta : I \times M \to \mathbb{R}^3$ defines the change of the domain.
- $\Omega_{\eta(t)}$ defines the changed domain: $\partial\Omega_{\eta(t)} = \Gamma \cup \eta(t, M)$.
- Inside the domain we assume a **compressible** fluid. Its motion is characterized by its **velocity**: $\mathbf{u} : I \times \Omega_{\eta(t)} \to \mathbb{R}^3$ and **density**: $\varrho : I \times \Omega_{\eta(t)} \to \mathbb{R}^+$.

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The PDE in the interior

The Fluid:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \qquad \text{in } I \times \Omega_\eta,$$
$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) = \mu \Delta \boldsymbol{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} - \nabla \varrho^\gamma + \mathbf{f} \qquad \text{in } I \times \Omega_\eta,$$

The Shell:

It is driven by the **Koiter-Energy**

$$K(\boldsymbol{\eta}) = \frac{1}{2}\varepsilon_0 \int_M \mathbf{C} : \boldsymbol{\sigma}(\boldsymbol{\eta}) \otimes \boldsymbol{\sigma}(\boldsymbol{\eta})\, dH^2 + \frac{1}{6}\varepsilon_0^3 \int_M \mathbf{C} : \boldsymbol{\theta}(\boldsymbol{\eta}) \otimes \boldsymbol{\theta}(\boldsymbol{\eta})\, dH^2.$$

The corresponding momentum equation is

$$\varepsilon_0 \varrho_S \partial_t^2 \boldsymbol{\eta} + K'(\boldsymbol{\eta}) = \mathbf{g},$$

K' is the L^2-gradient of K, ϱ_S is the density of the shell, ε_0 the thickness. Assuming that $\boldsymbol{\eta}(t,x) \equiv \eta(t,x)\nu(x)$ is moving in the fixed direction ν, the normal of $\partial\Omega$ one deduces

$$\varepsilon_0 \varrho_S \partial_t^2 \eta + \Delta^2 \eta + B\eta = g.$$

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The continuity equation

We assume $\eta : I \times M \to \mathbb{R}$
$\partial\Omega_{\boldsymbol{\eta}(t)} = \Gamma \cup \{x + \eta(t,x)\nu(x) : x \in M\}$.
And a coordinate map $\boldsymbol{\Psi}_\eta : \Omega \to \Omega_\eta$.
Reynolds transport theorem:

$$\frac{d}{dt}\int_{\Omega_{\eta(t)}} g\,dx = \int_{\Omega_{\eta(t)}} \partial_t g\,dx + \int_{\partial\Omega_{\eta(t)}} \partial_t \eta \circ \boldsymbol{\Psi}_\eta^{-1}\nu \cdot \nu_\eta g\,dH,$$

The weak continuity equation: Partial integration implies for $\psi \in C^\infty(I \times \overline{\Omega})$

$$\int_I \frac{d}{dt}\int_{\Omega_\eta} \varrho\psi\,dx\,dt - \int_I\int_{\Omega_\eta}\Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,dx\,dt = 0,$$

if $\mathbf{u}\circ\boldsymbol{\Psi}_\eta = \partial_t\eta\nu$ on $\partial\Omega_{\eta(t)}$. Testing with $\psi \equiv 1$ implies mass conservation.

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The coupled system:

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) &= 0, && \text{in } I \times \Omega_\eta,\\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) &= \mu \Delta \boldsymbol{u} + (\lambda+\mu)\nabla \operatorname{div} \mathbf{u} - \nabla \varrho^\gamma + \mathbf{f} && \text{in } I \times \Omega_\eta,\\
\mathbf{u}(t, x + \eta(x)\nu(x)) &= \partial_t \eta(t,x)\nu(x) && \text{on } I \times M,\\
\mathbf{u} &= 0 && \text{on } I \times \Gamma,\\
\varepsilon_0 \varrho_S \partial_t^2 \eta + K'(\eta) &= g + \nu \cdot \big(-\boldsymbol{\tau}\nu_\eta\big) \circ \boldsymbol{\Psi}_{\eta(t)} \lvert \det D\boldsymbol{\Psi}_{\eta(t)} \rvert && \text{on } I \times M,\\
\boldsymbol{\tau} &:= -\mu \nabla \mathbf{u} - (\lambda+\mu) \operatorname{div} \mathbf{u}\, \mathcal{I} + \varrho^\gamma \mathcal{I}. &&\\
\eta(t,x) &= 0 && \text{on } \partial M\\
\varrho(0) = \varrho_0, \quad (\varrho \boldsymbol{u})(0) &= \mathbf{q}_0 && \text{in } \Omega\\
\eta(0,x) = 0, \quad \partial_t \eta(0,x) &= \eta_1(x) && \text{in } M
\end{aligned}
$$

Here $g : [0, T] \times M \to \mathbb{R}$ and $\mathbf{f} : I \times \mathbb{R}^3 \to \mathbb{R}^3$ are given forces.

Weak formulation

We assume that $\varrho_S \varepsilon_0 = 1$.

"The momentum equation": For $(b, \varphi) \in C_0^\infty(M) \times C^\infty(\bar{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta \boldsymbol{\varphi} = b\nu$

$$\int_I \left(\frac{d}{dt} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \mathrm{d}x - \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \mathrm{d}x \right) dt$$

$$+ \int_I \int_{\Omega_\eta} \left(\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi} \, dx \, dt - a \varrho^\gamma \operatorname{div} \boldsymbol{\varphi} \right) dx \, dt$$

$$+ \int_I \frac{d}{dt} \int_M \partial_t \eta b \, dH - \int_M \partial_t \eta \, \partial_t b \, dH + \int_M K'(\eta) \, b \, dH \mathrm{d}t$$

$$= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, dx \, dt + \int_I \int_M g \, b \, dH \mathrm{d}t.$$

"The renormalized continuity equation": For $\psi \in C^\infty(I \times \overline{\Omega})$ and $\theta \in C^1(\mathbb{R}^+)$ positive

$$0 = \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \theta(\varrho) \, \psi \, dx \, dt - \int_I \int_{\Omega_\eta} \left(\theta(\varrho) \partial_t \psi + \theta(\varrho) \mathbf{u} \cdot \nabla \psi \right) dx \, dt$$

$$+ \int_I \int_{\Omega_\eta} (\varrho \theta'(\varrho) - \theta(\varrho)) \operatorname{div} \mathbf{u} \, \psi \, dx \, dt.$$

Weak formulation

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"The momentum equation": For $(b,\varphi) \in C_0^\infty(M) \times C^\infty(\bar{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta \varphi = b\nu$

$$\begin{aligned}
&\int_I \left(\frac{d}{dt} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \varphi \mathrm{d}x - \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \mathrm{d}x \right) dt \\
&\quad + \int_I \int_{\Omega_\eta} \left(\mu \nabla \mathbf{u} : \nabla \varphi + (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \varphi \, dx\, dt - a\varrho^\gamma \operatorname{div} \varphi \right) dx\, dt \\
&\quad + \int_I \frac{d}{dt} \int_M \partial_t \eta b \, dH - \int_M \partial_t \eta \, \partial_t b \, dH + \int_M K'(\eta) \, b \, dH \mathrm{d}t \\
&= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \varphi \, dx \, dt + \int_I \int_M g \, b \, dH \mathrm{d}t.
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&+ \int_I \int_{\Omega_\eta} (\varrho \theta'(\varrho) - \theta(\varrho)) \operatorname{div} \mathbf{u} \, \psi \, dx \, dt.
\end{aligned}$$

Main theorem

Theorem

Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). There is a weak solution $(\eta, \mathbf{u}, \varrho)$. The interval of existence is restricted only in case $\Omega_\eta(s)$ approaches a self intersection with $s \to T_*$.
The solution satisfies the energy estimate

$$\sup_{t\in I}\int_{\Omega_\eta} \varrho|\mathbf{u}|^2 \mathrm{d}x + \sup_{t\in I}\int_{\Omega_\eta} \varrho^\gamma \mathrm{d}x + \int_I\int_{\Omega_\eta} |\nabla \mathbf{u}|^2 \, dx\, dt$$
$$+ \sup_{t\in I}\int_M |\partial_t \eta|^2 \; d\mathcal{H}^2 + \sup_{t\in I}\int_M \left|\nabla^2 \eta\right|^2 \leq \; c(\mathbf{q}_0, \rho_0, \mathbf{f}, g, \eta_1)$$

provided that $\eta_1, \varrho_0, \mathbf{q}_0, \mathbf{f}$ and g are regular enough to give sense to the right-hand side.

The incompressible analogue was shown by Lengeler & Růžička, (ARMA, 2014). For non-Newtonian fluids of p-growth by Lengeler (SIMA, 2014).

Problems of the proof

- The system is highly coupled; a fixpoint argument is needed.
- The system is highly non-linear, compactness is needed.
- The regularity of the variable domain is **not Lipschitz**.

To be able to construct a solution we introduce a **four layer approximation**.
Once the fixpoint is established, we get a weakly converging subsequence **by uniform a-priori estimates**:

$$(\varrho_k, \mathbf{u}_k, \eta_k) \rightharpoonup (\varrho, \mathbf{u}, \eta).$$

Further, we need:

- Pass to the limit with $K'(\eta_k)$.
- Pass to the limit with the convective terms.
- Pass to the limit with the pressure: ρ_k^γ.

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Reconstruction of the pressure

The reconstruction of the pressure is (as usual in compressible problems) the major difficulty.

It splits up in three parts

- $\varrho_k^\gamma \chi_{\Omega_{\eta_k}} \rightharpoonup \overline{p}$ in $L^1(I \times \mathbb{R}^3)$, namely excluding concentrations.
- The effective viscous flux, i.e. exploiting some crucial structure of the momentum equation.
- Use the above to show that ϱ is a renormalized solution.

This can then be used to show the strong convergence by using the strictly convex quantity $\varrho \log \varrho$ satisfies a weak equation.

Outlook:

Open problems for (2-D,3-D, incompressible (Stokes), compressible):

- Allowing deformation in all directions $\boldsymbol{\eta} : M \to \mathbb{R}^3$.
- Strong solutions (short time, 2-D).
- The non-flexible case: K depends on the 1. Fundamental form only. Open problem: Long time weak solutions
- Regularity of the membrane. In particular: Exclude self intersections.
- The full Navier Stokes Fourier system.
- The low Mach number limit.
- Numerics. In particular constructive schemes. (Some work has been done by Mucha et all).

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