PARTICLE FREE BODIES

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Abstract. Few notions in mathematics and physics are as fundamental and useful as the notion of a "point". However, in addition to the concept of a "point" being far from apparent, the concept is not suitable for describing several important problems in natural philosophy. A far more tangible and sensible idea that is immediately grasped by our mind is that of a "chunk" (a solid object) which seems ideally suited to describe precisely and felicitously several problems which cannot be accurately described within the classical construct that exploits the conception of a "point"; in fact, the notion of a "point" not only comes in our way of providing an accurate identification of the problem, it obscures the quintessential features of the problem and hinders our attempts at finding a sensible to solution to it. In this short paper, I articulate the need for the use of topologies which are "point free" for the proper resolution of many important problems in natural philosophy.

1. **Introduction.**

I resisted the temptation to try and appear droll and title this work "Pointless bodies" as I remembered reading the witticism "*A celebrated reviewer once described a certain paper (in a phrase which never actually saw publication in Mathematical Reviews) as being concerned with the study of "valueless measures on pointless spaces*" in Johnstone's article titled "The point of pointless bodies" (see Johnstone (1983)); I was apprehensive that this paper might suffer a review which actually sees publication that expresses the sentiment that it is a "meaningless paper on pointless bodies".

The subject matter of this paper, namely what one means by a body and the space that it occupies and how one mathematically models it, has been the object of study from ancient to current time without any consensus with regard to the answer to these questions. These questions have attracted the attention and interest of philosophers concerned primarily with metaphysical issues, logicians interested in the foundations of natural philosophy, rigorous topologists and analysts, physicists, and engineers, and the conclusions of the various members of the group resembling Saxe's rendition of the Indian tale of the six blind Indians describing the form of the elephant. Even a cursory glance at the various writings on the subject, papers, review articles and books, reveals the insular interests of the members of the different groups that seem to be either ignorant of the interests and investigations of the other groups, or if aware obviously dismissive of the views of the other groups. I guess one should not be astonished at such ignorance amongst specialist scholars as this is true of even the best amongst us as evidenced by Collingwood's (1986) remarks that as distinguished a philosopher as Alfred North Whitehead was unaware of some of the most important works of Aristotle; such ignorance may have been all for the good as oftentimes pronouncements and writings of authorities can have a chilling effect on progress[1].

Before we get into a discussion of what one means by a body in continuum mechanics, which is the main thrust of this work, it is necessary to start with a discussion of the simplest model for a body that was used in Newtonian mechanics, namely the notion of a body in Newtonian Particle Mechanics. Within the construct of Particle Mechanics, a body is mathematically idealized to be a "point mass", that is it is assumed that a body can be represented as a "point"[2] mathematically, with the "point" having a finite mass.

The notion of a "particle" in physics, and a "point" in mathematics are idealizations, the former is supposedly an approximation of "reality" in physics and the latter a conceptualization of a primitive variable in geometry. Both these notions have served their respective fields so well that the scientific temper of a person that suggests that the applicability and usefulness of the concepts be examined, to be more accurate re-examined and carefully delineated, would be called into question. This notwithstanding, even at the cost of being ridiculed, in this short paper I argue that the applicability of the concept of "point", as it pertains to the problems of natural philosophy needs to be reassessed, especially in light of our inability to address many important issues within such a construct. In fact, the notion as it is conceived today is an impediment in the proper resolution of a variety of problems in natural philosophy, and I shall discuss these problems in the course of this article.

Whether it is classical continuum mechanics or more modern theories of physics like Quantum Mechanics[3], one has to deal with the notion of a body, and the equations that govern the motion of matter are given through field equations[4]. Since we are defining field equations that hold at every "point" in the space of interest, we are dealing with equations that apply at "points" belonging to a certain space (in classical mechanics) or space-time (in more modern aspects of physics). It is far from clear that one can in fact specify field equations for the quantities that we introduce in physics as matter does not occupy every point in space wherein the body seems to lie as far as the naked eye is concerned. In Quantum Mechanics, "probabilities" are associated with the physical variables that appear in the governing equations. Also, there is supposedly an "uncertainty" with regard to the precision with which certain pair of physical quantities, such as position and momentum, can

[1] Roger Bacon's (Roger Bacon (1267)) warning concerning the impediments to our grasping the truth is most appropriate: "*Now, there are four chief obstacles in grasping truth, which hinder everyman, however, learned, and scarcely allow anyone to win a clear title to learning, namely submission to faulty and unworthy authority, influence of custom, popular prejudice and the concealment of our own ignorance accompanied by an ostentatious display of our knowledge.*"

[2] We need to recognize that there are two questions that need investigation and need to be distinguished during the course of this article. The first question is concerned with what we mean by a "point" and the second questions concerns whether such a notion can be used in the construction of extended solid objects.

[3] It would not be unreasonable to question the validity of Quantum Mechanics as propounded by the Copenhagen school. While the proponents of Quantum Mechanics could point to the fact that its predictions are in agreement with numerous experimentally observed results, it is important to bear in mind that such successful comparisons concern very simple systems. To date, Quantum Mechanics has not been used to predict phenomena concerning sufficiently complex systems of molecules.

[4] Clifford (1882) suggested a theory of matter wherein the movement of matter through space is the propagation of the variation of the curvature of space.

be determined simultaneously. A few words about the role probability and "uncertainty" play in physics are in order. Whether there is "uncertainty" with regard to simultaneously knowing physical quantities with precision, and whether occurrence of natural phenomena need to be prescribed in a probabilistic manner is an issue that needs some discussion. Whether it is our ignorance[5] of the precise makeup of the natural world that is the ground spring for the apparent lack of certainty that prompts us to provide a probabilistic description for them and the laws governing them, is an unsettled question despite the legions that subscribe to the conventional wisdom of Quantum Mechanics. While most physicists cling to the point of view that natural phenomena are inherently random and non-deterministic, holding up the successes of their theories as validating and justifying them, the point of view for a more deterministic approach to Quantum Mechanics, which I subscribe to, is well articulated by the likes of Bohm (see Bohm (1957), Bohm and Hilley (1993),). If indeed there is no randomness or uncertainty[6] with regard to the occurrence of events, our assigning a probability and then carrying out a set of deductions based on a set of rules is no different than devising a devilishly clever game; and the game however demanding of one's intelligence is no more than a game. David Hilbert (see Rose (1888)) is supposed to have said "*Mathematics is a game played according to certain simple rules with meaningless marks on paper*"[7]. Whether Quantum Mechanics, as practiced by those that assign probabilities, is merely a very interesting and challenging game or whether it is in fact a true depiction of nature is a matter that is sub judice. It is also important to recognize that there will always be an "imprecision" with whatever is being measured as that which is being measured has to be sensed by "something" and the very act of measurement with "something" interferes with that which is being measured, and it is impossible to "measure" with "nothing".

When it comes to a body, it seems likely that the very approximation of the body as a field gives rise to the lack of precision in the specification of field quantities in modern physics rather than there being true uncertainty. That is, instead of assuming that there exist "smallest entities" that are discrete and have dimensions,

[5]Probabilities ascribed to quantities in physics are no more than "indices of ignorance". While it might sound anachronistic to espouse the views of Laplace (Laplace (1814)):"*We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes*", at this juncture in time, I feel that if we do not identify the "intellect" that Laplace refers to with Laplace's Demon or Laplace's Superhuman, but merely recognize that science at some point in time might make it possible for scientist to be aware of all the information that is necessary to make what is deemed to be random now, deterministic, then I feel that Laplace's sentiments are definitely as sensible a point of view , if not a more sensible view than that held by the Copenhagen School.

[6]The deterministic point of view has its own proponents; and the oft quoted comment "God does not play dice" by Einstein espouses such a point of view. Such a proponent is not expressing his religious faith, his intent is merely that "Nature does not play dice".

[7]Of course, mathematics is much more than merely a game. I believe mathematics to be a language, the best language in which to express the secrets of science. The great scientist Josiah Willard Gibbs is supposed to have said at a faculty meeting at Yale during a discussion on whether more time should be spent learning foreign languages at the expense of mathematics that "*Mathematics is a language*". We need to use the right terminology to express the secrets that nature hold and mathematics seems to be the language that is best at that.

and that a body is comprised of such entities and of gaps between such entities, with incertitude with regard to the precise location of such entities and the gaps, but assuming that one has a continuum comprised of "points" (or that one can define field equations), might be the source of uncertainties associated with field quantities that are assumed in modern theories. Whether it is classical continuum mechanics or more modern aspects of physics, the notion of "points" and "field equations" might be a source of the problems that inhibit and obscure our understanding of the true underlying physics.

One could justifiably argue that while matter might not fill up space without leaving holes, electrical and magnetic fields do fill up all space and hence using Maxwell's equations does not require the use of "point free" topology. Of course, we yet have to deal with the possibility that space is itself not a continuum. This is not a matter that can be dismissed offhand. In this work, I am not concerned with how the uncertainties that are associated with the specification of quantities arise. I am concerned about what one means by a body and how it ought to be approximated, and my main thesis is that there are several problems in mechanics wherein one needs to approximate a real body using the notion of "point free" topology, while it is a perfectly good approximation to use standard topology for other applications. With regard to the applicability of Maxwell's equations, one should recognize that it is also an approximation of reality; it is perfectly reasonable and useful to use it when such as approximation is felt to be reasonable and leads to reasonable results. I shall not get into a more extensive discussion of the applicability and use of modern theories of physics, the role of "uncertainty" or "randomness" in physics and the disquieting dichotomy concerning the "particle"/ "wave" nature of matter, and other contentious and debatable issues for the following reason; I am not qualified to talk about them and those that seem to be qualified feel that there is yet a lot that needs to be understood before one can even start to answer such questions with any degree of certainty.

In classical continuum mechanics, as it stand now, the notion of a "particle" plays a fundamental role in describing the make-up of the body and the notion of a dimensionless "point" supposedly convince the reader that the structure of a body comprised of "particles" is not suitable for describing and resolving many important problems that arise in continuum mechanics.

2. **The need for "point free" topology to describe a body.**

The history of natural philosophy bears witness to the fact that the notion of a "point" was a contentious one. A "point" in mathematics is understood to habit an abstract space[8], while in the case of a "particle" it is supposed to lie in "physical space". We first have to understand what is meant by an abstract mathematical space as well as what is meant by physical space. This is not as simple as it seems. In his seminal paper which has had a tremendous impact on the development of Mereology[9], Whitehead (1979) discusses the need to distinguish between four types of spaces which a natural philosopher has to reckon with. He identifies space as a "complex of relations between certain objects", and then discusses at length the distinction between "immediate apparent space", "complete apparent space",

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[8]In solid geometry this space is supposed to be the mathematical representation of the physical space in which solid objects lie.

[9]Mereology is the formal study of the relations between parts and wholes. The word was coined by the Polish logician Stanislaw Lesniewski.

"physical space" and "abstract mathematical space". I shall not get into a discussion of the distinction between these various concepts but refer the reader to the paper by Whitehead (1916). According to Whitehead, the first two spaces are observer dependent, "physical" space is the same for everyone and reflects a complex of relations between the objects in the universe. "Abstract space" is the space of abstract geometry where the rules of abstract geometry apply and such a space is not connected with any specific application. We are mainly interested in two of the four spaces that Whitehead (1916) introduces, the "abstract space" where the rules of abstract geometry are valid and "physical space" which is relevant to the space wherein the objects of this universe lie. At this juncture it suffices to say that the distinction between these two concepts, as well as what we mean by a "point" and a "particle" are far from being self-evident. Our understanding of these concepts hardly qualifies their being given the status of axioms[10].

I feel that the physical notion of "position", as well as the mathematical notion of a "point", pertains to the existence of an appropriate "smallest unit" in the physical and mathematical space respectively. The physical notion of a "particle" pertains to the "smallest unit" in a body, which lies in physical space. It is important to recognize that requiring a body to have a "smallest unit" is different from requiring that the "physical space" has a smallest element. Physical space might or might not have a smallest element[11], while the body does.

The concept of atomism[12] as espoused by the Carvakas[13] and the Jains[14] and Leucippus and Democritus seems to be a reasonable way to think of the building blocks of a body, though not in the exact form championed by them. That a real body has a smallest element seems to be incontestable and beyond questioning whether one is interested in describing a "Higgs-Boson" or something even smaller or a "String" whose existence no one has experimentally corroborated. Whether real space is atomistic might however be contested and debated. The idea that there is a smallest building block or a set of building blocks of finite size seems to be of interest to some modern physicist who subscribe to the view that there is such a smallest length that they refer to as "Planck length" and that space-time has a granular structure.

Most of us are perfectly willing to accept the special theory of relativity that claims that there is an upper bound for the speed of bodies, however small such bodies may be. Such an upper bound is not borne out of any physical compulsion,

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[10]Here, by an "axiom" we mean the usage of the terminology in the following sense: "*An Axiom, or a Maxim, is a self-evident proposition, requiring no formal demonstration to prove its truth; but received and assented to as soon as mentioned*" (See C. Hutton, O. Gregory). We choose this definition not because Hutton is regarded as an authority in mathematics, philosophy or etymology. This definition for an Axiom can be found in the Oxford English Dictionary (2000).

[11]Weyl's tile argument concerning this issue and the conclusion that there cannot be a smallest unit for space is far less compelling than it is made out to be.

[12]The notion that there exist "smallest" building blocks that make up all matter was an idea that was expressed by Indian and Greek philosophers. That idea was also used in Linguistics in India where the existence of a "smallest linguistic unit" was proposed. While some averred it was "words" others proposed that the basic building blocks were sentences.

[13]The Carvakas were atheists who did not accept the authority of the Vedas, which most of the popular Indian philosophical systems do. Carvakas rejected the role of inference in logic. The philosophy of the Carvakas influenced the logic used by Buddhists and Jains.

[14]Jainism is an Indian philosophical system that does not accept the authority of the Vedas. It is older than Buddhism and its most famous teacher, Mahavira is considered to be the 24th Tirtankara (Supreme Teacher).

the violation of which would be inconceivable on physical grounds or for that matter thought experiments. It is also not a bound that can be deduced on the basis of logical reasoning or inference based on uncontestable known physical facts. It is merely an assumption and in fact there are physicists that question such an upper bound. If one is willing to accept such an upper bound for speeds based on the special theory of relativity, why can we not entertain the possibility that there is a lower bound, that is, that there is an elementary unit, a monad, for real physical space? Of course this does not preclude the possibility of the mathematical idealization of a body to be endowed with the usual topology that allows for the existence of particles.

The main thesis of the paper is the need for the development of a theory which allows for the existence of a "smallest entity" as the building block for a body, and its use, not in Quantum Mechanics or Relativity Theory, but to applications in classical continuum mechanics.

A great deal has been written concerning the metaphysics of extended bodies, boundaries and that of contact. In his dissertation on whether a linear continuum can be an aggregate of "points" viewed as unextended elements, Grunbaum (1952)[15] is interested in delineating those aspects of Cantorean set theory that ensures that there is no inconsistency that arises from allowing an extended continuum being a consequence of the aggregation of unextended objects. Grunbaum (1952) is interested in determining the logical structures of set theory that will allow for consistency between the notion of "points" and the "real line". Grunbaum (1952) and most others interested in philosophical issues and logic are concerned with developing a consistent set theory wherein an extended continuum is built with dimensionless "points", they do not seem interested in the aggregation of discrete dimensional units leading to extended objects, and they are also not interested in topological issues and in particular in "point free" topology. For instance, the paper titled "Indivisible Parts and Extended Objects: Some Philosophical Episodes from Topology's Prehistory" (Zimmermann (1996a)) and references therein (see also Zimmermann (1996b)) there is no mention of Hausdorff, Stone, and others that are considered the founders of "point free" topology. My interests however are distinct from the efforts of the above investigators. I am not interested in trying to justify the use of dimensionless "points" to build extended solid objects. My interest concerns the use of objects which have dimension, which can be used as the "smallest building blocks" for extended solid objects. More importantly, these investigations into the construction of a continuum with dimensionless "points" are not concerned with difficulties that arise in physics by introducing the notion of dimensionless "points", the main concern that is addressed in this work.

I will mention just a few papers concerning the development of a basis for solid geometry, that does not rest on the "notion" of a point, that can be traced to the seminal work of Whitehead, titled the "Relational Theory of Space" (Whitehead (1916)). De Laguna (1922), in Part I of his paper on the "Nature of Space", states that his "*principal object is to exhibit the conception of space in its usual setting in experience*" and then he goes on to discuss the main differences between his approach to space and those of Whitehead's, the details of which we shall not get into here. De Laguna's object is to determine

[15]Grunbaum (1952) provides a long list of references concerning the paradox that arises from the early observations of Zeno with regard to an extended object being comprised of unextended dimensionless entities.

"*— the empirical foundations of geometry (so far as they may properly be held to lie within the limits of a physical, rather than a psychological, inquiry). The point and the peculiar relation between points, which in the mathematical part, figure as primary assumptions in terms of which explanation is to be made, are here themselves the goal that is to be attained.*"

The important question is what are the indefinables from which one starts to develop geometry and in this context, it seems that many chose to start with the notion of a "point" as the starting point. De Laguna (1922a) observes

"*What was not anticipated in the old days is the fact that a considerable amount of freedom of choice is possible in choosing the indefinables. The entity chosen is, indeed, the point.*"

and

"*In the case of geometry the deductive stage waited upon the conception of a point-of that which has position*[16] *and no magnitude... The existence of points is attested by general axioms, which recognize no distinction between possible points and actual points. Take away all bodies, and the system of points is conceived to be the same. Thus the system of points is space as such-space which may be full or empty without its characteristic properties being affected in any way.*"

However, De Laguna recognizes that the notion of a solid body precedes and is much more intuitive than the notion of a "point", and that the concept of a "point" ought to be derived from the notion of a solid body. I agree with De Laguna and those that hold such a point of view. As a starting point, as an indefinable, an axiom, the notion of a solid body is much more self-evident than the notion of an entity that is dimensionless. From times immemorial to the present, the most difficult concept to comprehend, even more difficult than that of infinity, is the concept of "nothingness" or the mathematical "zero".

De Laguna (1922b) observes that

"*First in order stands the physical solid. In some rough fashion this conception must have belonged to men since they began to reflect at all.*"

De Laguna (1922a)) starts with the notion of a "solid" and the relation "can connect" as the indefinables and derives the notion of a "point". He defines the notion of an abstractive set by requiring:

1. *Of every two members of the set one is contained in the other;*
2. *There is no member contained in every other member of the set; it is called an abstractive set*[17].

Finally he defines a "point" through "*A point is an abstractive element in which no other abstractive element lies.*" I shall not discuss his work in detail, but it

[16]This footnote is inserted by me. In the above comment of De Laguna as well as in the Oxford English Dictionary's definition of "point", the notion of a "position" is introduced as though it is a self-evident idea. If "position" is not supposed to have dimension, then it is no more self-evident than the notion of a "point" and equally problematic. The intent seems to be that the notion of "position" is intrinsic to physical space, but what if the space has a "smallest unit"? Why should one assume that physical space not have a "smallest unit".

[17]The following footnote is included in his definition of an abstractive set: "*This definition departs from Prof. Whitehead's by substituting the relation for containing for including as a part. There is the further difference, that the sets which he deals with are not set of solids but sets of four-dimensional events.*"

seems that the definition of a "point" as an abstractive element that is partless is more in keeping with his idea of what one might mean by a point. Interestingly, there is really no mention of the notion of "point free" topology or even endowing a body with a topology. De Laguna and other logicians and philosophers like him are not interested in topological issues. They are merely interested in the set theoretic structure that can be used for deriving the notion of a "point".

There have been rigorous attempts to deriving the notion of a "point" starting from the notion of "chunks". Tarski (1956) starting with the notion of sphere as a primitive[18] and introducing the notion of one sphere "being a part" of another, and of two spheres "being diametrical" and two spheres being internally and externally "tangent" to one another, and introducing an equivalence relation of "concentricity" shows that the notion of a point is an equivalence class:

"*A point is a class of all spheres which are concentric with a given sphere.*"

While Tarski (1956) is able to derive the notion of a "point" from the notion of spheres ("chunks"), his derivation of a "point" is in fact quite a complicated idea in that it stands for the an equivalence class, the purpose of this paper is to point out that such a definition is not in the least helpful to the problems which we wish to address in natural philosophy; these are problems wherein the notion of a "point" comes in the way of our mathematically defining the problem so as to faithfully describe the physics. Deriving the notion of the "point" as Tarski has done helps us in no way whatsoever in overcoming the infelicity to physics or the resolution of the mathematical singularities that this notion of a "point" entails. Giving meaning to the notion of a "point" using the equivalence relation and then using the notion of a "point" is a tremendous impediment to the proper resolution of many problems in natural philosophy. What is necessary in all these important technological problems is to do away with the notion of a "point" altogether, introduce a "point free" topology to describe the body, develop the attendant calculus, and proceed to resolve such problems. The above remarks do not apply to Tarski's work in general, it only pertains to this specific paper wherein he derives notion of a point. In his paper of 1956, Tarski is not interested in answering the question as to what ought to be the appropriate topology with which a body ought to be endowed. His aim was to show that the notion of a "point" is not primitive and can be derived from a more intuitive notion, that of a "solid body". As Johnstone (2001) remarks (see the quotation that follows), the work of Tarski (with McKinsey) can be viewed as one of the progenitors of "point free" topology, however, even in those works McKinsey and Tarski (1944), (1946) are not interested in whether a real body has to be described within the context of "point free" topology. Tarski (1956) and Mckinsey and Tarski (1944), (1946) are interested in mathematical issues, not mechanics.

A discussion of the development of geometry without using the notion of points can be found in the paper by Gerla and Volpe (1985) and the several references cited therein (see A. Grzegorczyk (1960), Huntington (1961), Sullivan (1971), and Clarke (1982)). For instance, Huntington (1961) also starts with the "sphere" as the fundamental concept and defines a "point" as a "sphere" in which no other "sphere" is contained, that is a point is a "part less sphere", a definition that is more in keeping with the early Greek notion of a "point". However, in none of these papers is the notion of using "point free" topology to describe a body articulated.

[18]While I find it impossible to accept the notion of a "point" as a primitive, I find it far easier to accept the notion a "sphere" or a "chunk" as a primitive.

As this work is mainly concerned with advancing the view that there are several problems in natural philosophy where it is necessary to use "point free" topology to describe a body, we shall not discuss all these papers in logic concerning the mathematical discussion of extended objects.

While a real body occupying real physical space can be thought of as being placed in an abstract mathematical space, and while the mathematical space may be endowed with a topology to which points belong, it is yet possible that the real body that is placed in it can have a "smallest element" that has non-zero volume. It might also be possible to define abstract mathematical spaces wherein the body that is placed could be endowed with "point-free" topologies. I feel that the framework that is appropriate for describing a real body[19] with regard to the description and resolution of many important problems in natural philosophy is that of a body that is endowed with a "point free" topology.

One could idealize and approximate such a real body in a variety of ways depending on the problem that we are interested in tackling, and thus it would be most reasonable to endow a real body with the usual topology for numerous applications in natural philosophy. However, I hasten to add that describing it by means of a "point free" topology is closer to reality than the usual description of a continuum with the usual topology that admits "points". Later, I discuss several problems in natural philosophy that are inadequately and inaccurately described when one uses the usual topology with the notion of "points". I do not try to provide answers as to how to develop the Calculus that goes hand in hand with "point free" topology so that one can define the notion of a derivative and the equations governing the response of bodies. That has to be left in the hands of those that are better trained in mathematics. It is my aim to identify the problems and suggest the basic framework that seems appropriate for overcoming the difficulties.

As mentioned earlier, in order to describe a body within the context of a continuum, according to the current mathematical thinking, it is to be endowed with a topology and a measure[20]. A topology, in its most general understanding, is a lattice[21] of open sets[22]. Usually a topological space is assumed to consist in a set of points, but this is not necessary. Different approximations of a real body can be captured and described mathematically by different topologies. It is perfectly reasonable, and as we shall show, even necessary, in order to describe important problems in natural philosophy to endow the body with different topologies, one of them being a "point free" topology. One should never lose sight of the fact that we merely approximate real physical objects, and express these approximate idealizations in the language of mathematics. Thus, it is perfectly reasonable to mathematically idealize the same body, in different ways, in order to study different problems. We need to choose the appropriate mathematical idealization that is merely an approximation of the body that fits the problem on hand. The fact

[19]Of course, it is essential to have a clear idea as to what one understands by a real body in physical space, and how one mathematically defines a real body. Unlike the Bishop of Cloyne to whom a body did not exist and was merely a "bundle of ideas" (see Berkeley (1710), (1713)), to most of us a body is a real physical object which we perceive in and around us.

[20]We will discuss the structure of a body in more detail later in the paper. The main thesis of the paper is not concerned with the precise topology or measure that is usually endowed on the body.

[21]A lattice is a set X with a partial order $\leq$ on which the operations of "join" and "meet" are defined (see McLane and Birkhoff (1967) for a discussion of lattices).

[22]In fact, one could also introduce topologies using the notion of closed sets.

that we need different approximations to study different problems is eloquently addressed by Feller (1947), who remarks

"*In applications of geometry and mechanics theoretical concepts are identified with certain physical objects, but the method is flexible and varies from occasion to occasion so that no general rules can be given. The concept of a rigid body is useful and essential to mechanics, and yet no physical objects meet the specifications. Only experience teaches us which bodies can, with a satisfactory approximation, be treated as rigid. Rubber is usually given as a typical example of a non-rigid body, but in discussing the motion of automobiles most textbooks treat the wheels, including rubber tires, as rigid bodies. This is an example of how theoretical models are chosen and varied according to convenience or needs. Depending on our purposes, we feel free to disregard atomic theories and treat the sun as a tremendous ball of continuous matter or, on another occasion, as a single mass point. We must always remember that mathematics deals with abstract models and that different models can describe the same empirical situation with various degrees of approximation and simplicity. The manner in which mathematical theories are applied does not depend on preconceived ideas and is not a matter of logic; it is a purposeful technique which depends on, and changes with, experience.*"

In fact, I prefer the following example to illustrate the role of idealization and approximation in natural philosophy. The Earth, depending on the problem of interest is approximated in a variety of ways. If one is interested in planetary motion, it suffices to model the Earth as a "point mass". If one is interested in explaining the eclipse, the Sun, Earth and the Moon could be viewed as "rigid spheres". To a person interested in constructing a building, even the size of the Empire State Building, it would be reasonable to think of the Earth as being an "elastic half-space" while to a geologist interested in the motion of the Earth's mantle, it could be regarded as a "viscoelastic fluid continuum" (a popular model to describe the Earth's mantle is the viscoelastic fluid model due to Burgers). Each of these approximations has an attendant mathematical representation. It is not that the Earth is any one of these idealizations. It is none of them and it is all of them within the context of an approximation. It is necessary to develop a whole host of mathematical arsenal to be able to capture, nay approximate, reality.

Even in abstract mathematics, that one could have a topology without introducing the notion of points was not recognized for a long time, and only a very small fraction of the mathematical community, even amongst topologists is interested in such a possibility even now. As Johnstone (2001) remarks

"*However, the idea that a topological space is a lattice of open sets, and that the points themselves are a secondary construct, took a much longer time to emerge. In part, this was because such an idea was dependent on the development of lattice theory as an autonomous branch of abstract algebra, which did not take place until the late 1930s with the work of Birkhoff and Stone, culminating in the first edition of Birkhoff [14]. Thus, although Vietoris's astonishing paper [95] of 1922 contains what is in effect an entirely lattice-theoretic description of how to construct the hyperspace (the space of all nonempty closed subsets) of a compact Hausdorff space, it could not be recognized in those terms because the concepts in abstract lattice theory, which were needed to formulate it as such, did not then exist. It was not until sixty years later [38], [41] that a purely lattice-theoretic translation of Vietoris's construction could be published.*"

The field of "point free" topology is relatively young. Marshall Stone, whom Johnstone (2001) refers to as the great grandfather of the field, did his work in the 1930s. Johnstone (2001) observes that

"*— the motivation for studying topological spaces at all had come entirely from geometric roots-the spaces one wished to study were those constructed by geometrical means from subsets of Euclidean space (or from infinite-dimensional generalizations thereof), even if one might wish to impose unusual topologies upon them in order to study nonstandard notions of continuity. The discovery that purely algebraic data could give rise to topologically interesting spaces was literally revolutionary; it forced topologists to take the algebraic aspects of the structure of a topological space seriously, and in so doing provided a powerful impetus towards freeing topologists form the preconception that classical Euclidean space could and should be described in terms of the totality of its points. Thus Stone was, if not the immediate begetter, at least the great-grandfather of pointless topology.*"

Johnstone (2001) also remarks that the works of Mckinsey and Tarski (1944), (1946) could "*be seen as the first attempt to do topology without mentioning points*". The few topologists that are interested today in the study of "point free" topologies seem to recognize the usefulness of such a concept with regard to applications in computer science, they have however not recognized the need for such tools in the oldest area of natural philosophy; mechanics. On the other hand, those concerned with research in mechanics use traditional ideas to attack problems in mechanics where such traditional ideas are invalid, akin to a person with a hammer who seems to perceive all problems as nails. My aim with regard to this short article is two-fold, the first is to clearly articulate the need for endowing bodies with "point free" topologies and impress those in the pursuit of problems in mechanics the need for such a structure, the second is to make the mathematicians aware that it is not merely sufficient to develop such "point free" topologies but to go a lot farther and develop a calculus and the analysis (partial differential and integral equations) that can be used to study problems in mechanics.

Loosely speaking, there are two types of mathematical difficulties that one runs into. In differential calculus, we have to define what one means by a derivative when dealing with functions defined on spaces endowed with "point free" topologies. When dealing with "chunks", no "chunk" would be of measure zero. Thus, problems that are currently swept under the carpet due to difficulties manifesting themselves on sets of measure zero will now have to be dealt with as they occur on the level of "chunks". But this might not be as onerous to deal with, as the singularities that appear due to the measure of the "chunk" shrinking to zero will disappear.

That even mathematicians need to be convinced of the need for a "point free" topology is rendered abundantly clear by the following remarks of Johnstone (1983)

"*I hope that by giving a historical survey of the subject known as "pointless topology" (i.e. the study of topology where open-set lattices are taken as the primitive notion) I shall succeed in convincing the reader that it does after all have some point to it. However, it is curious that the point (as I see it) is one which has emerged only relatively recently, after a substantial period during which the theory of pointless spaces has been developed without any very definite goal in view. I am sure there is a moral here; but I am not sure whether it shows that "pointless" abstraction for its own sake is a good thing (because it might one day turn out to be useful) or a* *bad thing (because it tends to obscure whatever point there might be in a subject). That much I shall leave for the reader to decide.*"

3. **Where the notion of a "point" hinders understanding certain problems of natural philosophy.**

Let us consider a few important problems in mechanics where the notion of a "point" comes in the way of clearly defining the problem and complicates issues rather than makes them tractable and amenable to meaningful analysis.

Philosophers have been interested in the question of the notion of a boundary, from the metaphysical as well as set theoretic point for a very long time. References to this literature can be found in the papers of Zimmermann (1996a), (1996b). Of interest to us here are the physical issues that lead to difficulties in defining a boundary. The notion of a boundary is a very difficult concept and it is not just a matter of defining appropriate elementary boundary units. If one considers a liquid in equilibrium with its ambient, say air, at the molecular level, there are constantly molecules leaving and entering the liquid, that is "chunks" exiting the body of interest into what is supposedly the ambient, and "chunks" of the ambient entering the body, with a fuzzy[23] ever changing region partly of the liquid molecules and partly of the air molecules. One does not have the sharp surface that encloses the liquid that can be defined as the boundary of fluid. We need to contend with boundaries that are "chunky" and ever changing. When the body and the ambient are in equilibrium[24], we might be able to get away by defining a boundary which has a thickness, the thickness associated with "the elementary boundary unit". Thus, at least for a class of problems it might be possible to define a boundary within the purview of "point free" topology. It is not clear to me how this is to be achieved, but the current definition of a boundary is clearly not well-suited for the study of many problems in natural philosophy.

An important and interesting problem that immediately comes to mind wherein the notion of "points" creates difficulties is the problem of "contact" between two bodies. Smith (2007) provides a very thorough discussion of the difficulties with the notion of "contact"[25] within the context of continuous bodies. One usually assumes that contact implies that the two bodies share a "point" in common, but this is impossible as this would imply interpenetration of the bodies which is not allowed, and if they do not share a "point", then they have no "contact" as we understand it's meaning. The question is whether this difficulty can be overcome by appealing to bodies that are "point free". The answer seems to be in the affirmative. Crudely speaking, suppose one has a body whose elementary units are put together in such a fashion that there are "holes" in it, and furthermore suppose that the boundary of the body has elementary boundary units that also have "gaps" in it (that is the boundary is like a mesh). Then it would be possible for one to place another body so that an elementary unit that belongs to its boundary fits in snugly or partially

[23]One could advocate the view that the boundary of a body is best described by a "fuzzy set" (see Rajagopal (2012)).

[24]The notion of "equilibrium" is an oft used idea in thermodynamics but I am yet to meet a person that can explain the notion in a convincing and satisfactory manner. One can of course define some mathematical condition that supposedly defines "equilibrium", but such definitions hardly help to understand the physical nature of "equilibrium".

[25]A very interesting discussion of "surfaces" and "contact" can be found in the influential work of Bolzano (1950), however the discussion would take us very far from the main intent of this work, which is meant merely to urge a rethinking of the usual topology that is associated with a body.

fills in, a "gap" in the boundary of the other body. Of course, one has to provide a rigorous mathematical definition for the body and its elementary unit and its boundary and its elementary unit. It is also possible that two bodies could be put together so that elementary units in the bulk of one body occupy the "holes" in the interior of a second body. Such a possibility is in fact at the heart of the process of making compounds and alloys from the constituent elements.

The notion of a "dislocation" provides a rather interesting example where "point free" topologies might prove useful. The terminology "dislocation" is inaccurate and the following explanation will render this transparent. When we think of the structure of a crystal, we envision a three dimensional grid in space, with atoms, in addition to occupying intersection points of the grids could also occupy the centers of the faces that are formed, or the center of a cube formed by the grid structure. It could happen that an intersection point, or a line in the grid network or a face in the grid network, is unoccupied by an atom which normally would have occupied such a position. For instance, a "point dislocation" is the situation wherein an atom is missing from the intersection of two lines in the grid. It then transpires that when one goes along a circuit wherein one travels, say east, north, west, and finally south by moving to the next available atom, one does not end up where one started. That is one that describes the circuit described above has been "dislocated". However, there is no entity that has caused this dislocation to take place. That is, "dislocation" is not an entity; in fact, it is the absence of an entity that has caused the dislocation. Thus one should not use the word "dislocation" as a noun. What is at play is the notion of the atomic structure that leads to one being dislocated. Since atoms themselves are made up of sub- atomic particles, the elementary unit is not the atom. However, for a class of problems, one could possibly have an approximation or an idealization that assumes that the atoms are the elementary units. One could then calculate the stresses in the body due to one of these chunks missing or being inserted into a body. When considering such structures, one has to take cognizance of the fact that we have discrete "chunks" (atoms for instance) and "gap" in between these "chunks".

One comes across another important related concept in crystal plasticity that relies heavily on the notion of a "point", when in fact it ought not to. This is the notion of "vacancy" that presumes that at a particular "point" in space an atom that was supposed to have been present is absent. Once again we see the notion of a "point" playing a critical role in the mathematical modeling. Since the atom has a dimension it does not occupy a "point" in space, the consequences of the absence of a "point" in the body versus the absence of a "chunk" can play an important role in describing and understanding the problem.

A problem that is at the heart of damage and failure of solids is the problem of fracture and crack propagation. The problem of fracture has been the object of study from very ancient times; it was studied by Galilei (1638) and before him it was discussed by Aristotle. Modern mathematical studies of cracks and their propagation, within the context of continua, assume that a crack ends at a "point", and such an assumption in the case of classical linearized elasticity leads to mathematical singularities at the tip of a crack. Of course, one can question the appropriateness of assuming that the body in question is elastic or investigating whether one could avoid such singularities by using different constitutive relations to model the response of the body[26], but this is not the real issue. A more fundamental question to grapple with is how reasonable is the assumption that the tip of crack is a point?

Another interesting example wherein one encounters singularities within the context of the classical theory of elasticity is the application of concentrated loads at a "point", as in the case of the Kelvin Problem in elasticity (see Thomson (1848)) the and Boussinesq problem (Boussinesq (1855)). The application of concentrated loads implies that the stresses at the point of application of the load are singular. Within the context of the classical linearized theory of elasticity this implies immediately singularities in the linearized strain, which could possibly be avoided in theories similar to those that have been recently used to study fracture; however the main difficulty yet remains unsolved. Can one really apply a "concentrated force" at a point? If the answer is in the negative and if one can only apply forces over non-zero area measures, which corresponds to the elementary unit of the boundary of a body, then we would not have a singularity to deal with, with regard to the stress. It is imperative to recognize that in the real physical world one does not encounter singularities due to the application of a load at a point; it is humanly impossible to do so. Introducing Dirac measures to represent such a load does not guarantee that we have exactitude in modeling the physical problem on hand.

One similarly finds exceedingly important problems in fluid mechanics, such as the creation of shocks (this is also a problem within the context of solids), initiation of cavitation and boiling of fluid, that are best studied within the context of "point free" bodies. A real shock is not without a thickness, what one encounters is a large change over a small length scale. What one has could possibly be described by a function that takes on "discontinuous" values over adjacent chunks. Of course, we now need to define what is meant by continuity as defined over the space of chunks. Also, the most important problem in fluid mechanics, possibly all of physics, the problem of turbulence could also be best explained within the context of "chunks" rather than field theories that assume a mean value for the velocity at each point and random and arbitrary fluctuations at such points. While description within the notion of field theories needs to take such fluctuations into account, defining quantities over "chunks" could provide a methodology that is more amenable to analysis. In fact, there have been attempts that assume a specific structure at a fine scale; however such ideas have not been cast within the context of "point free" topologies. I feel that rarefied gas dynamics and turbulence are not physical phenomena which are well described within the context of field theories. Interestingly, the notion of "point source" and "point sink" are singularities that have been used extensively in classical fluid mechanics. However, it is not difficult to immediately recognize that such artifices are not necessary and can be replaced by sets of finite measure, arbitrarily small, in compliance with the notion of a "smallest basic unit". In fact, an analysis with such a "basic unit" will immediately lead to finite velocities at the "sources' and "sinks" rather than the non-physical predictions of infinite velocities. A simple example would be the recasting of the Jefferey-Hamel flow within the context of the flow between two plates inclined with respect to one another but not intersecting each other at a "point" which is a physical impossibility.

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[26] Recently theories have been developed (see Rajagopal (2009), (2011), Rajagopal and Walton (2011), Bulíček et al. (2012)) wherein the strains at the crack tips are bounded. However, the stresses are yet unbounded at the tip as the area of the tip is of zero measure.

A serious difficulty that is encountered in many numerical studies of boundary value problems in mechanics is due to what is referred to as "corner singularities". Inordinate amount of time and effort is devoted to handling the singularities that are a consequence of the corners (which are mathematically represented by "points"). Corner singularities arise because we allow for corners; they do not reflect the physics of the problem. Similarly, in the study of partial differential equations in domains which have "corners" wherein boundary data that are prescribed are incompatible in the sense that different values are attained while arriving at the "corner" while traversing along different parts of the boundary, one finds that classical solutions are not possible. Within the construct of a boundary that has elementary units that are not points, one might be able to define functions such that there is no incompatibility. Of course, we have to first define a body and how it is comprised, define what is meant by the boundary of the body , define the topology, the notion of continuity for functions defined on the body, develop the appropriate calculus to arrive at the equations governing a physical problem, define what one means by a solution to the appropriate governing equations, etc., and this is no mean task.

In fact, in any problem, whether in fluid or solid mechanics, when the body is subject to loading (in general external stimuli), whether such loading is compatible or incompatible, the notion of a "point" undermines the comprehension of the true physics of the problem. The loads are never applied at a "point", they are spread over either a boundary element of non-zero measure or on a "chunk" in the interior of the body.

Noll and Virga (1988), (1990) have developed a frame-work, but within the context of classical topological ideas, to discuss issues concerning boundaries. These ideas, do not unfortunately address the sort of issues that we are concerned with here for which I propose the use of "point free" topology and the associated ideas of boundaries within such a construct. In view of this, I shall not discuss the work of Noll and Virga (1988), (1990) or related work here.

The necessity of "point free" topologies is not restricted to how we define bodies. It is also necessary to adopt such a point of view to mathematize the notion of "time". I shall not get into a discussion of what is meant by "time"; a great deal has been written about it. What concerns us is the fact that one cannot think of a "point" in time, that is, a particular instant of "time". In a later section wherein the various meanings attributed to a "point' are discussed we will see that one of the definitions uses the notion of time to motivate the notion of a "point". In classical mechanics, time is thought of as a one dimensional directed manifold. While such a definition has served us rather well, we will see below that there are several difficulties with such a definition.

When describing experimental procedures, say for instance stress relaxation or creep experiments, it is common to assume that one can apply a step load or a step strain or a step displacement. One then carries out the data reduction, especially for materials that are presumed to be described by linear theories, by appealing to standard methods like Laplace and Fourier transform, to study the response to such inputs. However, it is physically impossible to apply a step load or a step strain or step displacement. One can only "apply" things to a body over a small interval of time. We might not know precisely what is done to a body at an instant of time; all we know is in some average sense what we have done to the body over an interval of time. Once again we find that "point free" topologies can be of great assistance to us; we now have to define elementary units that are line segments of a certain non-zero length and this could be the equivalent within the context of "time" to the notion of the "Planck length" for space. I am not advocating here that one should do away with the notion of "points"; for most problems the classical mathematical structure provides meaningful and more importantly useful results. The refrain of this article is that to resolve certain physical problems we need to endow a different mathematical structure, a different topology, on the body, one that is not wedded to the concept of a "point".

There is however the deeper philosophical question that we have alluded to earlier, a question that has been debated since the earliest of times in natural philosophy, and that is the question whether there is a smallest unit of space. While the development of the Calculus that is used currently might give the idea that such a notion has been debated and dismissed, this would be an erroneous conclusion on our part. That the inability to develop a Calculus based on "atomism" does not mean that such a Calculus cannot be or need not be developed.

4. **Meaning of "point" in everyday usage.**

Most "definitions" of a "point" make a statement that articulates the following sentiment or meaning: "*A point is a precise location in space, and that it has no dimensions.*" Such a "definition" is most unsatisfactory as it does not convey any meaning whatsoever. If a "point" has no dimensions, how are we to ascertain if it is indeed at some location in space? What does it mean to be in some location is space? What is the meaning of "location"? If we are to conceptualize a "point" as a "dot", however small the dot, we cannot yet maintain that it has no dimension. It is fair to say that the concept is far from being self-evident; it is beyond the "stretch" or should one say the "contraction" of our imagination. Before we get into a discussion of what is meant by a "point" in mathematics, it is necessary to understand the history of the usage of the word and its various meanings.

Let us briefly consider some of the senses in which the word "point" is used. According to the Oxford English Dictionary (2000), the primary meaning of the word "point" is: "*A prick, a dot*": "*A minute hole or impression made by pricking*"[27], "*A minute mark on a surface, of the size or appearance of a fine puncture; a dot, a minute spot or speck; also anything excessively small or appearing as a speck*"[28], "*A dot or other small mark used in writing or in printing*"[29].

All the senses in which the word "point" is used above imply that it is something small, not that it has no dimension.

Another sense in which it was used that is relevant to what will be discussed is the following:

"*A minute part or particle of anything, the smallest unit of measurement", "The very least or very small part of something, a jot, whit, particle*"[30]. Once again we see

[27] The footnotes 18 through 22 are given for completeness and conformity with regard to the details of the definition that is found in the Oxford English Dictionary. I shall not give the references to the articles cited in these footnotes. The word "point" was used in this sense in "*Make a poynt bi the space of a litil fyngre from the oon ende of the wounde, & another point oper eende of the wounde (1400 Longfranc's Cirurg)*". "*A minute impression on the surface but not perforating it (1826 Kirby & Sp, Entomol. IV. 270; A point (Puntum)*".

[28] As in "*Which [Astrolabe] was of fin gold precious With pointz and cercles merveilous (1390, Gower Conf.III)*".

[29] As in "*And per a point, for ended is my tale (1386 Chaucer Can. Yeom. Prol. & T. 927)*".

[30] As in "*O poynt of ore pine to bate in the world ne is no leche (1300, Body & Soul in Map's Poems (Camden) 338)*".

that it is something that is small, in fact, it is a "***smallest unit of measurement***", the elementary unit.

Point also means:

"*The smallest or very small portion of time, a moment, an instant*", "*sensible point: the least discernible portion of matter or space*".

According to the definitions given above, we notice that a "point" has to be discernible; it is clearly identified as being the "smallest" and has to be sensible. This clearly disallows it being dimensionless as something which is dimensionless cannot be sensed.

The word "point" is also used in the following senses:

"*In medieval measurement of time the fourth (or according to some the fifth) part of an hour*",

and

"*The twelfth part of the side or radius of a quadrant*",

Yet another meaning of the word "point" which has relevance to how it is used with respect to the notion of time is the following:

"*In time, that which has position but not duration (as the beginning and end of a space of time), the precise time at which anything happens; an instant, moment, as the moment of noon, the moment of death*".

Once again we see its usage as an entity that has no magnitude.

In some meanings that have been attributed to a "point" thus far, we see the notion of it being the "smallest unit" of space or time, while in others we see it being used in the sense of being dimensionless.

However, the meaning that it seems to have acquired and in which it is commonly used, and which is of most consequence to our discussion in this work is:

"*Something having definite position, without extension, a position in space, time, succession, degree, order, etc.*"

It is impossible to envision an entity that has a "*definite position without extension*"[31], but we seem to either have no problem with such an entity or probably we do not recognize that there is a problem in perceiving such an entity. We also are told that the notion of a "point" in Geometry is supposedly understood in the sense:

"*That which is conceived as having position, but not magnitude (as the extremity of a Line, or the intersection of two lines*[32]*)*",

and

"*A place having definite spatial position but no extent, or of which the position alone is Considered; a spot.*"

[31] Of course, we are not given a precise definition of what is meant by "position" and thus the statement might be vacuous and convey nothing whatsoever.

[32] As in "*This forside cenyth is ymagened to ben the very point ouer the crowne of thyn heued (1391 Chaucer Astrol. 1 18).*"

The word point has various other meanings which have no relevance to this work and hence we shall not discuss them, the footnote below gives a few of them[33].

As we are interested in "Point free" (the terminology "point free" is being used instead of "pointless" for obvious reason) bodies, it is worthwhile to see the sense in which the word "pointless" is used[34]. According to the Oxford English Dictionary (2000) "pointless" means

"*Without a point; Having a rounded or blunt end; blunt*".

Of the different meanings mentioned above, the meaning that comes closest to our interest is "without a point". However, "without a point" refers to an object not having a tip and unless we assume that a tip has no magnitude, no dimension, "pointless" as it is used would not mean without having dimension.

Whether a point is a prick, a dot, a speck or a minute mark, it has dimension. Thus, its usage in Geometry given above as "*having position, but no magnitude*" is quite at odds with most of its other perceived meanings. The sense in which it is used in geometry is far from "self-evident" and I believe that its usage as an axiom in Geometry is on very shaky ground. That this is indeed so has been recognized by those working in the foundations of mathematics, especially solid geometry as considerable effort has been expended in deriving the notion of a "point" rather than accepting it as an axiom to start with.

5. The notion of a "point" in classical Euclidean Geometry.

Contrary to popular perception amongst mathematicians, Euclid did not perceive a "point" as being an entity without dimension. In Heath's authoritative translation of Euclid's Elements we find the following definition of the notion of "point"[35]: *Σημεῖόν έστιν, οὖ μέρος ούθέν*, which according to Heath means[36]

"*A point is that which has no part.*"

[33]Some examples of other usages of the notion of point are: "*— the point; the precise matter in discussion or to be discussed; the essential or important thing. Often in phr. To come to the point, to keep to the point, etc". "to make a point of: to treat or regard (something) as essential or indispensable , to make (it) a special subject". "That at which one aims or for which one strives, contends; aim, object, end. Also (the expression of) an important fact or truth; a noteworthy comment;— ; to have made a convincing or significant remark; to be correct*".

[34]As with the meaning of points, we shall not be concerned with the meaning of "pointless" as it is used in the sense "*Without point or force; ineffective, meaningless; Also without purpose or advantage; having no good effect", etc.*

[35]A very detailed quotation that discusses the various ways in which the notion was perceived by the early geometers, mathematicians and philosophers can be found in Heath (1921).

[36]Let me state without equivocation that I am totally ignorant in Greek and am in no position to decide whether Heath's translation is correct. I would also hasten to add that mere familiarity with Greek or for that matter having studied it for several years will not provide one with the authority to decide whether the translation that Heath is relying on is true to Euclid's intent or whether it lacks felicity. The comments of the Indian philosopher DasGupta (1975) concerning Sanskrit addresses the fundamental problem concerning translations:

"*It is therefore very difficult for a person unacquainted with Sanskrit to understand Indian philosophical thought in its true bearing from translations. –But no one from an acquaintance with Vedic or ordinary literary Sanskrit can have any idea of the difficulty of the logical and abstruse parts of Sanskrit philosophical literature. A man who can easily understand the Vedas, the Upanisads, the Puranas, the Law Books and the literary works, and is also well acquainted with European philosophical thought, may find it literally impossible to understand even small portions of a work of advanced Indian logic, or the dialectical Vedanta.*"

One has to understand the historical, geographical, social and various other contexts before translating a text. Practically most translation of texts in Sanskrit by so called experts in the west

Heath (1921) is very categorical about the meaning of point and is critical of its translation by Matianus Capella[37] which has become its accepted meaning in mathematics, namely "a point is that a part of which is nothing." Heath criticism, which I share, though I am incapable of judging its accuracy to the Greek text, is based on the logic of his argument. Heath remarks : "*If a part of a point is nothing, Euclid might as well have said that a point is itself "nothing," which of course he does not do.*". If one were to interprest Euclid as Heath does, then it is perfectly reasonable to think in terms of a smallest unit for space or time.

Even prior to Euclid, the Pythagoreans made a distinction between a "point" and position, and defined a "point" as a "monad with position" and Aristotle used the word "point" in this sense. Plato however did not subscribe to such a meaning and to him a "point" was the "*extremity of a line*"[38]. Aristotle states that a "*line is not made up of points*", "*an aggregation of points is also partless*"[39] and a "*point is not a body*"[40]. He also says that a "point" "*—is like the now in time: now is indivisible and is not a part of time, it is only the beginning or end, or a division, of time, and similarly a point may be an extremity, beginning or division of a line, but is not part of it or of magnitude* (cf. De caelo II. I, 300 a 14, Physics IV. 11, 220 a 1-21, VI. I 231 b 6 sqq.)." The notion of a point has been motivated in the discussion above as being "not divisible" or by saying that it "does not have magnitude". Such motivation by negation was criticized by several ancients as not providing a sensible definition, their contention being that one can never exhaust all the possible ways in which the concept can be negated[41].

We see a discussion with regard to the modern way of motivating the notion of a "point" via the procedure of taking limits of smaller and smaller sets in Weber and Wellstein in *Encyclopädie der elementaren Mathematik, II.*, 1905, p. 9. As they themselves remark, the notion of a point does not come from really understanding what it means or even comprehending the limiting process that supposedly leads to the concept. It is merely a belief, or as they put it, a question of willing it:

"*This notion is evolved from the notion of the real or supposed material point by the process of limits, i.e. by an act of the mind which sets a term to a series of presentations in itself unlimited. Suppose a grain of sand or a mote in a sunbeam, which continually becomes smaller and smaller. In this way vanishes more and more the possibility of determining still smaller atoms in the grain of sand, and there is evolved, so we say, with growing certainty, the presentation of the point as a definite position in space which is one and is incapable of further division. But this view is untenable; we have, it is true, some idea how the grain of sand gets smaller and smaller, but only so long as it remains just visible; after that we are completely in the dark, and we cannot see or imagine the further diminution. That this procedure*

as well as Sanskrit-English dictionaries written by western experts is full of errors. Unfortunately, most often we have nothing better to go by.

[37] Martianus Minneus Felix Capella was a writer from Africa of Late Antiquity, the fifth century, whose work *De nuptiis Philologiae et Mercurii influenced scholars of the Middle Ages.*

[38] How does one interpret what is meant by the extremity of a line? How is one to know how such a concept is envisioned by different persons? Do all of them envision it as having no dimension?

[39] That is an aggregation of points cannot be subdivided into points.

[40] This would mean that a body cannot be modeled as a particle.

[41] The "Neti" school believed that certain concepts could not be defined but only motivated through negation. For instance, they believed that all that one could do is to state what "God" was not, for example "God is not finite", "God is not transient", etc. (see S. DasGupta (1975)).

comes to an end is unthinkable; that nevertheless there exists a term beyond which it cannot go, we must believe or postulate without ever reaching it. . . It is a pure act of will, not of the understanding."

Max Simon observes similarly (see Heath (1921)):

"*'Point' then, according to our interpretation of Euclid, is the extremest limit of that which we can still think of (not observe) as a spatial presentation, and if we go further than that, not only does extension cease but even relative place, and in this sense the part is nothing.*"

Heath however takes strong exception to Simon's interpretation of Euclid. Heath states:

"*I confess I think that even the meaning which Simon intends to convey is better expressed by 'it has no part' than by 'the part is nothing'*",

My understanding from reading the relevant translations and comments concerning the motivation of the notion of "point" by Greeks is that a "point" is partless, it cannot be divided. There seems to be no intent on their part that it should have no dimension (magnitude). The notion of a "point" being the smallest element (elementary unit) of space is a perfectly reasonable interpretation of the intent of the Greeks.

Moving along to more recent workers on the foundations of mathematics, we find Whitehead (1916) making the following assertion:

"*Geometry as a mathematical theory, customarily takes its point of departure all or part of its fundamental spatial entities: points, straight or curved lines, surfaces, and volumes*[42]*. It takes them as simple primitive ideas, that is, in abstract language, as "variables", which are not logically functions of simpler variables. But if the relational theory of space is adopted, whether for the apparent world or for the physical world, this cannot be the first stage of research. For the relational theory of space it is essential that points, for example, be complex entities, logical functions of those relations between objects which constitute space. For, if a point is a simple thing, incapable of being logically defined by means of relations between objects, then points are in fact absolute positions. Thus, the relation of "being at a point" must be a primitive relation incapable of definition, and thus, we must take as the sole ultimate fact of geometry the primitive relations of the objects to their absolute positions. But this is none other than the absolute theory of space, which nominally at least, has been universally abandoned. Thus the primary object of geometricians investigating the foundation of their science is to define points as a function of relation between objects.*

Likewise, in physical space, a point is (practically speaking) an area or volume sufficiently small so that a division is useless in the actual state of science."

"*Some years ago Lesniewski suggested the establishment of the foundation of the geometry of solids, understanding by this term a system of geometry destitute of such geometrical figures as points, lines,surfaces, and admitting as figures only solids-the*

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[42]The following footnote appears in Whitehead's article: "*Cf. for example: Staudt, Geometry der Lage "Geometry proceeds from the concept of unbounded space; a completely bounded section of it is called a body. Bodies, and, in general, physical space, will be divided and limited by planes, planes by lines, and lines by points, . . . a point is indivisible.*"

intuitive correlates of open (or closed) regular sets of three dimensional Euclidean geometry[43]"

A rather extreme view that dismisses the main thrust of my thesis is propounded by Boyer (1959) who avers that:

"*Mathematics knows no minimum interval of continuous magnitudes-and distance and time may be considered as such, inasmuch as there is no evidence which would lead one to regard them otherwise. Attempts to supply a logical definition of such an infinitesimal minimum which shall be consistent with the body of mathematics as a whole have failed.*"

However, the fact that mathematics has failed thus far to work with such a minimum is more an admission of an inability to develop an appropriate language for expressing the possibility that there exists a "minimum interval" rather than such a minimum indivisible interval that the Greeks referred to as being a partless object not existing[44]. In fact, it is an aporia that begs to be deciphered, untangled. Also, it is not clear what Boyer means by "continuous magnitudes" and why one should expect continuity as traditionally defined in mathematics has to hold or for that matter why the notions of continuity and derivatives cannot be generalized.

We should not be complacent with the progress that has been made in Calculus, nor should we forget what Boyer (1959) himself states in the same book on the development of calculus that:

"*The fundamental definitions of the calculus, those of the derivative and integral, are now so clearly stated in textbooks on the subject, and the operations involving them so easily mastered, that it is easy to forget the difficulty with which these basic concepts have been developed.*"

"*Some twenty-five hundred years of effort to explain a vague instinctive feeling for continuity culminated thus in precise concepts which are logically defined but which represent extrapolations beyond the world of sensory experience.*"

It is my hope that it would not take another twenty five hundred years to develop the Calculus that goes with the notion of "point free" topology.

6. **Notion of Points in Mechanics.**

The earliest and simplest description of a body within the purview of mechanics is that of a particle, or more precisely a "point mass", and much of the early studies in natural philosophy were devoted to the investigation of the motion of projectiles and planets within the context of such a description. While the mathematical notion of a point could be considered as an indispensable notion in mathematics, bereft of which it would be no exaggeration to say there would not be much point to mathematics, as we have seen one could also lay at the feet of the notion of

[43]This reference is found in Tarski "*By a regular open set, or an open domain, we understand a point set which coincides with the interior of its closure; by a regular closed set, or a closed domain, we understand a point set which coincides with the closure of its interior. These notions have been defined by Kuratowski C. (42) and (41), pp 37 f.*".

[44]Unfortunately, few, in fact a mere handful in each century, are the mathematical physicts that seem to be interested in developing "what it takes" to describe and analyze a problem, the overwhelming majority of them are merely interested in using "what they know" to try to study problems and even if it is fitting a square peg in a round hole they seem to have no qualms about hammering it in. The engineer or physicist seems absolutely content with using "what it pleases" them while applying mathematical techniques to study physical problems.

"point" the inability to resolve many of the important unresolved issues in natural philosophy.

The notion of a point is taken for granted by Newton in his immortal Principia (Newton (1687)), the concept does not seem to give him cause for concern. He starts with concepts of space, time, and place, and then starts drawing figures wherein lines are drawn from point A to point B, etc., there is no pause or any attempt whatsoever to explore what it means to locate a point in space. In the Chapter titled "AXIOMS, OR LAWS OF MOTION", in order to describe the resultant force on a body, acted upon by [two] forces acting jointly Newton uses a parallelogram and states that the resultant force is along the diagonal of the parallelogram. He names the vertices, as points A, B, C and D.

Boscovich (1966) introduced an atomic theory wherein the atoms are indivisible and without dimension or shape which are "points of force". Though he assumed the notion of indivisible atoms, he assumed that they occupied a "point" in space and this becomes obvious in his discussion of the nature of matter. In his very influential treatise "A Theory of Natural Philosophy" Boscovich (1966) states

"*The primary elements of matter are in my opinion perfectly indivisible and non-extended points; they are so scattered in an immense vacuum that every two of them are separated from on another by a definite interval; this interval can be indefinitely increased or diminished, but can never vanish altogether without compenetration of the points themselves; for I do not admit as possible any immediate contact between them. On the contrary I consider that it is a certainty that, if the distance between two points of matter should become absolutely nothing, then the very same indivisible point of space, according to the usual idea of it, must be occupied by both together, and we have true compenetration in every way. Therefore indeed I do not admit the idea of vacuum interspersed amongst matter, but matter is interspersed in a vacuum and floats in it.*"

The following remarks of Russell concern the critical roles that "points", "instants" and "particles" play in Newtonian mechanics (Russell (2007)):

"*The Newtonian system, stated with schematic simplicity, as, e.g., Boscovich is as follows. There is an absolute space composed of points, and an absolute time composed of instants; there are particles of matter each of which subsists through all time and occupies a point at each instant. Each particle exerts forces on other particles, the effect of which is to produce accelerations. Each particle is associated with a certain quantity, its "mass" which is inversely proportional to the acceleration produced in the particle by a given force.*"

In the above explanation provided by Russell, one is supposed to understand what is meant by a "point", an "instant", and a "particle", but none of these ideas are accessible to one's intuition. Russell himself observes in the same book, earlier, that

"*Moreover, the word "entities" which we have been using, is too narrow if used with any metaphysical implication. The "entities", concerned may, in a given application of a deductive system, be complicated logical structures. Of this we have some examples in pure mathematics in the definition of cardinal numbers, ratios, real numbers, etc. We must be prepared for the possibility of a similar result in physics, in the definition of point of space time, and even in the definition of an electron or a proton.*"

While Russell identifies the notion of a "point" in space-time as a complicated logical structure, it is odd that he does not mention (I am sure he recognizes it) the notion of a "point" in geometry (as a part of mathematics), is also a complicated logical structure far from intuitive recognition. Unfortunately, Russell (2007) makes a lot of other statements in the book concerning physics that I do not subscribe to, but as they are not germane to this work I shall not discuss them here.

As far as the evolution of mechanics in the eighteenth and nineteenth century is concerned, there seems to have been no qualms with regard to using the notion of a "point" or a "particle".

Tremendous progress was made in science using "particle" and rigid body mechanics and given the success of science based on the concept there was not the slightest inclination to question the basic tenets. Continuum Mechanics adopted the notion of "points" and based on this notion built a grand edifice that inspired in turn various sub-fields in mathematics.

7. **Abstract definitions of a body in continuum mechanics.**

The remarks of Fransescoe De Oviedo, that can be found in the translation that appears in "Discourse of things above reason" in the Selected Philosophical Papers of Robert Boyle, edited by M. A. Stewart, while somewhat harsh seems to have some merit even today, though most mathematicians will tend to view it as being without merit. Oviedo remarks (Stewart (1991)):

"*We come to the composition of a continuum, whose hitherto unsurmounted difficulty has sorely taxed the wits of all the learned, and everyone without exception acknowledges it to be virtually insurmountable. Most of them mask it in obscure terminology with repeated and tortuous distinctions and sub-distinctions, so that no-one may openly catch them despairing of other means of solution which might yield to the light of reason; but they must necessarily conceal it in the darkness of confusion, so that it may not be laid bare by perspicuous argument.*"

The notion of a continuum is not easy to grasp either intuitively or from a very rigorous mathematical perspective as such understanding requires acceptance of many fundamental assumptions (axioms) of mathematics which are far from settled or accepted by all mathematicians (see Kline (1980) for a detailed and interesting discussion of the ambiguous and tenuous foundations of mathematics).

Most investigators in mathematics adopt the view that a body is a differentiable manifold. Few of them have shown an inclination to delve deeply into how a body might be described mathematically. Noll and his co-workers are possibly an exception. Noll has provided definitions of a body at various levels of abstraction (see Noll (1958), (1972), (1992), (1993); see also Noll and Seguin (2010)). While the definition of a body by Noll is in terms of a Boolean algebra[45] with the underlying lattice structure, the underlying topological structure does not consider the possibility of a body having a "point free" topology.

I shall start with the definition that can be found in a book by Truesdell (1972), in which he closely follows Noll's definition, as this is a standard reference in continuum mechanics. Truesdell introduces bodies in the following manner:

[45]See MacLane and Birkhoff (1967) or Sikorsky (1986) for a definition of Lattice, Boolean Algebra, etc.

"*Most collections of bodies $\mathcal{A}, \mathcal{B}$, conform with the mathematical structure of a Boolean lattice, or complemented distributive lattice, often called a Boolean algebra.*"

He then goes on to motivate this abstract collection by making the following observation:

"*To picture the relation amongst bodies, it may help to consider Ω as being the collection of all open sets in the Euclidean plane and to take $\prec$ as being the sign of inclusion, $\subset$, so that the suggestive sketches often called "Venn diagrams" are easy to draw. This illustration is only one of many. Others, including the universes commonly presumed in mechanics, will be presented in the next section.*"

Later in the book he introduces placers that map the abstract body into a three dimensional Euclidean space with the usual topology and points, thus endowing a body with the classical topology.

Nowhere in the book does one find even the possibility of "point free" topologies for bodies being discussed.

I will now turn to a brief discussion of the definition of a body due to Noll and how it evolved, as his definition is usually adopted in most rigorous treatments of a continuum. In one of his early rigorous discussions of a body, Noll (1958) provides the following definition:

"*A body $\mathcal{B}$ is a smooth manifold of elements $X, Z \ldots$ called particles. The configurations $\varphi, \theta \ldots$ are elements of a set of one to one mappings of $\mathcal{B}$ into a three dimensional Euclidean point space $\mathcal{E}$.*"

He goes on to define the measure structure given to the body etc. We notice that in this early definition of the body, the topological structure with "points" is fundamental. Later, in a paper titled "Lectures on the foundations of continuum mechanics and thermodynamics", after motivating the notion of a body through:

"*A material system $\mathcal{B}$ is a set of material points $X, Y, \ldots$ These are idealizations of material objects or marking in or on material objects.*"

He provides the following rigorous definition:

We say a material system $\mathcal{B}$ is a continuous body of type **D** if it is endowed with a structure defined by a non-empty class **P** of mappings subject to the following conditions:

(B1) The members of **P** *are invertible mappings from $\mathcal{B}$ onto open subsets of Euclidean spaces.*

We once again see that the body is endowed with the usual topology with points. In much more recent work with Seguin, we yet find that the body is endowed with a topology induced by the classical Euclidean topology (see Noll and Seguin (2010)). There are several terms in the definition of a body given below that have not been defined; we shall not do so as this would require introducing numerous ideas that are not germane to our intent, namely to recognize that the definition while quite complicated and intricate yet hinges on the fact that a body has the topological structure of that of a differentiable manifold. They start by motivating a continuum thus:

"*In order to define a continuous body system two classes must be specified. One being the class Fr of all subsets of three-dimensional Euclidean spaces that are possible regions that a body system can occupy. Intuitively, the term "body" suggests that the regions it can occupy are connected. We do not assume this but we will use the term "body" rather than "body system" from now on. The other being the class Tp of mappings which are possible changes of placement of a body.*

Then they go on to define a body rigorously that is given below:

"*A continuous body* $\mathcal{B}$ *is a set endowed with structure by the specification of a non-empty set* $\operatorname{Conf}\mathcal{B}$*, whose elements are called configurations of* $\mathcal{B}$*, satisfying the following requirements:*

(B1) *Every* $\delta \in \operatorname{Conf}\mathcal{B}$ *is a fit Euclidean metric.*
(B2) *For all* $\delta \in \operatorname{Conf}\mathcal{B}$ *the mapping* $\lambda := \operatorname{imb}_\delta \circ \operatorname{imb}_\varepsilon \leftarrow$ *is a transplacement, with* $\operatorname{Dsp}\lambda = \mathcal{E}_\varepsilon$ *and* $\operatorname{Rsp}\lambda = \mathcal{E}_\delta$*.*
(B3) *For every* $\delta \in \operatorname{Conf}\mathcal{B}$ *and every transplacement* λ *such that* $\operatorname{Dom}\lambda = \operatorname{Rng}\operatorname{imb}_\delta$ *the function* $\varepsilon : \mathcal{B} \times \mathcal{B} \to \mathbf{P}$*, defined by*

$$\varepsilon(X, Y) := \operatorname{dist}(\lambda(\operatorname{imb}_\delta(X)), \lambda(\operatorname{imb}_\delta(Y)) \text{ for all } X, Y \in \mathcal{B},$$

is a fit Euclidean metric that belongs to $\operatorname{Conf}\mathcal{B}$*. The elements of* $\mathcal{B}$ *are called material points."*

They go on to observe that the body which they so define is endowed with the structure of a twice continuously differentiable three dimensional manifold. Thus, the body clearly has a topology in which the classical notion of a "point" plays a critical role.

Thus, while it is true that Noll, Truesdell and their co-workers provide the mathematical structure of a Boolean algebra to a body, the mathematical object that they have in mind are those sets whose topology has the notion of "points" associated with them as the bodies are always identified with the help of an underlying Euclidean space which is populated by "points" and endowed with the classical Euclidean metric, and the notion of distance associated between "points". In fact, all the examples and counter examples to explain the notions of meet and join as opposed to the operations of unions and intersections, and the motivation for the body in Truesdell's book (see Truesdell (1972)) are within the context of sets that contain "points". Likewise, Noll's definition of a fit body and sets with finite perimeter are also concerned with sets whose members are "points". Nowhere in the writings in continuum mechanics, is the need for "point free" bodies articulated.

I believe that it is necessary to use the language of "point free" topology if we are to describe certain problems of natural philosophy and resolve them satisfactorily. While there has been considerable effort in developing "point free" topology, there has been no effort in developing a Calculus where one can talk sensibly about the derivatives of functions that are defined on such topological spaces, the development ofordinary, partial differential and integral equations within such a context, etc. This is however not surprising as the subject of "point free" topology has been the dominion of pure mathematicians, who might not even be aware of, and if aware of, not interested in applications[46] in natural philosophy.

[46] Most pure mathematicians yet seem to cling to the view of Plato, as espoused by Plutarch (Plutarch's lives translated by John Dryden (1934)): "*Eudoxus and Archytas had been the originators of this far-famed and highly prized art of mechanics, which they employed as an elegant*

To those that have worked all their scientific life accepting the concept of a "point" as an article of faith, no different from an adherent's faith in a religious commandment, any suggestion that the applicability of the notion be reexamined would be perceived as nothing short of tergiversation and apostasy. Most of us think we understand the notion of a "point" since we have been using the term from our elementary school days, we were too young and uninformed to question the notion when we were first introduced to it, and our familiarity with the word from a very young age gives us a false sense that we do comprehend the underlying concept[47].

It might very well be that my advocacy for the use of "point free" bodies is uncalled for, but I am convinced otherwise. I hope I have articulated the need for developing "point free" topologies and the attendant calculus to describe bodies. The development of such a structure is most daunting; it would not only be a great achievement in mathematics, it would prove to be of great use in natural philosophy.

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illustration of geometrical truths, and as a means of sustaining experimentally, to the satisfaction of the senses, conclusions too intricate for proof by words and diagram. As for example, to solve the problem, given the two extremes, to find the two mean lines of a proportion, both these mathematicians had recourse to the aid of instruments, adapting to their purpose certain curves and sections of lines. But what with Plato's indignation at it, and his invective against it as mere corruption and annihilation of the good in geometry, which was thus shamefully turning its back upon the unembodied objects of pure intelligence to recur to sensation, and to ask help (not to be obtained without base supervisions and depravation) from matter; so it was that mechanics came to be so separated from geometry, and repudiated and neglected by philosophers, took its place as a military art."

[47]Though most scientists and mathematicians would like to think otherwise, Dryden's comments (Dryden (1687)) apply to them as much as it applies to those that subscribe to specific religious beliefs:

"*By education most have been misled;*
So they believe, because so they were bred,
The priest continues what the nurse began,
And thus the child imposes on the man."

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