POTENTIAL ESTIMATES FOR THE p-LAPLACE SYSTEM WITH DATA IN DIVERGENCE FORM

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Abstract. A pointwise bound for local weak solutions to the p-Laplace system is established in terms of data on the right-hand side in divergence form. The relevant bound involves a Havin-Maz'ya-Wulff potential of the datum, and is a counterpart for data in divergence form of a classical result of [KiMa], that has recently been extended to systems in [KuMi2]. A local bound for oscillations is also provided. These results allow for a unified approach to regularity estimates for broad classes of norms, including Banach function norms (e.g. Lebesgue, Lorentz and Orlicz norms), and norms depending on the oscillation of functions (e.g. Hölder, BMO and, more generally, Campanato type norms). In particular, new regularity properties are exhibited, and well-known results are easily recovered.

1. Introduction and main results

The present paper deals with the regularity of local weak solutions to the p-Laplace system

$$-\mathrm{div}(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}) = -\mathrm{div}\mathbf{F} \qquad \text{in } \Omega. \tag{1.1}$$

Here, Ω is an open set in $\mathbb{R}^n$, with $n \geq 2$, the exponent $p \in (1,\infty)$, the function $\mathbf{F} : \Omega \to \mathbb{R}^{N\times n}$, with $N \geq 1$, is assigned, and $\mathbf{u} : \Omega \to \mathbb{R}^N$ is the unknown.

We shall assume that $\mathbf{F} \in L^{p'}_{\rm loc}(\Omega)$ throughout, where p' stands for the Hölder conjugate of p. This assumption guarantees that weak solutions to system (1.1) are well defined. Any weak solution $\mathbf{u}$ to system (1.1) belongs, by definition, to the Sobolev space $W^{1,p}_{\rm loc}(\Omega)$. Basic regularity properties of $\mathbf{u}$, such as membership in Lebsegue or Hölder spaces, according to whether $p \leq n$ or $p > n$, can be immediately derived from this piece of information, via the standard Sobolev embedding theorem. Additional regularity of $\mathbf{F}$, beyond local $L^{p'}$– integrability, is reflected into stronger regularity properties of any solution $\mathbf{u}$.

The regularity theory of solutions to p-Laplace type equations and systems has been the subject of a vast literature, starting from the second part of the last century. Classical fundamental contributions to this theory include [Ur], [Uh], [Iw1], [Ev], [Di], [Le1] [To], [ChDiB] [DiBMa].

Instead of focusing on estimates for specific norms, here we offer a precise pointwise estimate for any weak solution $\mathbf{u}$ to (1.1), and an estimate for its oscillations in integral form. They provide us with versatile tools for the proof of norm bounds in a wide range of function spaces, as shown in the last part of the paper.

Our first main result is contained in Theorem 1.1, and amounts to a pointwise bound for $\mathbf{u}$ in terms of a Havin-Maz'ya-Wulff potential of $\mathbf{F}$. This can be regarded as a version, for right-hand sides in divergence form, of the pointwise bound established in [KiMa] for equations whose right-hand side is a function (or, more generally, a measure), and of its recent extension to systems from [KuMi2]. The latter paper also contains a parallel pointwise estimate for the gradient via a Riesz potential of the datum, which carries over to systems a result from [KuMi1]. Earlier contributions along a similar line of research are [Mi] and [DuMi2]. Pointwise gradient estimates for solutions to systems with right-hand side in the form of (1.1) are proved in [BCDKS1] – see also [BCDKS2]. Let us add that rearrangement bounds for solutions to boundary value problems for p-Laplace type elliptic equations

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with the same kind of right-hand side can be found in [BFM]. Pointwise estimates, in rearrangement form, for the gradient are in [AFT], [ACMM] and [CiMa].

Recall that, given $s > 1$ and $\sigma > 0$, the truncated Havin-Maz'ya-Wulff potential $\mathbf{W}^R_{\alpha,s}\mathbf{f}$ of an $\mathbb{R}^m$-valued function $\mathbf{f} \in L^1_{\rm loc}(\Omega)$, with $m \geq 1$, is defined as

$$\mathbf{W}^R_{\alpha,s}\mathbf{f}(x) = \int_0^R \left(r^{\alpha s} \fint_{B_r(x)} |\mathbf{f}|\, dy \right)^{\frac{1}{s-1}} \frac{d\,r}{r} \tag{1.2}$$

for every $x \in \Omega$ and $R > 0$ such that $B_R(x) \subset \Omega$. Here, $B_r(x)$ denotes the ball centered at x, with radius r, and $\fint_E$ stands for the averaged integral $\frac{1}{|E|}\int_E$, where $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$.

Theorem 1.1. *Let $n \geq 2$, $N \geq 1$ and $p \in (1, \infty)$. Assume that Ω is an open set in $\mathbb{R}^n$, and that $\mathbf{F} \in L^{p'}_{\rm loc}(\Omega)$. Let $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ be a local weak solution to system* (1.1). *There exists a constant $C = C(n, N, p)$ such that*

$$|\mathbf{u}(x)| \leq C\mathbf{W}^R_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)(x) + C \fint_{B_R(x)} |\mathbf{u}|\, dy \tag{1.3}$$

for a.e. $x \in \Omega$ and every $R > 0$ such that $B_R(x) \subset \Omega$. Moreover, a point $x \in \Omega$ is a Lebesgue point of $\mathbf{u}$ whenever the right-hand side of (1.3) *is finite for some $R > 0$.*

Remark 1.2. Assume that $\Omega = \mathbb{R}^n$, and that $\mathbf{u}$ is a weak solution to system (1.1) decaying so fast at infinity that

$$\lim_{R\to\infty} \fint_{B_R(x)} |\mathbf{u}|\, dy = 0\,. \tag{1.4}$$

Then, passing to the limit as $R \to \infty$ in inequality (1.3) yields

$$|\mathbf{u}(x)| \leq C\mathbf{W}_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.5}$$

Here, $\mathbf{W}_{\alpha,s}\mathbf{f}$ denotes the potential of a function $\mathbf{f} \in L^1_{\rm loc}(\mathbb{R}^n)$ given by the integral on the right-hand side of (1.2), with R replaced by ∞. Namely,

$$\mathbf{W}_{\alpha,s}\mathbf{f}(x) = \int_0^\infty \left(r^{\alpha s} \fint_{B_r(x)} |\mathbf{f}|\, dy \right)^{\frac{1}{s-1}} \frac{dr}{r} \tag{1.6}$$

for $x \in \mathbb{R}^n$. Note that condition (1.4) is fulfilled, for instance, if $p \in (1, n)$, $\nabla\mathbf{u} \in L^p(\mathbb{R}^n)$ and $|\{x \in \mathbb{R}^n : |\mathbf{u}(x)| > t\}| < \infty$ for every $t > 0$. Indeed, (1.4) follows via the Hölder and the classical Sobolev inequality, which holds under these assumptions on $\mathbf{u}$.

Remark 1.3. In the case when $p = 2$, system (1.1) reduces to the classical Poisson system

$$-\Delta\mathbf{u} = -\mathrm{div}\mathbf{F}\,, \tag{1.7}$$

and inequality (1.3) reads

$$|\mathbf{u}(x)| \leq c \int_0^R \left(\fint_{B_r(x)} |\mathbf{F}|^2\, dy \right)^{\frac{1}{2}} dr + c \fint_{B_R(x)} |\mathbf{u}|\, dy. \tag{1.8}$$

The operator acting on $|\mathbf{F}|$ in (1.8) bounds, via Hölder's inequality, the operator $\int_0^R \fint_{B_r(x)} |\mathbf{F}|\, dy\, dr$, which is, in turn, equivalent to a truncated Riesz potential of order 1 of $|\mathbf{F}|$. In fact, if $\Omega = \mathbb{R}^n$, and F is regular enough, then the solution $\mathbf{u}$ to (1.7), that decays to 0 near infinity, admits the representation formula

$$\mathbf{u}(x) = c \int_{\mathbb{R}^n} \mathbf{F}(y) \frac{x-y}{|x-y|^n}\, dy \qquad \text{for } x \in \mathbb{R}^n,$$

for a suitable constant $c = c(n)$. The reason why Theorem 1.1 cannot yield inequality (1.8), with $\int_0^R \big(\fint_{B_r(x)} |\mathbf{F}|^2\,dz\big)^{\frac{1}{2}}\,dr$ replaced by $\int_0^R \fint_{B_r(x)} |\mathbf{F}|\,dy\,dr$, is that data $\mathbf{F}$, that do not belong to $L^{p'}_{\rm loc}(\Omega)$, are not included in our analysis, which is nonlinear in nature. This is related to a gap between the linear Calderón-Zygmund theory for the Laplacian, and its nonlinear counterpart for the p-Laplacian, and hence to a well known open problem in the latter about right-hand sides $\mathbf{F} \in L^q_{\rm loc}(\Omega)$ with $q < p'$. See [Iw2], [Le2] and [KiLe] in this connection, and also the recent contributions [BDS, BuSch].

Theorem 1.1 applies, in principle, to regularity estimates for solutions to (1.1) in any norm depending only on the size of functions, or, more precisely, in any norm that is monotone under poitwise domination of functions. They are usually called Banach function norms in the literature [BeSh]. This class of norms includes various customary instances, such as the (possibly weighted) Lebesgue norms, the Orlicz norms and the Lorentz norms.

Bounds in norms depending on oscillations of functions, such as Hölder, BMO and, more generally, Campanato type norms can be derived from the next result. Its content is an estimate for the oscillation of solutions to (1.1) on balls. The latter is also our first step in the proof of Theorem 1.1. In the statement, and in what follows, the notation $\langle \mathbf{f}\rangle_E$ is used, when convenient, to denote the integral average $\fint_E \mathbf{f}(x)\,dx$ of a function $\mathbf{f}$ over a set E.

Theorem 1.4. *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Assume that Ω is an open set in $\mathbb{R}^n$, and that $\mathbf{F} \in L^{p'}_{\rm loc}(\Omega)$. Let $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ be a local weak solution to system (1.1). There exists a constant $C = C(n,N,p)$ such that*

$$\fint_{B_r(x)} |\mathbf{u}(y) - \langle \mathbf{u}\rangle_{B_r(x)}|dy \leq Cr\bigg(\int_r^R \bigg(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy\bigg)^{\frac{1}{p'}} \frac{d\rho}{\rho}\bigg)^{\frac{1}{p-1}} + Cr\fint_{B_R(x)} |\nabla \mathbf{u}|\,dy \tag{1.9}$$

for a.e. $x \in \Omega$, every $R > 0$ such that $B_R(x) \subset \Omega$ and every $r \in (0,R]$.

Remark 1.5. A close inspection of the proof of Theorem 1.1 will reveal that inequality (1.3) can be slightly improved, in that the Wulff potential $\mathbf{W}^R_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)$ on its right-hand side can be replaced with a smaller nonstandard potential, introduced in [BCDKS1]. The relevant potential involves the oscillation of $\mathbf{F}$ on balls, as in (1.9), instead of just its $L^{p'}$ averages, as in $\mathbf{W}^R_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)$. The resulting inequality reads

$$|\mathbf{u}(x)| \leq C\int_0^R \bigg(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy\bigg)^{\frac{1}{p}} d\rho + C\fint_{B_R(x)} |\mathbf{u}|\,dy \tag{1.10}$$

for a.e. $x \in \Omega$ and every $R > 0$ such that $B_R(x) \subset \Omega$. Clearly, inequality (1.10) implies (1.3), since dropping the expression $\langle \mathbf{F}\rangle_{B_\rho(x)}$ in the first integral of its right-hand side results in $\mathbf{W}^R_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)(x)$.

The proofs of Theorems 1.1 and 1.4 are accomplished in the next section. Tools playing a role in our approach include a recent pointwise estimate for a sharp maximal function of the gradient of solutions to system (1.1) from [BCDKS1], suitable versions of classical results in the regularity theory of the p-Laplacian system, as well as certain weighted inequalities of Hardy type for functions of one variable. In Section 3, Theorem 1.1 is exploited, in combination with apropos results on the boundedness of nonlinear potentials, to establish regularity bounds for solutions in Lorentz and Orlicz spaces. Bounds in Campanato type spaces, and ensuing estimates in spaces of uniformly continuous functions are derived in the same section, via Theorem 1.4.

2. Proofs of Theorems 1.1 and 1.4

The Hardy type inequalities contained in the next lemma will be needed in our proof of Theorem 1.1. In what follows, a function $\varphi : (0,L) \to [0,\infty)$, with $L \in (0,\infty]$ will be called quasi-increasing if there exists a constant $k \geq 1$ such that

$$\varphi(r) \leq k\varphi(s) \qquad \text{if } 0 < r \leq s < L. \tag{2.1}$$

Note that, if the function φ fulfills inequality (2.1), then the function $\psi : (0, L) \to [0, \infty)$ associated with φ as

$$\psi(s) = \sup_{0<r<s} \varphi(r) \quad \text{for } s \in (0, L),$$

is non-decreasing, and satisfies the inequalities

$$\varphi(s) \leq \psi(s) \leq k\varphi(s) \quad \text{for } s \in (0, L). \tag{2.2}$$

Lemma 2.1. *Let $\alpha \in \mathbb{R}$.*
(i) *Assume that $q \in [1, \infty)$. Then there exists a constant $C = C(q, \alpha)$ such that*

$$\left(\int_0^\infty \left(\int_s^\infty \varphi(r) r^\alpha \, dr \right)^q ds \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \varphi(s)^q s^{q(\alpha+1)} \, ds \right)^{\frac{1}{q}} \tag{2.3}$$

for every measurable function $\varphi : (0, \infty) \to [0, \infty)$.
(ii) *Assume that $q \in (0, 1)$. If $\alpha < -1 - \frac{1}{q}$, then there exists a constant $C = C(q, \alpha, k)$ such that inequality* (2.3) *holds for every quasi-increasing function $\varphi : (0, \infty) \to [0, \infty)$ with constant k as in* (2.1)*. If $-1 - \frac{1}{q} \leq \alpha < -1$ and $0 < a < \infty$, then there exists a constant $C = C(q, \alpha, k)$ such that*

$$\left(\int_0^a \left(\int_s^a \varphi(r) r^\alpha \, dr \right)^q ds \right)^{\frac{1}{q}} \leq C \left(\int_0^{2a} \varphi(s)^q s^{q(\alpha+1)} \, ds \right)^{\frac{1}{q}} \tag{2.4}$$

for every quasi-increasing function $\varphi : (0, 2a) \to [0, \infty)$ with constant k as in (2.1)*.*

Proof. *Part (i).* This is a classical Hardy inequality – see e.g. [Ma, Theorem 1.3.2/3].
Part (ii). By inequalities (2.2), it suffices to prove the statement for non-decreasing functions φ.
Let $0 < a \leq b \leq \infty$. A characterization of weighted Hardy type inequalities for monotone functions tells us that the inequality

$$\left(\int_0^a \left(\int_s^a \varphi(r) r^\alpha \, dr \right)^q ds \right)^{\frac{1}{q}} \leq C \left(\int_0^b \varphi(s)^q s^{q(\alpha+1)} \, ds \right)^{\frac{1}{q}} \tag{2.5}$$

holds for every non-decreasing function $\varphi : (0, b) \to [0, \infty)$ if and only if

$$\left(\int_0^a \left(\int_{\max\{t,s\}}^b \chi_{(0,a)}(r) r^\alpha \, dr \right)^q ds \right)^{\frac{1}{q}} \leq C \left(\int_t^b s^{q(\alpha+1)} \, ds \right)^{\frac{1}{q}} \quad \text{for every } t \in (0, b), \tag{2.6}$$

with the same constant C (see e.g. [HeMa, Theorem 3.3]). If $\alpha < -1 - \frac{1}{q}$, then

$$\frac{\left(\int_0^\infty \left(\int_{\max\{t,s\}}^\infty r^\alpha \, dr \right)^q ds \right)^{\frac{1}{q}}}{\left(\int_t^\infty s^{q(\alpha+1)} \, ds \right)^{\frac{1}{q}}} = C \qquad \text{for } t > 0, \tag{2.7}$$

for some constant $C = C(\alpha, q)$. Thus, inequality (2.6), and hence also inequality (2.5), follows with $a = b = \infty$.
Suppose next that $\alpha = -1 - \frac{1}{q}$ and $b = 2a < \infty$. Straightforward computations show that there exist

constants C and C', depending on q, such that

$$\frac{\big(\int_0^a \big(\int_{\max\{t,s\}}^a \chi_{(0,a)}(r) r^\alpha \, dr\big)^q \, ds\big)^{\frac{1}{q}}}{\big(\int_t^{2a} s^{q(\alpha+1)} \, ds\big)^{\frac{1}{q}}} = C \frac{\big[t(t^{-\frac{1}{q}} - a^{-\frac{1}{q}})^q + \int_{t/a}^1 (s^{-\frac{1}{q}} - 1)^q \, ds\big]^{\frac{1}{q}}}{\big(\log \frac{2a}{t}\big)^{\frac{1}{q}}} \tag{2.8}$$

$$\leq C \frac{\big[1 + \int_{t/a}^1 \frac{ds}{s}\big]^{\frac{1}{q}}}{\big(\log \frac{2a}{t}\big)^{\frac{1}{q}}} = C \frac{\big[1 + \log \frac{a}{t}\big]^{\frac{1}{q}}}{\big(\log \frac{2a}{t}\big)^{\frac{1}{q}}}$$

$$\leq C' \left[\frac{1}{\big(\log \frac{2a}{t}\big)^{\frac{1}{q}}} + \left(\frac{\log \frac{a}{t}}{\log \frac{2a}{t}}\right)^{\frac{1}{q}}\right] \leq C' \left[\frac{1}{(\log 2)^{\frac{1}{q}}} + 1\right]$$

for $t \in (0, a)$. Since the leftmost side of the chain (2.8) vanishes for $t \in [a, 2a)$, inequality (2.6), and hence also inequality (2.5), follows with $b = 2a$.

Finally, assume that $-1 - \frac{1}{q} < \alpha < -1$ and $b = 2a < \infty$. Then, there exist constants $C = C(\alpha, q)$ and $C' = C'(\alpha, q)$ such that

$$\frac{\big(\int_0^a \big(\int_{\max\{t,s\}}^a \chi_{(0,a)}(r) r^\alpha \, dr\big)^q \, ds\big)^{\frac{1}{q}}}{\big(\int_t^{2a} s^{q(\alpha+1)} \, ds\big)^{\frac{1}{q}}} = C a^{\alpha+1+\frac{1}{q}} \frac{\big[(\frac{t}{a})^{q(\alpha+1)+1}(1 - (\frac{a}{t})^{\alpha+1})^q + \int_{t/a}^1 (s^{\alpha+1} - 1)^q \, ds\big]^{\frac{1}{q}}}{\big[(2a)^{q(\alpha+1)+1} - t^{q(\alpha+1)+1}\big]^{\frac{1}{q}}} \tag{2.9}$$

$$\leq C' a^{\alpha+1+\frac{1}{q}} \frac{\big[(\frac{t}{a})^{q(\alpha+1)+1}(1 - (\frac{a}{t})^{\alpha+1})^q + (1 - (\frac{t}{a})^{q(\alpha+1)+1})\big]^{\frac{1}{q}}}{\big[(2a)^{q(\alpha+1)+1} - t^{q(\alpha+1)+1}\big]^{\frac{1}{q}}}$$

$$\leq C' \frac{a^{\alpha+1+\frac{1}{q}} 2^{\frac{1}{q}}}{a^{\alpha+1+\frac{1}{q}}} = C' 2^{\frac{1}{q}}$$

for $t \in (0, a)$. The leftmost side of (2.9) vanishes for $t \in [a, 2a)$, and therefore inequalities (2.6) and (2.5), hold with $b = 2a$. ☐

In what follows, we shall make repeatedly use of the inequality

$$\int_E |\mathbf{f} - \langle \mathbf{f} \rangle_E| \, dx \leq 2 \int_E |\mathbf{f} - \mathbf{c}| \, dx \tag{2.10}$$

for every measurable set $E \subset \mathbb{R}^n$, every function $\mathbf{f}$ in E, and every vector $\mathbf{c} \in \mathbb{R}^n$. In particular, it plays a role in the proof of the next lemma, whose objective is an estimate for the difference between the mean values of a function over balls. In its statement and proof, all balls are concentric. Since the center is irrelevant, it will be dropped in the notation.

Lemma 2.2. *Let $R > 0$ and let $\mathbf{f} \in L^1(B_R)$. Then*

$$|\langle \mathbf{f} \rangle_{B_r} - \langle \mathbf{f} \rangle_{B_R}| \leq 2^{2n+2} \int_r^R \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_\rho}| \, dx \frac{d\rho}{\rho} \qquad \text{for } r \in (0, R], \tag{2.11}$$

and

$$|\langle |\mathbf{f}| \rangle_{B_r} - \langle |\mathbf{f}| \rangle_{B_R}| \leq 2^{2n+3} \int_r^R \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_\rho}| \, dx \frac{d\rho}{\rho} \qquad \text{for } r \in (0, R]. \tag{2.12}$$

Proof. Since inequalities (2.11) and (2.12) are scale-invariant, we may assume, without loss of generality, that $R = 1$. To begin with, observe that

$$|\langle \mathbf{f} \rangle_{B_{\frac{1}{2}}} - \langle \mathbf{f} \rangle_{B_1}| \leq \fint_{B_{\frac{1}{2}}} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_1}| dx \leq 2^n \fint_{B_1} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_1}| \, dx \, .$$

Applying this inequality with B_1 replaced by $B_{2^{-i}}$, for $i = 0, \dots, m-1$, and adding the resulting inequalities yield

$$|\langle \mathbf{f}\rangle_{B_{2^{-m}}} - \langle \mathbf{f}\rangle_{B_1}| \le 2^n \sum_{i=0}^{m-1} \fint_{B_{2^{-i}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{-i}}}|\,dx. \tag{2.13}$$

Let $\theta \in (2^{-m}, 2^{1-m}]$. Owing to inequality (2.10),

$$\fint_{B_{2^{-m}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{-m}}}|\,dx \le 2^{n+1} \fint_{B_\theta} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\theta}|\,dx \le 2^{2n+2} \fint_{B_{2^{1-m}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{1-m}}}|\,dx. \tag{2.14}$$

On exploiting inequality (2.14), one can show that

$$\fint_{B_{2^{-m}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{-m}}}|\,dx \le 2^{n+1} \fint_{2^{-m}}^{2^{1-m}} \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\rho}|\,dx\,d\rho \le 2^{n+2} \int_{2^{-m}}^{2^{1-m}} \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\rho}|\,dx \frac{d\rho}{\rho}. \tag{2.15}$$

Now, given $r \in (0,1]$, let $m \in \mathbb{N}$ be such that $r \in (2^{-m}, 2^{1-m}]$. Thanks to inequality (2.15), we have that

$$\begin{aligned}
|\langle \mathbf{f}\rangle_{B_r} - \langle \mathbf{f}\rangle_{B_1}| &\le |\langle \mathbf{f}\rangle_{B_r} - \langle \mathbf{f}\rangle_{B_{2^{1-m}}}| + |\langle \mathbf{f}\rangle_{B_{2^{1-m}}} - \langle \mathbf{f}\rangle_{B_1}| \\
&\le 2^n \fint_{B_{2^{1-m}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{1-m}}}|\,dx + 2^n \sum_{i=0}^{m-2} \fint_{B_{2^{-i}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{-i}}}|\,dx \\
&\le 2^n \sum_{i=0}^{m-1} \fint_{B_{2^{-i}}} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_{2^{-i}}}|\,dx \le 2^{2n+2} \int_{2^{1-m}}^{1} \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\rho}|\,dx \frac{d\rho}{\rho} \\
&\le 2^{2n+2} \int_r^1 \fint_{B_\rho} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_\rho}|\,dx \frac{d\rho}{\rho}.
\end{aligned}$$

This concludes the proof of inequality (2.11). Inequality (2.12) follows via the same argument, combined with the fact that, by inequality (2.10),

$$\fint_{B_1} ||\mathbf{f}| - \langle |\mathbf{f}|_{B_1}\rangle|\,dx \le 2 \fint_{B_1} ||\mathbf{f}| - |\langle \mathbf{f}\rangle_{B_1}||\,dx \le 2 \fint_{B} |\mathbf{f} - \langle \mathbf{f}\rangle_{B_1}|\,dx. \tag{2.16}$$ □

We are now in a position to prove our main results.

Proof of Theorem 1.4. Let $\mathbf{u}$ be a local weak solution to system (1.1). This means that $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_{\Omega'} |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi}\,dx = \int_{\Omega'} \mathbf{F} \cdot \nabla \boldsymbol{\varphi}\,dx \tag{2.17}$$

for every function $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega')$, and every open set $\Omega' \subset\subset \Omega$. Here, the dot " · " stands for scalar product. Let us set

$$\mathbf{A}(\nabla \mathbf{u}) = |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}.$$

Fix any $x \in \Omega$ and $R > 0$ such that $B_R(x) \subset \Omega$. Let $r \in (0, R/2]$. By a standard Poincaré inequality on balls, and Hölder's inequality,

$$\begin{aligned}(2.18)\qquad \Big(\fint_{B_r(x)} |\mathbf{u}(y) - \langle \mathbf{u}\rangle_{B_r}(x)|\, dy\Big)^{p-1} &\lesssim \Big(r \fint_{B_r(x)} |\nabla \mathbf{u}(y)|\, dy\Big)^{p-1} \\ &\lesssim r^{p-1}\Big(\fint_{B_r(x)} |\mathbf{A}(\nabla \mathbf{u}(y))|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}}.\end{aligned}$$

Here, and in what follows, the relation " $\lesssim$ " between two expressions means that the former is bounded by the latter, up to multiplicative constants independent of the relevant variables involved. Inequality [BCDKS1, (3.16)] tells us that

$$\begin{aligned}(2.19)\qquad &\sum_{i=0}^{k}\Big(\fint_{B_{\theta^i \frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\theta^i \frac{R}{2}}(x)}|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}} \\ &\lesssim \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\frac{R}{2}}(x)}|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}} + \sum_{i=0}^{k-1}\Big(\fint_{B_{\theta^i \frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_{\theta^i \frac{R}{2}}(x)}|^{p'}\, dy\Big)^{\frac{1}{p'}}\end{aligned}$$

for every $\theta \in (0,1)$ and $k \in \mathbb{N}$. Now, choose $k \in \mathbb{N}$, such that $\theta^{k+1}R \le r \le \theta^k R$ in (2.19). Via a telescope sum argument, one can infer that

$$\begin{aligned}(2.20)\qquad \Big(\fint_{B_r(x)} |\mathbf{A}(\nabla \mathbf{u})|^{\min\{p',2\}} dy\Big)^{\frac{1}{\min\{p',2\}}} &\lesssim \Big(\fint_{B_{\theta^k \frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u})|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}} \\ &\lesssim \Big(\fint_{B_{\theta^k \frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\theta^k \frac{R}{2}}(x)}|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}} \\ &\quad + \sum_{i=1}^{k} |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\theta^{i-1} \frac{R}{2}}(x)} - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\theta^i \frac{R}{2}}(x)}| + |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\frac{R}{2}}(x)}| \\ &\lesssim \sum_{i=0}^{k}\Big(\fint_{B_{\theta^i \frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\theta^i \frac{R}{2}}(x)}|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}} + |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\frac{R}{2}}(x)}| \\ &\lesssim \sum_{i=0}^{k-1}\Big(\fint_{B_{\theta^i \frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_{\theta^i \frac{R}{2}}(x)}|^{p'}\, dy\Big)^{\frac{1}{p'}} + \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u})|^{\min\{p',2\}}\, dy\Big)^{\frac{1}{\min\{p',2\}}},\end{aligned}$$

where the second inequality holds by an iterated use of the triangle inequality, the third one by Hölder's inequality, and the last one by (2.19). A standard reverse Hölder's inequality ensures that

$$(2.21)\qquad \Big(\fint_{B_{\frac{R}{2}}(x)} |\nabla \mathbf{u}|^p\, dx\Big)^{\frac{1}{p}} \lesssim \fint_{B_R(x)} |\nabla \mathbf{u}|\, dx + \Big(\fint_{B_R(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_R(x)}|^{p'}\, dx\Big)^{\frac{1}{p}},$$

see e.g. [DuMi1, Lemma 3.2], or [DKS, Lemma 3.2, Lemma 3.3.]. An application of Hölder's inequality and of (2.21) yields

$$
\Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{A}(\nabla \mathbf{u})|^{\min\{p',2\}}\,dy \Big)^{\frac{1}{\min\{p',2\}}} \lesssim \Big(\fint_{B_R(x)} |\nabla \mathbf{u}|\,dy \Big)^{p-1} + \Big(\fint_{B_R(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_R(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}}. \tag{2.22}
$$

Finally, note that, by (2.15)

$$
\sum_{i=0}^{k} \Big(\fint_{B_{\theta^i \frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_{\theta^i \frac{R}{2}}}|^{p'}\,dy \Big)^{\frac{1}{p'}} + \Big(\fint_{B_R(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_R(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \lesssim \int_r^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \frac{d\rho}{\rho}. \tag{2.23}
$$

If $r \in (0, R/2]$, inequality (1.9) follows from (2.18), (2.20), (2.22) and (2.23). Inequality (1.9) continues to hold for $r \in (R/2, R]$, thanks to the Sobolev-Poincaré inequality. $\square$

Proof of Theorem 1.1. Let $x \in \Omega$ and $R > 0$ be such that $B_R(x) \subset \Omega$. Lemma 2.2 and inequality (1.9) yield

$$
\begin{aligned}
\Big(\fint_{B_r(x)} |\mathbf{u}|dy \Big)^{p-1} &\lesssim \Big(|\langle |\mathbf{u}|\rangle_{B_r(x)} - \langle |\mathbf{u}|\rangle_{B_{\frac{R}{4}}(x)}| + \fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}|dy \Big)^{p-1} \\
&\lesssim \Big(\int_r^{\frac{R}{4}} \fint_{B_\rho(x)} |\mathbf{u} - \langle \mathbf{u}\rangle_{B_\rho(x)}|dy \frac{d\rho}{\rho} \Big)^{p-1} + \Big(\fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}|dy \Big)^{p-1} \\
&\lesssim \Big(\int_0^{\frac{R}{4}} \Big(\int_\rho^{\frac{R}{4}} \Big(\fint_{B_s(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_s(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \frac{ds}{s} \Big)^{\frac{1}{p-1}} d\rho \Big)^{p-1} \\
&\quad + \Big(\int_0^{\frac{R}{4}} \fint_{B_{\frac{R}{4}}(x)} |\nabla \mathbf{u}|\,dy\,dr \Big)^{p-1} + \Big(\fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}|dy \Big)^{p-1},
\end{aligned} \tag{2.24}
$$

for $\rho \in (0, R]$. Next, observe that

$$
\Big(\int_{B_r(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_r(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \le 2 \Big(\int_{B_r(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \le 2 \Big(\int_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \tag{2.25}
$$

if $0 < r \le \rho < R$. As a consequence, the function

$$
(0, R) \ni r \mapsto \Big(\int_{B_r(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_r(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}}
$$

is quasi-increasing, with constant $k = 2$, according to definition (2.1). Hence, by Lemma 2.1,

$$
\Big(\int_0^{\frac{R}{4}} \Big(\int_\rho^{\frac{R}{4}} \Big(\fint_{B_s(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_s(x)}|^{p'}\,dy \Big)^{\frac{1}{p'}} \frac{ds}{s} \Big)^{\frac{1}{p-1}} d\rho \Big)^{p-1} \lesssim \Big(\int_0^{\frac{R}{2}} \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_r(x)}|^{p'}\,dy \Big)^{\frac{1}{p}} d\rho \Big)^{p-1}. \tag{2.26}
$$

From inequalities (2.24) and (2.26) one deduces that

$$\fint_{B_r(x)} |\mathbf{u}| dy \lesssim \int_0^{\frac{R}{2}} \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} dy \Big)^{\frac{1}{p}} dr + R \fint_{B_{\frac{R}{4}}(x)} |\nabla \mathbf{u}| \, dy + \fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}| \, dy \,. \tag{2.27}$$

An inequality of Caccioppoli type tells us that

$$\Big(\fint_{B_{\frac{R}{4}}(x)} |\nabla \mathbf{u}|^p \, dy \Big)^{\frac{1}{p}} \lesssim \frac{1}{R} \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{u} - \langle \mathbf{u} \rangle_{B_{\frac{R}{2}}(x)}|^p \, dy \Big)^{\frac{1}{p}} + \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}} , \tag{2.28}$$

see, for instance, [DKS, Lemma 3.2]. Inequality (2.28), coupled with the Sobolev-Poincaré inequality, yields

$$\Big(\fint_{B_{\frac{R}{4}}(x)} |\mathbf{u} - \langle \mathbf{u} \rangle_{B_{\frac{R}{4}}(x)}|^{qp} \, dy \Big)^{\frac{1}{qp}} \lesssim \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{u} - \langle \mathbf{u} \rangle_{B_{\frac{R}{2}}(x)}|^p \, dy \Big)^{\frac{1}{p}} + R \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}},$$

for every $q \in (1, \frac{n}{n-p}]$. Thus,

$$\Big(\fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}|^{qp} \, dy \Big)^{\frac{1}{qp}} \lesssim \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{u}|^p \, dy \Big)^{\frac{1}{p}} + R \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}},$$

whence, by and interpolation argument as in [Gi, Remark 6.12], one can deduce that

$$\Big(\fint_{B_{\frac{R}{4}}(x)} |\mathbf{u}|^p \, dy \Big)^{\frac{1}{p}} \lesssim \fint_{B_{\frac{R}{2}}(x)} |\mathbf{u}| \, dy + R \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}}. \tag{2.29}$$

Inequalities (2.28) and (2.29) entail that

$$R \fint_{B_{\frac{R}{4}}(x)} |\nabla \mathbf{u}| \, dy \lesssim \fint_{B_{\frac{R}{2}}(x)} |\mathbf{u}| \, dy + R \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}}. \tag{2.30}$$

Owing to equation (2.25)

$$R \Big(\fint_{B_{\frac{R}{2}}(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_{\frac{R}{2}}(x)}|^{p'} \, dy \Big)^{\frac{1}{p}} \lesssim \int_{\frac{R}{2}}^{R} \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} \, dy \Big)^{\frac{1}{p}} d\rho. \tag{2.31}$$

Combining inequalities (2.27), (2.30), (2.31), and passing to the limit as $r \to 0^+$ yield inequality (1.3) for a.e. $x \in \Omega$, inasmuch as

$$\int_0^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} \, dy \Big)^{\frac{1}{p}} d\rho \le c \mathbf{W}^R_{\frac{p}{p+1}, p+1}(|\mathbf{F}|^{p'})(x) \quad \text{for } x \in \Omega. \tag{2.32}$$

It remains to show that each point $x \in \Omega$ such that

$$\mathbf{W}^R_{\frac{p}{p+1}, p+1}(|\mathbf{F}|^{p'})(x) < \infty \tag{2.33}$$

for some $R > 0$ is a Lebesgue point of $\mathbf{u}$. Owing to inequality (2.32), assumption (2.33) implies that

$$\bigg(\int_0^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'}\,dy\Big)^{\frac{1}{p}} d\rho\bigg)^{p-1} < \infty$$

for some R. Let $0 < s \le r \le R$. By inequalities (2.11) and (1.9),

$$\begin{aligned} (2.34)\quad |\langle \mathbf{u}\rangle_{B_r(x)} - \langle \mathbf{u}\rangle_{B_s(x)}| &\lesssim \int_s^r \fint_{B_\rho(x)} |\mathbf{u} - \langle \mathbf{u}\rangle_{B_\rho(x)}|\,dy\,\frac{d\rho}{\rho} \\ &\lesssim (r-s)\bigg[\bigg(\int_0^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy\Big)^{\frac{1}{p'}}\frac{d\rho}{\rho}\bigg)^{\frac{1}{p-1}} + \fint_{B_R(x)} |\nabla \mathbf{u}|\,dy\bigg]. \end{aligned}$$

Inequality (2.34) ensures that the function $s \mapsto \langle \mathbf{u}\rangle_{B_s(x)}$ satisfies Cauchy's criterion, and hence converges to a finite limit $\mathbf{u}(x)$ as $s \to 0^+$. Next, by inequalities (1.9) and (2.34),

$$\begin{aligned} (2.35)\quad \fint_{B_r(x)} |\mathbf{u} - \langle \mathbf{u}\rangle_{B_s(x)}|\,dy &\le \fint_{B_r(x)} |\mathbf{u} - \langle \mathbf{u}\rangle_{B_r(x)}|\,dy + |\langle \mathbf{u}\rangle_{B_r(x)} - \langle \mathbf{u}\rangle_{B_s(x)}| \\ &\lesssim r\bigg(\int_0^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy\Big)^{\frac{1}{p'}}\frac{d\rho}{\rho}\bigg)^{\frac{1}{p-1}} \\ &\quad + r\bigg[\bigg(\int_0^R \Big(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_\rho(x)}|^{p'}\,dy\Big)^{\frac{1}{p'}}\frac{d\rho}{\rho}\bigg)^{\frac{1}{p-1}} + \fint_{B_R(x)} |\nabla \mathbf{u}|\,dy\bigg]. \end{aligned}$$

Passing to the limit in (2.35) first as $s \to 0^+$, and then as $r \to 0^+$ tells us that

$$\lim_{r\to 0^+} \fint_{B_r(x)} |\mathbf{u} - \mathbf{u}(x)|\,dy = 0\,,$$

namely, x is a Lebesgue point for $\mathbf{u}$. □

3. Norm estimates

The Havin-Maz'ya-Wulff potential $\mathbf{W}_{\alpha,s}\mathbf{f}$, and hence $\mathbf{W}^R_{\alpha,s}\mathbf{f}$ for every $R > 0$, admit an estimate from above, independent of $\mathbf{f}$, in terms of the nonlinear potential $\mathbf{V}_{\alpha,s}\mathbf{f}$ given, for $s \in (1,\infty)$ and $\alpha \in (0,n)$, by

$$(3.1)\qquad \mathbf{V}_{\alpha,s}\mathbf{f}(x) = \mathbf{I}_\alpha\big((\mathbf{I}_\alpha|\mathbf{f}|)^{\frac{1}{s-1}}\big)(x) \qquad \text{for } x \in \mathbb{R}^n.$$

Here, $\mathbf{I}_\alpha$ stands for the standard Riesz potential of order α. If the domain of $\mathbf{f}$ is just an open set $\Omega \subset \mathbb{R}^n$, then $\mathbf{W}_{\alpha,s}\mathbf{f}$ and $\mathbf{V}_{\alpha,s}\mathbf{f}$ are defined by continuing $\mathbf{f}$ by 0 outside Ω.

The potentials $\mathbf{V}_{\alpha,s}$ extend the Riesz potentials, since, if $2\alpha < n$, then $\mathbf{V}_{\alpha,2}\mathbf{f} = c\mathbf{I}_{2\alpha}|\mathbf{f}|$ for a suitable constant $c = c(\alpha, n)$, and for every measurable function $\mathbf{f}$. They were introduced by V.P.Havin and V.G.Maz'ya [HaMa], and extensively investigated in the framework of nonlinear capacity theory. By [AdMe, Theorem 3.1], if $\alpha s < n$, then there exists a constant $C = C(\alpha, s, n)$ such that

$$(3.2)\qquad \mathbf{W}_{\alpha,s}\mathbf{f}(x) \le C\mathbf{V}_{\alpha,s}\mathbf{f}(x) \qquad \text{for } x \in \mathbb{R}^n.$$

Incidentally, let us mention that a reverse pointwise inequality only holds for s, n and α in a suitable range – see [HaMa]. However, a lower bound in integral form for $\mathbf{W}_{\alpha,p}\mathbf{f}$ in terms of $\mathbf{V}_{\alpha,s}\mathbf{f}$ is always available, as proved in [HeWo].

As a consequence of inequalities (1.3) and (3.2), local estimates for quasi-norms of solutions to system (1.1), that are monotone with respect to pointwise ordering, are reduced to boundedness properties of the nonlinear potential $\mathbf{V}_{\frac{p}{p+1},p+1}$ with respect to the same quasi-norms. We shall exhibit

the estimates in question for the Lorentz quasi-norms and the Orlicz norms.
Recall that, if $|\Omega| < \infty$, and either $q \in (1, \infty]$, $\varrho \in (0, \infty]$, $\beta \in \mathbb{R}$, or $q = 1$, $\varrho \in (0, 1]$, $\beta \in [0, \infty)$, the Lorentz-Zygmund space $L^{q,\varrho}(\log L)^\beta(\Omega)$ is defined as the set of all measurable functions $\mathbf{f}$ on Ω such that the quasi-norm given by

$$\|\mathbf{f}\|_{L^{q,\varrho}(\log L)^\beta(\Omega)} = \|s^{\frac{1}{q}-\frac{1}{\varrho}}(1 + \log(|\Omega|/s)^\beta |\mathbf{f}|^*(s)\|_{L^\varrho(0,|\Omega|)}$$

is finite. Here, $|\mathbf{f}|^* : [0, \infty) \to [0, \infty]$ denotes the decreasing–rearrangement of $|\mathbf{f}|$. When $\beta = 0$, the space $L^{q,\varrho}(\log L)^0(\Omega)$ is a standard Lorentz space, denoted by $L^{q,\varrho}(\Omega)$. In particular, $L^{q,q}(\log L)^0 = L^{q,q}(\Omega) = L^q(\Omega)$ for $q \in [1, \infty]$. The notation $L^{q,\varrho}(\log L)^\beta_{\mathrm{loc}}(\Omega)$ stands for the set of functions which belong to $L^{q,\varrho}(\log L)^\beta(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Theorem 3.1. *Assume that* $1 < p < n$. *Let* $\mathbf{u}$ *be a local weak solution to system* (1.1).
(i) *Let* $0 < \varrho \le \infty$ *and* $1 < q < \frac{n}{p}$. *If* $\mathbf{F} \in L^{qp',\varrho p'}_{\mathrm{loc}}(\Omega)$, *then* $\mathbf{u} \in L^{\frac{qnp}{n-qp},\varrho p}_{\mathrm{loc}}(\Omega)$.
(ii) *Let* $\varrho > \frac{1}{p}$. *If* $\mathbf{F} \in L^{\frac{n}{p-1},\varrho p'}_{\mathrm{loc}}(\Omega)$, *then* $\mathbf{u} \in L^{\infty,\varrho p}(\log L)^{-1}_{\mathrm{loc}}(\Omega)$.
(iii) *Let* $0 < \varrho \le \frac{1}{p}$. *If* $\mathbf{F} \in L^{\frac{n}{p-1},\varrho p'}_{\mathrm{loc}}(\Omega)$, *then* $\mathbf{u} \in L^\infty_{\mathrm{loc}}(\Omega)$.

Remark 3.2. Under the assumptions of Theorem 3.1, Part (ii), one has that $\mathbf{u} \in \exp L^{(\varrho p)'}_{\mathrm{loc}}(\Omega)$, thanks to the inclusion of the Lorentz-Zygmund space $L^{\infty,\varrho}(\log L)^{-1}_{\mathrm{loc}}(\Omega)$, with $\varrho > 1$, into the Orlicz space of exponential type $\exp L^{\varrho'}_{\mathrm{loc}}(\Omega)$, whose definition is recalled below.

A proof of Theorem 3.1 combines Theorem 1.1 with the following characterization of boundedness properties of the potential $\mathbf{V}_{\alpha,s}\mathbf{f}$ in Lorentz type spaces from [Ci].

Theorem A. *Let* $s > 1$, $n \ge 2$, *and* $0 < \alpha < \frac{n}{s}$. *Let* Ω *be a measurable subset of* $\mathbb{R}^n$.
(i) *If* $0 < \varrho \le \infty$ *and* $1 < \sigma < \frac{n}{\alpha s}$, *then there exists a constant* $C = C(\alpha, s, n, \sigma, \varrho)$ *such that*

$$\|\mathbf{V}_{\alpha,s}\mathbf{f}\|_{L^{\frac{\sigma n(s-1)}{n-\sigma\alpha s},\varrho(s-1)}(\Omega)} \le C\|\mathbf{f}\|^{\frac{1}{s-1}}_{L^{\sigma,\varrho}(\Omega)} \tag{3.3}$$

for every $\mathbf{f} \in L^{\sigma,\varrho}(\Omega)$.
(iii) *If* $|\Omega| < \infty$ *and* $\varrho > \frac{1}{s-1}$, *then there exists a constant* $C = C(\alpha, s, n, \varrho, |\Omega|)$ *such that*

$$\|\mathbf{V}_{\alpha,s}\mathbf{f}\|_{L^{\infty,\varrho(s-1)}(\log L)^{-1}(\Omega)} \le C\|\mathbf{f}\|^{\frac{1}{s-1}}_{L^{\frac{n}{\alpha s},\varrho}(\Omega)} \tag{3.4}$$

for every $\mathbf{f} \in L^{\frac{n}{\alpha p},\varrho}(\Omega)$.
(iv) If $0 < \varrho \le \frac{1}{s-1}$, *then there exists a constant* $C = C(\alpha, s, n)$ *such that*

$$\|\mathbf{V}_{\alpha,s}\mathbf{f}\|_{L^\infty(\Omega)} \le C\|\mathbf{f}\|^{\frac{1}{s-1}}_{L^{\frac{n}{\alpha s},\varrho}(\Omega)} \tag{3.5}$$

for every $\mathbf{f} \in L^{\frac{n}{\alpha s},\varrho}(\Omega)$.

Proof of Theorem 3.1. (i) Given any ball $B_R \subset \Omega$, we have, by Theorem A, Part (i),

$$\begin{aligned}\big\|\mathbf{V}_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)\big\|_{L^{\frac{qnp}{n-qp},\varrho p}(B_R)} &= \big\|\mathbf{V}_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)\big\|_{L^{\frac{qnp}{n-\frac{qp}{p+1}(p+1)},\varrho p}(B_R)} \\ &\lesssim \||\mathbf{F}|^{p'}\|^{\frac{1}{p}}_{L^{q,\varrho}(B_R)} = \|\mathbf{F}\|^{\frac{1}{p-1}}_{L^{qp',\varrho p'}(B_R)}.\end{aligned} \tag{3.6}$$

The claim hence follows, via (1.3) and (3.2).
(ii) Theorem A, Part (ii), ensures that

$$\big\|\mathbf{V}_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)\big\|_{L^{\infty,\varrho p}(\log L)^{-1}(B_R)} \lesssim \||\mathbf{F}|^{p'}\|^{\frac{1}{p}}_{L^{\frac{n}{p},\varrho}(B_R)} = \|\mathbf{F}\|^{\frac{1}{p-1}}_{L^{\frac{n}{p-1},\varrho p'}(B_R)}. \tag{3.7}$$

The conclusion is now again a consequence of (1.3) and (3.2).
(iii) By Theorem A, Part (iii),

$$\big\|\mathbf{V}_{\frac{p}{p+1},p+1}\big(|\mathbf{F}|^{p'}\big)\big\|_{L^\infty(B_R)} \lesssim \big\||\mathbf{F}|^{p'}\big\|_{L^{\frac{n}{p},\varrho}(B_R)}^{\frac{1}{p}} = \big\|\mathbf{F}\big\|_{L^{\frac{n}{p-1},\varrho p'}(B_R)}^{\frac{1}{p-1}}. \tag{3.8}$$

A combination of inequalities (1.3) and (3.2) completes the proof. □

Let us next focus on estimates involving norms in Orlicz spaces. Orlicz spaces are built upon Young functions, namely functions $A : [0,\infty) \to [0,\infty]$ that are convex (non trivial), left-continuous, and vanish at 0. The Orlicz space $L^A(\Omega)$ consists of all measurable function $\mathbf{f}$ on Ω for which the Luxemburg norm

$$\|\mathbf{f}\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|\mathbf{f}(x)|}{\lambda}\right)\,dx \le 1\right\}$$

is finite. The local Orlicz space $L^A_{\rm loc}(\Omega)$ is defined accordingly. Observe that $L^A(\Omega) = L^q(\Omega)$ if $A(t) = t^q$ for some $q \in [1,\infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = 0$ for $t \in [0,1]$, and $A(t) = \infty$ if $t \in (1,\infty)$. The Orlicz space associated with a Young function $A(t)$ which behaves like $t^p(\log t)^\alpha$ as $t \to \infty$, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \ge 0$, agrees with the Zygmund space denoted by $L^p(\log L)^\alpha(\Omega)$. The exponential type space $\exp L^\beta(\Omega)$, with $\beta > 0$, is the Orlicz space built upon the Young function $A(t) = e^{t^\beta} - 1$. The double exponential space $\exp\exp L^\beta(\Omega)$ is defined analogously.

Given two Young functions A and B, and $p \in (1,n)$, define the functions E_p and F_p from $[0,\infty)$ into $[0,\infty]$ as

$$E_p(t) = \left(\int_0^t \left(\frac{\tau^{\frac{1}{p}-\frac{1}{n'}}}{A(\tau)^{\frac{1}{n}}}\right)^{n'} d\tau\right)^{\frac{p}{n'}} \tag{3.9}$$

and

$$F_p(t) = \left(\int_0^t \frac{B(\tau)}{\tau^{1+\frac{n}{n-p}}}\,d\tau\right)^{\frac{n-p}{n}} \tag{3.10}$$

for $t \ge 0$.
Our regularity result in Orlicz spaces reads as follows.

Theorem 3.3. *Let $1 < p < n$, and let A and B be Young functions such that the functions E_p and F_p defined as in* (3.9) *and* (3.10) *are finite-valued. Assume that there exist $\gamma > 0$ and $t_0 > 0$ such that*

$$F_p\left(\frac{E_p(t)}{\gamma}\right) \le \gamma\frac{A(t)}{t} \qquad \text{for } t > t_0. \tag{3.11}$$

If $|\mathbf{F}|^{p'} \in L^A_{\rm loc}(\Omega)$, and $\mathbf{u}$ is any local weak solution to system (1.1), *then $|\mathbf{u}|^p \in L^B_{\rm loc}(\Omega)$.*

Remark 3.4. The assumption that the functions E_p and F_p are finite-valued, namely that the integrals on the right-hand sides of equations (3.9) and (3.10) are finite for $t \ge 0$, is immaterial in Theorem 3.3. Indeed, A and B can be replaced, if necessary, by Young functions that are equivalent to A and B near infinity, and make the integrals in question converge. Indeed, such a replacement leaves the spaces $L^A_{\rm loc}(\Omega)$ and $L^B_{\rm loc}(\Omega)$ unchanged.

Example 3.5. We specialize here Theorem 3.3 to the case when $\mathbf{F} \in L^q(\log L)^\beta_{\rm loc}(\Omega)$. Assume that $q > p'$ and $\beta \in \mathbb{R}$. An application of Theorem 3.3 tells us that, if $\mathbf{u}$ is any local weak solution to system (1.1), then

$$\mathbf{u} \in \begin{cases} L^{\frac{nq(p-1)}{n-q(p-1)}}(\log L)^{\frac{n\beta}{n-q(p-1)}}_{\rm loc}(\Omega) & \text{if } p' < q < \frac{n}{p-1} \text{ and } \beta \in \mathbb{R},\\ \exp L^{\frac{n}{n-1-\beta}}_{\rm loc}(\Omega) & \text{if } q = \frac{n}{p-1}, \text{ and } \beta < n-1,\\ \exp\exp L^{n'}_{\rm loc}(\Omega) & \text{if } q = \frac{n}{p-1}, \text{ and } \beta = n-1\\ L^\infty_{\rm loc}(\Omega) & \text{if either } q > \frac{n}{p-1} \text{ and } \beta \in \mathbb{R}, \text{ or } q = \frac{n}{p-1} \text{ and } \beta > n-1. \end{cases}$$

Theorem 3.3 is a consequence of Theorem 1.1 and of (a special case of) a characterization of boundedness properties of the potential $(\mathbf{V}_{\alpha,s})^{p-1}$ in Orlicz spaces established in [Ci], and recalled in Theorem B below. Its statement involves the functions $E_{\alpha,s}$ and $F_{\alpha,s}$ associated with the Young functions A and B as follows. Let $0 < \alpha < \min\{\frac{n}{s}, \frac{n}{s'}\}$. Define the functions $E_{\alpha,s}$ and $F_{\alpha,s} : [0,\infty) \to [0,\infty]$ as

$$E_{\alpha,s}(t) = \left(\int_0^t \left(\frac{\tau^{\frac{1}{s-1}-1+\frac{\alpha s'}{n}}}{A(\tau)^{\frac{\alpha s'}{n}}} \right)^{\frac{n}{n-\alpha s'}} d\tau \right)^{s-1-\frac{\alpha s}{n}}, \tag{3.12}$$

and

$$F_{\alpha,s}(t) = \left(\int_0^t \frac{B(\tau)}{\tau^{1+\frac{n}{n-\alpha s}}} d\tau \right)^{\frac{n-\alpha s}{n}} \tag{3.13}$$

for $t \geq 0$.

Theorem B. *Let $s > 1$, $n \geq 2$, and $0 < \alpha < \min\{\frac{n}{s}, \frac{n}{s'}\}$. Let Ω be a measurable subset of $\mathbb{R}^n$ with $|\Omega| < \infty$. Let A and B be Young functions such that the functions $E_{\alpha,s}$ and $F_{\alpha,s}$, defined as in* (3.12) *and* (3.13)*, are finite-valued. Assume that there exist $\gamma > 0$ and $t_0 > 0$ such that*

$$F_{\alpha,s}\left(\frac{E_{\alpha,s}(t)}{\gamma} \right) \leq \gamma \frac{A(t)}{t} \qquad \text{for } t > t_0. \tag{3.14}$$

Then there exists a constant $C = C(\alpha, s, n, \gamma, A)$ such that

$$\|(\mathbf{V}_{\alpha,s}\mathbf{f})^{s-1}\|_{L^B(\Omega)} \leq C\|\mathbf{f}\|_{L^A(\Omega)} \tag{3.15}$$

for every $\mathbf{f} \in L^A(\Omega)$.

Remark 3.6. The assumption on the finiteness of the functions $E_{\alpha,s}$ and $F_{\alpha,s}$ in Theorem B is irrelevant, by a reason analogous to that explained in Remark 3.4.

In the remaining part of this section, we present a few applications of Theorem 1.4 to the regularity of solutions to (1.1) in spaces of Campanato type, and, as a consequence, in BMO and in spaces of uniformly continuous functions.

Given an exponent $q \geq 1$ and a continuous function $\omega : (0,\infty) \to (0,\infty)$, the Campanato type space $\mathcal{L}^{\omega(\cdot),q}(\Omega)$ is the space of those functions $\mathbf{f} \in L^q_{\rm loc}(\Omega)$ for which the semi-norm

$$\|\mathbf{f}\|_{\mathcal{L}^{\omega(\cdot),q}(\Omega)} = \sup_{B_r \subset \Omega} \frac{1}{\omega(r)} \left(\fint_{B_r} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_r}|^q \, dx \right)^{\frac{1}{q}} \tag{3.16}$$

is finite. The space $\mathcal{L}^{\omega(\cdot),q}_{\rm loc}(\Omega)$ is defined accordingly, as the set of all functions $\mathbf{f}$ such that $\|\mathbf{f}\|_{\mathcal{L}^{\omega(\cdot),q}(\Omega')} < \infty$ for every open set $\Omega' \subset\subset \Omega$. Observe that $\mathcal{L}^{\omega(\cdot),q}_{\rm loc}(\Omega)$ depends only on the behavior of ω in a right neighborhood of 0. The same is true for $\mathcal{L}^{\omega(\cdot),q}(\Omega)$, if Ω is bounded.

When $q = 1$, we simply denote $\mathcal{L}^{\omega(\cdot),1}(\Omega)$ by $\mathcal{L}^{\omega(\cdot)}(\Omega)$, and similarly for local spaces. Let us notice that, if ω is non-decreasing, then

$$\mathcal{L}^{\omega(\cdot),q}_{\rm loc}(\Omega) = \mathcal{L}^{\omega(\cdot)}_{\rm loc}(\Omega)$$

for every $q \geq 1$. This is an easy consequence of the John-Nirenberg theorem, asserting that, if $\omega(r) = 1$ and $q \geq 1$, then

$$\mathcal{L}^{\omega(\cdot),q}_{\rm loc}(\Omega) = {\rm BMO}_{\rm loc}(\Omega), \tag{3.17}$$

the space of functions of (locally) bounded mean oscillation. Another standard choice is

$$\omega(r) = r^\beta \quad \text{for } r \geq 0, \tag{3.18}$$

for some $\beta \le 1$, in which case we also make use of the notation $\mathcal{L}^{\beta}(\Omega)$ instead of $\mathcal{L}^{\omega(\cdot)}(\Omega)$.
We next denote by $C^{\omega(\cdot)}(\Omega)$ the space of those functions $\mathbf{f}$ in Ω such that

$$\sup_{\substack{x,\, y \,\in\, \Omega \\ x \neq y}} \frac{|\mathbf{f}(x) - \mathbf{f}(y)|}{\omega(|x-y|)} < \infty\,.$$

The notation $C^{\omega(\cdot)}_{\rm loc}(\Omega)$ is employed accordingly. Thus, if $\lim_{r\to 0^+}\omega(r) = 0$, then $C^{\omega(\cdot)}(\Omega)$ is the space of uniformly continuous functions, with modulus of continuity not exceeding ω. In particular, if ω has the form (3.18) for some $\beta \in (0,1]$, then $C^{\omega(\cdot)}(\Omega)$ agrees with the space of Hölder continuous functions in Ω, with exponent β. The latter is denoted by $C^{\beta}(\Omega)$ when $\beta \in (0,1)$, and by ${\rm Lip}(\Omega)$ when $\beta = 1$. More generally, it is easily verified that

$$C^{\omega(\cdot)}_{\rm loc}(\Omega) \subset \mathcal{L}^{\omega(\cdot)}_{\rm loc}(\Omega)\,. \tag{3.19}$$

However, the reverse inclusion in (3.19) need not hold for an arbitrary ω, as shown, for example, by equation (3.17). Results from [Sp] ensure that, if ω decays at 0 so fast that

$$\int_0 \frac{\omega(r)}{r}\, dr < \infty, \tag{3.20}$$

then

$$\mathcal{L}^{\omega(\cdot)}_{\rm loc}(\Omega) \subset C^{\varpi(\cdot)}_{\rm loc}(\Omega)\,, \tag{3.21}$$

where the function $\varpi : [0,\infty) \to [0,\infty)$ is defined as

$$\varpi(r) = \int_0^r \frac{\omega(\rho)}{\rho}\, d\rho \qquad \text{for } r \ge 0. \tag{3.22}$$

For instance, if ω is given by (3.18), then (3.21) recovers Campanato's representation theorem asserting that

$$\mathcal{L}^{\beta}_{\rm loc}(\Omega) = \begin{cases} C^{\beta}_{\rm loc}(\Omega) & \text{if } \beta \in (0,1) \\ {\rm Lip}_{\rm loc}(\Omega) & \text{if } \beta = 1. \end{cases} \tag{3.23}$$

On the other hand, if (3.20) fails, and the function $r \mapsto \frac{\omega(r)}{r}$ is non-increasing near 0, then the space $\mathcal{L}^{\omega(\cdot)}_{\rm loc}(\Omega)$ is not even contained in $L^{\infty}_{\rm loc}(\Omega)$.
The Morrey type space $\mathcal{M}^{\omega(\cdot),q}(\Omega)$ is defined as the space of those functions $\mathbf{f} \in L^q_{\rm loc}(\Omega)$ such that the norm

$$\|\mathbf{f}\|_{\mathcal{M}^{\omega(\cdot),q}(\Omega)} = \sup_{B_r \subset \Omega} \frac{1}{\omega(r)} \bigg(\int_{B_r} |\mathbf{f}|^q\, dx \bigg)^{\frac{1}{q}} \tag{3.24}$$

is finite. The local Morrey space $\mathcal{M}^{\omega(\cdot),q}_{\rm loc}(\Omega)$ is defined with the usual modification. When ω has the form (3.18), we also make use of the alternate notation $\mathcal{M}^{\beta,q}(\Omega)$ instead of $\mathcal{M}^{\omega(\cdot),q}(\Omega)$.

The regularity of solutions $\mathbf{u}$ to system (1.1) in Campanato spaces, for data $\mathbf{F}$ in Morrey spaces, is described in the following result.

Theorem 3.7. *Let $\omega : (0,\infty) \to (0,\infty)$ be a continuous function, and let $\mu : (0,\infty) \to (0,\infty)$ be a function given by*

$$\mu(r) = r \bigg(\int_r^1 \frac{\omega(\varrho)}{\varrho^{\frac{n}{p'}+1}}\, d\varrho \bigg)^{\frac{1}{p-1}}$$

for r near 0. If $\mathbf{F} \in \mathcal{M}^{\omega(\cdot),p'}_{loc}(\Omega)$, and $\mathbf{u}$ is any local weak solution to system (1.1), then $\mathbf{u} \in \mathcal{L}^{\mu(\cdot)}_{loc}(\Omega)$.

Proof. Inequality (1.9) ensures that there exists a constant C such that

$$\frac{1}{\mu(r)} \fint_{B_r(x)} |\mathbf{u}(y) - \langle \mathbf{u} \rangle_{B_r(x)}| \, dy \le \frac{Cr}{\mu(r)} \bigg(\int_r^R \bigg(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} \, dy \bigg)^{\frac{1}{p'}} \frac{d\rho}{\rho} \bigg)^{\frac{1}{p-1}} + \frac{Cr}{\mu(r)} \fint_{B_R(x)} |\nabla \mathbf{u}| \, dy \tag{3.25}$$

for every $x \in \Omega$ and $R > 0$ such that $B_R(x) \subset \Omega$, and $r \in (0, R]$. On the other hand,

$$\begin{aligned} &\frac{r}{\mu(r)} \bigg(\int_r^R \bigg(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} \, dy \bigg)^{\frac{1}{p'}} \frac{d\rho}{\rho} \bigg)^{\frac{1}{p-1}} \\ &\le \frac{Cr}{\mu(r)} \bigg(\int_r^R \bigg(\int_{B_\rho(x)} |\mathbf{F}|^{p'} \, dy \bigg)^{\frac{1}{p'}} \frac{d\rho}{\rho^{\frac{n}{p'}+1}} \bigg)^{\frac{1}{p-1}} \\ &\le \frac{Cr}{\mu(r)} \bigg(\int_r^R \frac{\omega(\rho)}{\rho^{\frac{n}{p'}+1}} \, d\rho \bigg)^{\frac{1}{p-1}} \bigg(\sup_{\rho \in (0,R)} \frac{1}{\omega(\rho)} \bigg(\int_{B_\rho(x)} |\mathbf{F}|^{p'} \, dy \bigg)^{\frac{1}{p'}} \bigg)^{\frac{1}{p-1}} \end{aligned} \tag{3.26}$$

for some constant C and for every $r \in (0, R]$. Hence, the conclusion follows. □

Example 3.8. Assume that $\mathbf{F} \in \mathcal{M}^{\beta,p'}_{\mathrm{loc}}(\Omega)$ for some $\beta \ge (n-p)/p'$. An application of Theorem 3.7 tells us the following. If $\beta = (n-p)/p'$, then

$$\mathbf{u} \in \mathrm{BMO}_{\mathrm{loc}}(\Omega);$$

if $(n-p)/p' < \beta < n/p'$, then

$$\mathbf{u} \in C^{1-(\frac{n}{p} - \frac{\beta}{p-1})}_{\mathrm{loc}}(\Omega);$$

if $\beta = n/p'$, then

$$\mathbf{u} \in C^{\nu(\cdot)}_{\mathrm{loc}}(\Omega)$$

with $\nu(r) \approx r(\log 1/r)^{\frac{1}{p-1}}$ for r near 0; if $\beta > n/p'$, then

$$\mathbf{u} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega).$$

Corollary 3.9. *Assume that* $\mathbf{F} \in L^q_{\mathrm{loc}}(\Omega)$ *for some* $q > \max\{p', \frac{n}{p-1}\}$. *If* $\mathbf{u}$ *is any local weak solution to system* (1.1), *then* $\mathbf{u} \in C^{1-\frac{n}{q(p-1)}}_{\mathrm{loc}}(\Omega)$.

Proof. Hölder's inequality ensures that, if $\mathbf{F} \in L^q_{\mathrm{loc}}(\Omega)$, then $\mathbf{F} \in \mathcal{M}^{\frac{n(q-p')}{qp'},p'}_{\mathrm{loc}}(\Omega)$. Theorem 3.7 then entails that $u \in \mathcal{L}^{1-\frac{n}{q(p-1)}}_{\mathrm{loc}}(\Omega)$. Hence, the conclusion follows, owing to equation 3.23. □

Our last result concerns the Lipschitz continuity of weak solutions to system (1.1).

Theorem 3.10. *Let* $\omega : (0,\infty) \to (0,\infty)$ *be a continuous function fulfilling condition* (3.20). *If* $\mathbf{F} \in \mathcal{L}^{\omega(\cdot),p'}_{\mathrm{loc}}(\Omega)$, *and* $\mathbf{u}$ *is any local weak solution to system* (1.1), *then* $\mathbf{u} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

Proof. By inequality (1.4), there exists a constant C such that

$$\begin{aligned} \frac{1}{r} \fint_{B_r(x)} |\mathbf{u} - \langle \mathbf{u} \rangle_{B_r(x)}| \, dy &\le C \bigg(\int_r^R \bigg(\fint_{B_\rho(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_\rho(x)}|^{p'} \, dy \bigg)^{\frac{1}{p'}} \frac{d\rho}{\rho} \bigg)^{\frac{1}{p-1}} + C \fint_{B_R(x)} |\nabla \mathbf{u}| \, dy \\ &\le C \|\mathbf{F}\|_{\mathcal{L}^{\omega(\cdot),p'}_{\mathrm{loc}}(B_R(x))} \bigg(\int_0^R \frac{\omega(\rho)}{\rho} \, d\rho \bigg)^{\frac{1}{p-1}} + C \fint_{B_R(x)} |\nabla \mathbf{u}| \, dy \end{aligned} \tag{3.27}$$

for every $x \in \Omega$ and $R > 0$ such that $B_R(x) \subset \Omega$, and $r \in (0, R]$. Thus, $\mathbf{u} \in \mathcal{L}^{\omega(\cdot)}_{\rm loc}(\Omega)$, with $\omega(r) = r$, whence $\mathbf{u} \in \mathrm{Lip}_{\rm loc}(\Omega)$, by (3.23). □

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