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STABILITY OF THE ALE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS IN TIME-DEPENDENT DOMAINS

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Abstract. The paper is concerned with the analysis of the space-time discontinuous Galerkin method (STDGM) applied to the numerical solution of nonstationary nonlinear convection-diffusion initial-boundary value problem in a time-dependent domain. The problem is reformulated using the arbitrary Lagrangian-Eulerian (ALE) method, which replaces the classical partial time derivative by the so-called ALE derivative and an additional convective term. The problem is discretized with the use of the ALE-space time discontinuous Galerkin method (ALE-STDGM). In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The main attention is paid to the proof of the unconditional stability of the method. An important step is the generalization of a discrete characteristic function associated with the approximate solution and the derivation of its properties.

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Dedicated to Professor Chi-Wang Shu on the occasion of his 60th birthday

Introduction

Most of results on the solvability and numerical analysis of nonstationary partial differential equations(PDEs) are obtained under the assumption that a space domain Ω is independent of time t. However, problems in time-dependent domains Ω_t are important in a number of areas of science and technology. We can mention, for example, problems with moving boundaries, when the motion of the boundary $\partial\Omega_t$ is prescribed, or free boundary problems, when the motion of the boundary $\partial\Omega_t$ should be determined together with the solution of the PDEs in consideration. This is particularly the case of fluid-structure interaction (FSI), when the flow is solved in a domain deformed due to the coupling with an elastic structure.

There are various approaches to the solution of problems in time-dependent domains as, for example, fictitious domain method ([43]), or imersed boundary method ([10]). Very popular technique is the arbitrary Lagrangian-Eulerian (ALE) method based on a suitable one-to-one ALE mapping of the reference configuration Ω_0 onto the

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current configuration Ω_t. It is usually applied in connection with conforming finite element space discretization and combined with the time discretization by the use of a backward difference formula (BDF). From a wide literature we mention, e.g., the works [21], [39], [41], [42]. This method is analyzed theoretically for linear parabolic convection-diffusion initial-boundary value problems. The paper [35] investigates the stability of the ALE-conforming finite element method. In [4] and [36] error estimates for the ALE-conforming finite element method are derived.

In the numerical solution of compressible flow, it is suitable to apply the discontinuous Galerkin method(DGM) for the space discretization. It is based on piecewise polynomial approximations over finite element meshes, in general discontinuous on interfaces between neighbouring elements. This method was applied to the solution of compressible flow first in [8] and then in [9]. It allows a good resolution of boundary and internal layers (including shock waves and contact discontinuities) and has been used for the solution of various types of flow problems ([19], [26], [32]). Theory of the space DGM is a subject of a number of works. We cite only some of them: [2], [3], [13], [18], [46], [20], [21], [34], [38], [40], [45], [50]. It is also possible to refer to the monograph [20] containing a number of references.

In the cited works, the time discretization is carried out with the aid of the BDF of the first or second order. One possibility how to construct a higher order method in time is the application of the DGM in time. This technique uses a piecewise polynomial approximation in time, in general discontinuous at discrete time instants that form a partition in a time interval. This method was used for time discretization combined with conforming finite elements for the space discretization of linear parabolic equations in [1], [17], [47], [48], [49], [23], [24] and [25].

By the combination of the DGM in space and time we get the space-time discontinuous Galerkin method (STDGM). This method was theoretically analyzed in [7], [14], [29], [33], [51] and [20]. In [28] and [44], the BDF-DGM and STDGM is applied to linear and nonlinear dynamic elasticity problems. One of the advantages of the STDGM is the possibility to use different meshes on different time levels.

The mentioned methods have also been extended to the numerical solution of initial-boundary value problems in time-dependent domains using the ALE method. The ALE method combined with the space DGM and BDF in time (ALE-DGM-BDF) was applied with success to interaction of compressible flow with elastic structures in [15], [30], [37] and [44]. In [16], the ALE-STDGM is applied to the simulation of flow induced airfoil vibrations and the results are compared with the ALE-DGM-BDF approach. It appears that the ALE-STDGM is more robust and accurate.

The ALE-time discontinuous Galerkin semidiscretization of a linear parabolic convection-diffusion problem is analyzed in [11] and [12]. Both papers assume that the transport velocity is divergence free and consider homogeneous Dirichelt boundary condition. In [11], the stability of the ALE-time DGM is proved and [12] is devoted to the error estimation. The papers [5] and [6] are concerned with the stability analysis of the ALE-STDGM applied to a linear convection-diffusion initial-boundary value problem ([6]) as well as to the case with nonlinear convection and diffusion ([5]) with nonhomogeneous Dirichlet boundary condition, using piecewise linear DG time discretization.

In the present paper we extend the results from [5]. We deal with the stability analysis of the ALE-STDGM with arbitrary polynomial degree in space as well as in time, applied to a scalar nonstationary nonlinear convection-diffusion problem equipped with initial condition and nonhomogeneous Dirichlet boundary condition. This problem can be considered as a simplified prototype of the compressible Navier-Stokes system. The ALE-STDGM analyzed here corresponds to the technique used in [16] and [28] for the numerical simulation of airfoil vibrations induced by compressible flow. This means that the ALE mapping is constructed successively from one time slab to the next one.

The presented stability theory is based on estimates of forms from the definition of an approximate solution. An important tool is the concept of the discrete characteristic function introduced in [17] in the framework of the time DGM applied to a linear parabolic problem. The discrete characteristic function was generalized in connection with the STDGM for nonlinear parabolic problems in fixed domains ([7], [14]). Here we extend the concept of the discrete characteristic function and prove its important properties in the case of the ALE-STDGM in time dependent domains. On the basis of a technical analysis we obtain an unconditional stability of this method represented by a bound of the approximate solution in terms of data without any limitation of the time step in dependence on the size of the triangulations.

In Section 2 we formulate the continuous problem. Section 3 is devoted to the ALE-space time discretization. We describe here triangulations and ALE mappings and introduce important function spaces and concepts. Then an approximate solution is defined. Section 4 deals with the stability analysis. First some auxiliary results are presented. Then we introduce important estimates and the generalized concept of the discrete characteristic function. An important part is devoted to the derivation of its properties. Finally, the last part presents the proof of unconditional stability of the ALE-STDGM.

1. Formulation of the continuous problem

In what follows, we shall use the standard notation $L^2(\omega)$ for the Lebesgue space, $W^{k,p}(\omega)$, $H^k(\omega) = W^{k,2}(\omega)$ for the Sobolev spaces over a bounded domain $\omega \subset I\!R^d$, $d = 2, 3$, and the Bochner spaces $L^\infty(0, T; X)$ with a Banach space X and

$$\begin{aligned} &W^{1,\infty}(0,T;W^{1,\infty}(\Omega_t)) \\ &= \left\{ f \in L^\infty(0,T;W^{1,\infty}(\Omega_t));\ df/dt \in L^\infty(0,T;W^{1,\infty}(\Omega_t)) \right\}, \end{aligned}$$

where df/dt denotes here the distributional derivative.

If X is a Banach (Hilbert) space, then its norm (scalar product) will be denoted by $\|\cdot\|_X$ $((\cdot,\cdot)_X)$. By $|\cdot|_X$ we denote a seminorm in X. For simplicity we use the notation $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_\omega$, $(\cdot,\cdot)_{L^2(\omega)} = (\cdot,\cdot)_\omega$ and $\|\cdot\|_{L^2(\partial\omega)} = \|\cdot\|_{\partial\omega}$.

We shall be concerned with an initial-boundary value nonlinear convection-diffusion problem in a time-dependent bounded domain $\Omega_t \subset I\!R^d$, where $t \in [0, T]$, $T > 0$: Find a function $u = u(x, t)$ with $x \in \Omega_t$, $t \in (0, T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \mathrm{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{3}$$

We assume that $f_s \in C^1(I\!R)$, $f_s(0) = 0$,

$$|f_s'| \le L_f, \quad s = 1, \dots, d, \tag{4}$$

and function β is bounded and Lipschitz-continuous:

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{5}$$

$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{6}$$

Problem (1)–(3) can be reformulated with the aid of the so called arbitrary Lagrangian-Eulerian (ALE) method. It is based on a regular one-to-one ALE mapping of the reference configuration Ω_0 onto the current configuration Ω_t:

$$\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \quad X \in \overline{\Omega}_0 \to x = x(X,t) = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0,T]. \tag{7}$$

We can also write $\mathcal{A}(X,t) = \mathcal{A}_t(X)$, $X \in \overline{\Omega}_0$, $t \in [0,T]$. Usually it is supposed that the ALE mapping is sufficiently regular, e.g., $\mathcal{A} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega_t))$. In further considerations more general property will appear. Now we introduce the domain velocity

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X),\ \boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t),\ t \in [0,T],\ X \in \Omega_0,\ x \in \Omega_t, \tag{8}$$

and define the ALE derivative $D_t f = Df/Dt$ of a function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t), \tag{9}$$

where $\tilde{f}(X,t) = f(\mathcal{A}_t(X),t)$, $X \in \Omega_0$, and $x = \mathcal{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f, \tag{10}$$

which allows us to reformulate problem (1)–(3) in the ALE form:
Find $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z} \cdot \nabla u - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{11}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{12}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{13}$$

In what follows we shall be concerned with the numerical solution of the ALE problem (11)-(13) by the space-time discontinuous Galerkin method. In the theoretical analysis a number of various constants will appear. Some important constants in main assertions will be denoted by $C_{L1},\ C_{L2}$, etc. in Lemma 1, Lemma 2, etc. and $C_{T1},\ C_{T2}$, etc. in Theorem 1, Theorem 2, etc. Inside proofs, constants are denoted locally by c, c_1, c_2, c^* etc.

2. ALE-space time discretization

In the time interval $[0,T]$ we consider a partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1},\ I_m = (t_{m-1}, t_m),\ \overline{I}_m = [t_{m-1}, t_m]$ for $m = 1,\ldots,M,\ \tau = \max_{m=1,\ldots,M}\tau_m$. We assume that $\tau \in (0,\overline{\tau})$, where $\overline{\tau} > 0$. The space-time discontinuous Galerkin method (STDGM) has an advantage that on every time interval $\overline{I}_m = [t_{m-1}, t_m]$ it is possible to consider a different space partition (i. e. triangulation) – see, e. g. [20], [14]. Here we also use this possibility for the application of the STDGM in the framework of the ALE method. It allows to consider an ALE mapping separately on each time interval $[t_{m-1}, t_m)$ for $m = 1,\ldots,M$ and the resulting ALE mapping in $[0,T]$ may be discontinuous at time instants $t_m,\ m = 1,\ldots,M-1$. This means that one-sided limits $\mathcal{A}(t_m-) \neq \mathcal{A}(t_m+)$ in general. Similarly the same may hold for the approximate solution. Such situation appears in the numerical solution of fluid-structure interaction problems, when both the ALE mapping and the approximate flow solution are constructed successively on the time intervals $I_m,\ m = 1,\ldots,M$, by the space-time discontinuous Galerkin method (see, e.g., [16], [44]).

2.1. **ALE mappings and triangulations**

For every $m = 1,\ldots,M$ we consider a standard conforming triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$ in $\Omega_{t_{m-1}}$, where $h \in (0,\overline{h})$ and $\overline{h} > 0$. This triangulation is formed by a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$) with disjoint interiors. We assume that the domain $\Omega_{t_{m-1}}$ is polygonal (polyhedral). Further, for each $m = 1,\ldots,M$ we introduce a one-to-one ALE mapping

$$\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \overset{\text{onto}}{\longrightarrow} \overline{\Omega}_t \text{ for } t \in [t_{m-1}, t_m),\ h \in (0,\overline{h}). \tag{14}$$

We assume that $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping on the triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$, continuous in space variable $X \in \Omega_{t_{m-1}}$ and in time $t \in [0, t_m)$ and $\mathcal{A}_{h,t_{m-1}}^{m-1} = \mathrm{Id}$ (identical mapping). Hence, we assume that all domains Ω_t are polygonal (polyhedral). For every $t \in [t_{m-1}, t_m)$ we define the conforming triangulation

$$\mathcal{T}_{h,t} = \left\{ K = \mathcal{A}_{h,t}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\} \text{ in } \Omega_t. \tag{15}$$

At $t = t_m$ we define the one-sided limit $\mathcal{A}_{h,t_m-}^{m-1}$, introduce the triangulation

$$\mathcal{T}_{h,t_m-} = \{\mathcal{A}_{h,t_m-}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}\} \text{ in } \overline{\Omega}_{t_m}$$

and suppose that

$$\mathcal{A}_{h,t_m}^{m-1}\left(\overline{\Omega}_{t_{m-1}}\right) = \overline{\Omega}_{t_m}. \tag{16}$$

We have $\mathcal{T}_{h,t_{m-1}} = \hat{\mathcal{T}}_{h,t_{m-1}}$, but in general, $\mathcal{T}_{h,t_m-} \neq \hat{\mathcal{T}}_{h,t_m}$.

As we see, for every $t \in [0, T]$ we have a family $\{\mathcal{T}_{h,t}\}_{h \in (0,\overline{h})}$ of triangulations of the domain Ω_t.

Remark 1. *In general, the triangulations may be even nonconforming with hanging nodes (and hanging edges in 3D) and the ALE mapping may be nonaffine in the domain $\Omega_{t_{m-1}}$. However, the analysis would be rather complicated and, therefore, we are not concerned with such a situation.*

2.2. Discrete function spaces

In what follows, for every $m = 1, \dots, M$ we consider the space

$$S_h^{p,m-1} = \left\{ \varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\}, \tag{17}$$

where $p \geq 1$ is an integer and $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\leq p$. Now for every $t \in \overline{I}_m$ we define the space

$$S_h^{t,p,m-1} = \left\{ \varphi \in L^2(\Omega_t);\ \varphi \circ \mathcal{A}_{h,t}^{m-1} \in S_h^{p,m-1} \right\}. \tag{18}$$

It is possible to see that

$$S_h^{t,p,m-1} = \left\{ \varphi \in L^2(\Omega_t);\ \varphi|_K \in P^p(K)\ \forall K \in \mathcal{T}_{h,t} \right\}. \tag{19}$$

Of course, $S_h^{t_m,p,m-1} \neq S_h^{p,m}$ in general.

Further, let p, $q \geq 1$ be integers. By $P^q(I_m; S_h^{p,m-1})$ we denote the space of mappings of the time interval I_m into the space $S_h^{p,m-1}$ which are polynomials of degree $\leq q$ in time. We set

$$S_{h,\tau}^{p,q} = \left\{ \varphi;\ \varphi(t) \circ \mathcal{A}_{h,t}^{m-1}|_{I_m} \in P^q(I_m; S_h^{p,m-1}),\ m = 1, \dots, M \right\}. \tag{20}$$

This means that if $\varphi \in S_{h,\tau}^{p,q}$, then

$$\varphi\left(\mathcal{A}_{h,t}^{m-1}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \tag{21}$$
$$\vartheta_i \in S_h^{p,m-1},\ X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \dots, M,\ h \in (0, \overline{h}).$$

An approximate solution of problem (11)–(13) and test functions will be elements of the space $S_{h,\tau}^{p,q}$.

By D_t we denote the ALE derivative defined by (9) for $t \in \bigcup_{m=1}^{M} I_m$.

2.3. Some notation and important concepts

Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space

$$H^k(\Omega_t, \mathcal{T}_{h,t}) = \{v;\, v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega_t,\mathcal{T}_{h,t})} = \left(\sum_{K\in\mathcal{T}_{h,t}} |v|^2_{H^k(K)} \right)^{1/2},$$

where $|\cdot|_{H^k(K)}$ denotes the seminorm in the space $H^k(K)$.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}^I_{h,t}$ and the set of all boundary faces $\mathcal{F}^B_{h,t}$: $\mathcal{F}_{h,t} = \mathcal{F}^I_{h,t} \cup \mathcal{F}^B_{h,t}$. Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K^{(L)}_\Gamma$ and $K^{(R)}_\Gamma \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}^I_{h,t}$. Moreover, for $\Gamma \in \mathcal{F}^B_{h,t}$ the element adjacent to this face will be denoted by $K^{(L)}_\Gamma$. We shall use the convention, that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K^{(L)}_\Gamma$.

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v^{(L)}_\Gamma$ and $v^{(R)}_\Gamma$ will denote the traces of v on Γ from the side of elements $K^{(L)}_\Gamma$ and $K^{(R)}_\Gamma$, respectively. We set $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$, $h(\Gamma) = \operatorname{diam}\Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$ and $\langle v\rangle_\Gamma = \frac{1}{2}\left(v^{(L)}_\Gamma + v^{(R)}_\Gamma\right)$, $[v]_\Gamma = v^{(L)}_\Gamma - v^{(R)}_\Gamma$, for $\Gamma \in \mathcal{F}^I_{h,t}$. Moreover, by ρ_K we denote the diameter of the largest ball inscribed into $K \in \mathcal{T}_{h,t}$.

2.4. Discretization

First we introduce the space semidiscretization of problem (11)–(13). We assume that u is a sufficiently smooth solution of our problem. If we choose an arbitrary but fixed $t \in (0,T)$, multiply equation (11) by a test function $\varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, integrate over any element K and finally sum over all elements $K \in \mathcal{T}_{h,t}$, then for $t \in I_m$ we get

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{h,t}} \int_K \frac{Du}{Dt}\varphi \,\mathrm{d}x + \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d \frac{\partial f_s(u)}{\partial x_s}\varphi \,\mathrm{d}x \\
&- \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d z_s \frac{\partial u}{\partial x_s}\varphi \,\mathrm{d}x - \sum_{K\in\mathcal{T}_{h,t}} \int_K \operatorname{div}(\beta(u)\nabla u)\varphi \,\mathrm{d}x = \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi \,\mathrm{d}x.
\end{aligned} \tag{22}$$

Applying Green's theorem to the convection and diffusion terms, introducing the concept of a numerical flux and suitable expressions mutually vanishing, after some manipulation we arrive at the identity

$$(D_t u, \varphi) + A_h(u,\varphi,t) + b_h(u,\varphi,t) + d_h(u,\varphi,t) = l_h(\varphi,t), \tag{23}$$

where the forms appearing here are defined for $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, $\theta \in I\!R$ and $c_W > 0$ in the following way

$$a_h(u,\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u\cdot\nabla\varphi\,\mathrm{d}x \tag{24}$$

$$- \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left(\langle\beta(u)\nabla u\rangle\cdot\mathbf{n}_\Gamma\,[\varphi] + \theta\,\langle\beta(u)\nabla\varphi\rangle\cdot\mathbf{n}_\Gamma\,[u]\right)\mathrm{d}S$$

$$- \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \left(\beta(u)\nabla u\cdot\mathbf{n}_\Gamma\,\varphi + \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u - \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u_D\right)\mathrm{d}S,$$

$$J_h(u,\varphi,t) = c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} h(\Gamma)^{-1}\int_\Gamma [u]\,[\varphi]\,\mathrm{d}S \tag{25}$$

$$+ c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S,$$

$$J_h^B(u,\varphi,t) = c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S, \tag{26}$$

$$A_h(u,\varphi,t) = a_h(u,\varphi,t) + \beta_0\,J_h(u,\varphi,t), \tag{27}$$

$$b_h(u,\varphi,t) = -\sum_{K\in\mathcal{T}_{h,t}}\int_K \sum_{s=1}^d f_s(u)\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \tag{28}$$

$$+ \sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma)\,[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma)\,\varphi\,\mathrm{d}S,$$

$$d_h(u,\varphi,t) = -\sum_{K\in\mathcal{T}_{h,t}}\int_K \sum_{s=1}^d z_s\frac{\partial u}{\partial x_s}\varphi\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_{h,t}}\int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x, \tag{29}$$

$$l_h(\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}}\int_K g\varphi\,\mathrm{d}x + \beta_0\,c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S. \tag{30}$$

Let us note that in integrals over faces we omit the subscript Γ of $\langle\cdot\rangle$ and $[\cdot]$. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

In (28), H is a numerical flux with the following properties:

(H1) $H(u,v,\mathbf{n})$ is defined in $\mathbb{R}^d\times B_1$, where $B_1 = \{\mathbf{n}\in\mathbb{R}^d;\ |\mathbf{n}| = 1\}$, and is Lipschitz-continuous with respect to u, v: there exists $L_H > 0$ such that
$|H(u,v,\mathbf{n}) - H(u^*,v^*,\mathbf{n})| \le L_H(|u-u^*| + |v-v^*|)$, for all $u, v, u^*, v^* \in\mathbb{R}$.

(H2) H is consistent: $H(u,u,\mathbf{n}) = \sum_{s=1}^d f_s(u)n_s, \quad u\in\mathbb{R},\ \mathbf{n}\in B_1$,

(H3) H is conservative: $H(u,v,\mathbf{n}) = -H(v,u,-\mathbf{n}), \quad u,v\in\mathbb{R},\ \mathbf{n}\in B_1$.

In what follows, in the stability analysis we shall use the properties **(H1)** and **(H2)**. (Assumption **(H3)** is important for error estimation, but here it is not necessary.)

For a function φ defined in $\bigcup_{m=1}^M I_m$ we denote

$$\varphi_m^\pm = \varphi(t_m\pm) = \lim_{t\to t_m\pm}\varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-), \tag{31}$$

if the one-sided limits $\varphi_m^\pm$ exist.

Now we define an ALE-STDG approximate solution of problem (11)–(13).

Definition 1. *A function U is an approximate solution of problem (11)–(13), if $U \in S_{h,\tau}^{p,q}$ and*

$$\int_{I_m} \big((D_t U, \varphi)_{\Omega_t} + A_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t)\big)\,\mathrm{dt} \tag{32}$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^+)_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{dt} \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1,\dots,M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S_h^{p,0}. \tag{33}$$

(For $m = 1$ we set $\{U\}_{m-1} = \{U\}_0 := U_0^+ - U_0^-$ with U_0^- given by (33)).

The ALE-STDG numerical method (32)–(33) was applied in [16] and [44] to the numerical simulation of a compressible flow in time-dependent domains and fluid-structure interaction.

3. Analysis of the stability

3.1. Some auxiliary results

As was mentioned in Section 2.1, for each $t \in [0,T]$ we consider a system of triangulations $\{\mathcal{T}_{h,t}\}_{h\in(0,\overline{h})}$. We assume that these systems are uniformly shape regular. This means that there exists a positive constant c_R, independent of K, t and h such that

$$\frac{h_K}{\rho_K} \le c_R \quad \text{for all} \quad K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}), t \in [t_{m-1}, t_m], \tag{34}$$
$$\tau_m \le \tau \in (0,\overline{\tau}),\ m = 1,\dots,M.$$

By $(\mathcal{A}_{h,t}^{m-1})^{-1}$ we denote the inverse to the mapping $\mathcal{A}_{h,t}^{m-1}$. The symbols $\frac{d\mathcal{A}_{h,t}^{m-1}}{dX}$ and $\frac{d(\mathcal{A}_{h,t}^{m-1})^{-1}}{dx}$ denote the Jacobian matrices of $\mathcal{A}_{h,t}^{m-1}$ and $(\mathcal{A}_{h,t}^{m-1})^{-1}$, respectively. The entries of $\frac{d\mathcal{A}_{h,t}^{m-1}}{dX}$ and $\frac{d(\mathcal{A}_{h,t}^{m-1})^{-1}}{dx}$ are constant on every element $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and $K \in \mathcal{T}_{h,t}$, respectively. Moreover, we define the Jacobians $J(X,t) = \det\frac{d\mathcal{A}_{h,t}^{m-1}(X)}{dX}$, $X \in \Omega_{t_{m-1}}$, and $J^{-1}(x,t) = \det\frac{d(\mathcal{A}_{h,t}^{m-1}(x))^{-1}}{dx}$, $x \in \Omega_t$. The Jacobians J and J^{-1} are piecewise constant over $\hat{\mathcal{T}}_{h,t_{m-1}}$ and $\mathcal{T}_{h,t}$, respectively. The constant value of J on $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and of J^{-1} on $K \in \mathcal{T}_{h,t}$ will be denoted by $J_{\hat{K}}$ and J_K^{-1}, respectively. Of course, these terms depend on t and, hence, $J_{\hat{K}} = J_{\hat{K}}(t)$ and $J_K^{-1} = J_K^{-1}(t)$.

In what follows, we assume that

$$\mathcal{A}_{h,t}^{m-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_{t_{m-1}})), \quad m = 1,\dots,M,\ h \in (0,\overline{h}) \tag{35}$$

and

$$(A_{h,t}^{m-1})^{-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_t)), \quad m = 1,\dots M,\ h \in (0,\overline{h}). \tag{36}$$

Obviously, we have $J \in W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}}))$, $J^{-1} \in W^{1,\infty}(I_m; L^\infty(\Omega_t))$. Since $\mathcal{A}_{h,t_{m-1}}^{m-1}$ is the identical mapping and, hence, $J(X, t_{m-1}) = 1$, we assume that there exist constants C_J^-, $C_J^+ > 0$ such that the Jacobians satisfy the conditions

$$C_J^- \le J(X,t) \le C_J^+, \quad X \in \overline{\Omega}_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1,\dots,M,\ h \in (0,\overline{h}), \tag{37}$$
$$(C_J^+)^{-1} \le J^{-1}(x,t) \le (C_J^-)^{-1}, \quad x \in \overline{\Omega}_t,\ t \in \overline{I}_m,\ m = 1,\dots,M,\ h \in (0,\overline{h}).$$

Finally, there exist constants $C_A^-, C_A^+ > 0$ such that

$$\left\| \frac{d\mathcal{A}_{h,t}^{m-1}(X)}{dX} \right\| \le C_A^+, \; X \in \overline{\Omega}_{t_{m-1}}, \; t \in \overline{I}_m, \; m = 1, \dots, M, \; h \in (0, \overline{h}), \tag{38}$$

$$\left\| \frac{d(\mathcal{A}_{h,t}^{m-1})^{-1}(x)}{dx} \right\| \le C_A^-, \; x \in \overline{\Omega}_t, \; t \in \overline{I}_m, \; m = 1, \dots, M, \; h \in (0, \overline{h}), \tag{39}$$

where $\| \cdot \|$ is the matrix norm induced by the Euclidean norm $| \cdot |$ in $I\!R^d$.

The above assumptions imply the following properties of the domain velocity: There exists a constant $c_z > 0$ such that

$$|\boldsymbol{z}(x,t)|, \; |\mathrm{div} \boldsymbol{z}(x,t)| \le c_z \quad \text{for } x \in \Omega_t, \; t \in (0,T). \tag{40}$$

In what follows, for the sake of simplicity, we use the notation $\mathcal{A}_t$ for the ALE mapping defined in $\bigcup_{m=1}^M I_m$ so that

$$\mathcal{A}_t(X) = \mathcal{A}_{h,t}^{m-1}(X) \quad \text{for } X \in \overline{\Omega}_{t_{m-1}}, \; t \in \overline{I}_m, \; m = 1, \dots, M, \; h \in (0, \overline{h}). \tag{41}$$

The symbol $\mathcal{A}_t^{-1}$ will denote the inverse to $\mathcal{A}_t$. This means that $\mathcal{A}_t^{-1} : \overline{\Omega}_t \overset{\text{onto}}{\longrightarrow} \overline{\Omega}_{t_{m-1}}$ for $t \in \overline{I}_m, \; m = 1, \dots, M$.

Under assumption (34), the multiplicative trace inequality and the inverse inequality hold: There exist constants $c_M, c_I > 0$ independent of v, h, t and K such that

$$\begin{aligned} &\|v\|_{L^2(\partial K)}^2 \le c_M \left(\|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \|v\|_{L^2(K)}^2 \right), \\ &v \in H^1(K), \; K \in \mathcal{T}_{h,t}, \; h \in (0, \overline{h}), \; t \in [0,T], \end{aligned} \tag{42}$$

and

$$\begin{aligned} &|v|_{H^1(K)} \le c_I \, h_K^{-1} \|v\|_{L^2(K)}, \\ &v \in P^p(K), \; K \in \mathcal{T}_{h,t}, \; h \in (0, \overline{h}), \; t \in [0,T]. \end{aligned} \tag{43}$$

In the space $H^1(\Omega_t, \mathcal{T}_{h,t})$ we define the norm

$$\|\varphi\|_{DG,t} = \left(\sum_{K \in \mathcal{T}_{h,t}} |\varphi|_{H^1(K)}^2 + J_h(\varphi, \varphi, t) \right)^{1/2}. \tag{44}$$

Moreover, over $\partial\Omega$ we define the norm

$$\|u_D\|_{DGB,t} = \left(c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \, \mathrm{d}S \right)^{1/2} = \left(J_h^B(u_D, u_D, t) \right)^{1/2}. \tag{45}$$

If we use $\varphi := U$ as a test function in (32), we get the basic identity

$$\begin{aligned} &\int_{I_m} \left((D_t U, U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + d_h(U,U,t) \right) \mathrm{d}t \\ &+ (\{U\}_{m-1}, U_{m-1}^+)_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U,t) \, \mathrm{d}t. \end{aligned} \tag{46}$$

3.2. Important estimates

Here we estimate the forms from the definition of the approximate solution. The proofs can be carried out in a similar way as in [5]. For a sufficiently large constant c_W we obtain the coercivity of the diffusion and penalty terms.

Lemma 1. *Let*

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad for \quad \theta = -1 \ (NIPG), \tag{47}$$

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad for \quad \theta = 0 \ (IIPG), \tag{48}$$

$$c_W \geq \frac{16\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad for \quad \theta = 1 \ (SIPG). \tag{49}$$

Then

$$\begin{aligned} &\int_{I_m} \left(a_h(U,U,t) + \beta_0 \, J_h(U,U,t)\right) \mathrm{d}t \\ &\geq \frac{\beta_0}{2} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t - \frac{\beta_0}{2} \int_{I_m} \|u_D\|_{DGB,t}^2 \, \mathrm{d}t. \end{aligned} \tag{50}$$

Further, we estimate the convection terms:

Lemma 2. *For each $k_1 > 0$ there exists a constant $c_b > 0$ such that we have the inequality*

$$\int_{I_m} |b_h(U,U,t)| \mathrm{d}t \leq \frac{\beta_0}{2k_1} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t + c_b \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t. \tag{51}$$

Lemma 3. *For each $k_2 > 0$ there exists a constant $c_d > 0$ such that we have the inequality*

$$\int_{I_m} |d_h(U,U,t)| \, \mathrm{d}t \leq \frac{\beta_0}{2k_2} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t + \frac{c_d}{2\beta_0} \int_{I_m} \|U\|_{\Omega_t}^2 \, \mathrm{d}t. \tag{52}$$

We also need to estimate the right-hand side form:

Lemma 4. *For any $k_3 > 0$ we have*

$$\begin{aligned} \int_{I_m} |l_h(U,t)| \, \mathrm{d}t \leq &\frac{1}{2} \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|U\|_{\Omega_t}^2\right) \, \mathrm{d}t \\ &+ \frac{\beta_0 k_3}{2} \int_{I_m} \|u_D\|_{DGB,t}^2 \, \mathrm{d}t + \frac{\beta_0}{2k_3} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t. \end{aligned} \tag{53}$$

Finally we need to estimate the term with the ALE derivative:

Lemma 5. *It holds that*

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t \geq \frac{1}{2}\left(\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t \right), \tag{54}$$

$$\left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} = \frac{1}{2}\left(\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \right), \tag{55}$$

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} \geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t - \left(U_{m-1}^-, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}. \tag{56}$$

Proof. We start with the first inequality. We have

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t = \sum_{K\in\mathcal{T}_{h,t}} \int_{I_m} (D_t U, U)_K\,\mathrm{d}t. \tag{57}$$

By virtue of relation (15), the Reynolds transport theorem (see, e.g. [27] or [1]) and relation (10), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_K U^2(x,t)\,\mathrm{d}x \\
&= \int_K \left(\frac{\partial U^2(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla(U^2(x,t)) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t) \right)\mathrm{d}x \\
&= \int_K \left(2U(x,t)\left(\frac{\partial U(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla U(x,t) \right) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t) \right)\mathrm{d}x \\
&= 2(D_t U, U)_K + (U^2, \mathrm{div}\,\boldsymbol{z})_K.
\end{aligned} \tag{58}$$

Expressing $(D_t U, U)_K$, summing over $K \in \mathcal{T}_{h,t}$ and integrating over I_m together with assumption (40) yield

$$\begin{aligned}
&\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{I_m} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} U^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{I_m} (U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\
&= \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m} (U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\
&\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t,
\end{aligned} \tag{59}$$

which is (54).

Further, by a simple manipulation we find that

$$\begin{aligned}
&2(U_{m-1}^+ - U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} \\
&= \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}},
\end{aligned}$$

which immediately implies (55).

Concerning inequality (56), from (59) we get

$$\begin{aligned}
&\int_{I_m} (D_t U, U)_{\Omega_t} \mathrm{d}t + \big(\{U\}_{m-1}, U_{m-1}^{+}\big)_{\Omega_{t_{m-1}}} \\
&= \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \\
&\quad -\frac{1}{2}\int_{I_m} (U^2, \operatorname{div} \boldsymbol{z})_{\Omega_t} \mathrm{d}t + \|U_{m-1}^+\|_{\Omega_{t_{m-1}}} - (U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} \\
&\geq \frac{1}{2}\left(\|U_m^-\|^2_{\Omega_{t_m}} + \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t\right) - \big(U_{m-1}^-, U_{m-1}^+\big)_{\Omega_{t_{m-1}}},
\end{aligned}$$

which proves the lemma. □

3.3. Discrete characteristic function

In our further considerations, the concept of a discrete characteristic function will play an important role. The discrete characteristic function was introduced in [17] in the framework of the time discontinuous Galerkin method combined with conforming finite elements applied to a linear parabolic problem. The discrete characteristic function was generalized in connection with the STDGM for nonlinear parabolic problems in [7], [14], [20]. Here it is generalized to time-dependent domains.

For $m = 1, \ldots, M$ we use the following notation:
$U = U(x,t)$, $x \in \Omega_t, t \in I_m$ will denote the approximate solution in Ω_t, and
$\tilde{U} = \tilde{U}(X,t) = U(\mathcal{A}_t(X), t)$, $X \in \Omega_{t_{m-1}}, t \in I_m$ denotes the approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$.
For $s \in I_m$ we denote $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t)$, $X \in \Omega_{t_{m-1}}, t \in I_m$, the discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$. It is defined as $\tilde{\mathcal{U}}_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$\int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}} \mathrm{d}t = \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}} \mathrm{d}t \quad \forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}), \tag{60}$$

$$\tilde{\mathcal{U}}_s(X, t_{m-1}^+) = \tilde{U}(X, t_{m-1}^+), \ X \in \Omega_{t_{m-1}}. \tag{61}$$

Further, we introduce the discrete characteristic function $\mathcal{U}_s = \mathcal{U}_s(x,t)$, $x \in \Omega_t, t \in I_m$ to $U \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$\mathcal{U}_s(x,t) = \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x), t), \ x \in \Omega_t, \ t \in I_m. \tag{62}$$

Hence, in view of (20), $\mathcal{U}_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$\mathcal{U}_s(X, t_{m-1}+) = U(X, t_{m-1}+). \tag{63}$$

In what follows, we prove some important properties of the discrete characteristic function. Namely, we prove that the discrete characteristic function mapping $U \to \mathcal{U}_s$ is continuous with respect of the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$. In the proof we use a result from [7] for the discrete characteristic function on a reference domain: There exists a constant $\tilde{c}_{CH}^{(1)} > 0$ depending on q only such that

$$\int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \leq \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t, \tag{64}$$

for all $m = 1, \ldots, M$ and $h \in (0, \overline{h})$.

Lemma 6. *There exist constants* $C_{L6}^{*},\ C_{L6}^{**} > 0$ *such that*

$$C_{L6}^{*}\, h(\hat{\Gamma})^{-1} \le h(\Gamma)^{-1} \le C_{L6}^{**}\, h(\hat{\Gamma})^{-1} \tag{65}$$

for all $\hat{\Gamma} \in \mathcal{F}_{h,t_{m-1}}, \Gamma = \mathcal{A}_t(\hat{\Gamma}) \in \mathcal{F}_{h,t}$ *and all* $t \in \overline{I}_m,\ m = 1,\ldots,M,\ h \in (0,\overline{h})$.

Proof. We use the relation between Γ and $\hat{\Gamma}$ and the properties (38) and (39) of the mappings $\mathcal{A}_t$ and $\mathcal{A}_t^{-1}$. We also take into account that $\hat{\Gamma} \subset \hat{K}$ for some $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, $\Gamma \subset K = \mathcal{A}_t(\hat{K}) \in \mathcal{T}_{h,t}$ and that the Jacobian matrices $\frac{d\mathcal{A}_t}{dX}$ and $\frac{d\mathcal{A}_t^{-1}}{dx}$ are constant on $\hat{K}$ and K, respectively. Then we can write

$$\begin{aligned} h(\Gamma) &= \operatorname{diam}(\Gamma) = \max_{x,x^*\in\Gamma} |x - x^*| = \max_{X,X^*\in\hat{\Gamma}} |\mathcal{A}_t(X) - \mathcal{A}_t(X^*)| \\ &\le \max_{X\in\hat{\Gamma}} \left\| \frac{d\mathcal{A}_t(X)}{dX} \right\| \max_{X,X^*\in\hat{\Gamma}} |X - X^*| \le C_A^{+} \max_{X,X^*\in\hat{\Gamma}} |X - X^*| = C_A^{+}\, h(\hat{\Gamma}). \end{aligned}$$

Similarly, we get $h(\hat{\Gamma}) \le C_A^{-}\, h(\Gamma)$. These inequalities immediately imply (65) with $C_{L6}^{*} = (C_A^{+})^{-1}$ and $C_{L6}^{**} = C_A^{-}$. □

Theorem 1. *There exist constants* $c_{CH}^{(1)}, c_{CH}^{(2)} > 0$, *such that*

$$\int_{I_m} \|\mathcal{U}_s\|_{\Omega_t}^2 \,\mathrm{d}t \le c_{CH}^{(1)} \int_{I_m} \|U\|_{\Omega_t}^2 \,\mathrm{d}t \tag{66}$$

$$\int_{I_m} \|\mathcal{U}_s\|_{DG,t}^2 \,\mathrm{d}t \le c_{CH}^{(2)} \int_{I_m} \|U\|_{DG,t}^2 \,\mathrm{d}t \tag{67}$$

for all $s \in I_m,\ m = 1,\ldots,M$ *and* $h \in (0,\overline{h})$.

Proof. We begin with the proof of the first inequality. We have

$$\begin{aligned} &\|\mathcal{U}_s(t)\|_{\Omega_t}^2 = \int_{\Omega_t} |\mathcal{U}_s(x,t)|^2 \,\mathrm{d}x = \int_{\Omega_t} |\tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t)|^2 \,\mathrm{d}x \\ &= \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 J(X,t) \,\mathrm{d}X \le C_J^{+} \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 \,\mathrm{d}X \\ &= C_J^{+} \|\tilde{\mathcal{U}}_s(t)\|_{\Omega_{t_{m-1}}}^2 \end{aligned}$$

Integrating over I_m and using (64), we obtain

$$\begin{aligned}
\int_{I_m} \|\mathcal{U}_s(t)\|^2_{\Omega_t}\,\mathrm{d}t &\le C_J^+ \int_{I_m} \|\tilde{\mathcal{U}}_s(t)\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t \\
&\le C_J^+ \tilde{c}^{(1)}_{CH} \int_{I_m} \|\tilde{U}(t)\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t \\
&= C_J^+ \tilde{c}^{(1)}_{CH} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2\,\mathrm{d}X \right) \mathrm{d}t \\
&= C_J^+ \tilde{c}^{(1)}_{CH} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |U(\mathcal{A}_t(X),t)|^2\,\mathrm{d}X \right) \mathrm{d}t \\
&= C_J^+ \tilde{c}^{(1)}_{CH} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 J^{-1}(x,t)\,\mathrm{d}x \right) \mathrm{d}t \\
&\le C_J^+ \tilde{c}^{(1)}_{CH} C_J^- \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2\,\mathrm{d}x \right) \mathrm{d}t \\
&= C_J^+ \tilde{c}^{(1)}_{CH} C_J^- \int_{I_m} \|U(t)\|^2_{\Omega_t}\,\mathrm{d}t.
\end{aligned}$$

Setting $c^{(1)}_{CH} = C_J^+ \tilde{c}^{(1)}_{CH} C_J^-$, we get (66).

Now we pay our attention to the proof of the second inequality in the theorem. From the definition of the DG-norm we have

$$\begin{aligned}
&\int_{I_m} ||\mathcal{U}_s||^2_{DG,t}\,\mathrm{d}t \qquad (68)\\
&= \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s|^2_{H^1(K)}\,\mathrm{d}t + \int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2\,\mathrm{d}S \right) \mathrm{d}t \\
&+ \int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}^B_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma |\mathcal{U}_s|^2\,\mathrm{d}S \right) \mathrm{d}t,
\end{aligned}$$

where $\mathcal{F}^I_{h,t} = \{\mathcal{A}^{m-1}_{h,t}(\hat{\Gamma});\ \hat{\Gamma}\in\mathcal{F}^I_{h,t_{m-1}}\}$ and similarly $\mathcal{F}^B_{h,t} = \{\mathcal{A}^{m-1}_{h,t}(\hat{\Gamma});\ \hat{\Gamma}\in\mathcal{F}^B_{h,t_{m-1}}\}$.

Further, we estimate each term on the right-hand side of (68). From [20], relation (6.161), it follows that

$$\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \le \tilde{c}^{(2)}_{CH} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{U}(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t, \qquad (69)$$

with a constant $\tilde{c}^{(2)}_{CH} > 0$ depending on q only. For simplicity let us denote

$$B_t = B_t(X) = \frac{d\mathcal{A}^{m-1}_{h,t}(X)}{dX}, \quad B_t^{-1} = B_t^{-1}(x) = \frac{d(\mathcal{A}^{m-1}_{h,t})^{-1}(x)}{dx}.$$

Then it follows from (38) and (39) that $\|B_t\| \le C_A^+$ and $\|B_t^{-1}\| \le C_A^-$.
Now, for $K \in \mathcal{T}_{h,t}$, $K = \mathcal{A}_t(\hat{K})$ with $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, using that $\|B_t|_{\hat{K}}\|$ and $\|B_t^{-1}|_{\hat{K}}\|$ are constant, we have

$$
\begin{aligned}
|\mathcal{U}_s(t)|^2_{H^1(K)} &= \int_K |\nabla \mathcal{U}_s(x,t)|^2 \,\mathrm{d}x = \int_K \left|\nabla \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t)\right|^2 \mathrm{d}x \\
&\le \int_{\hat{K}} \left|B_t^{-1}|_K \nabla \tilde{\mathcal{U}}_s(X,t)\right|^2 J(X,t)\,\mathrm{d}X \le (C_A^-)^2 C_J^+ \,|\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}.
\end{aligned}
\tag{70}
$$

The summation over all $K \in \mathcal{T}_{h,t}$, integration over I_m and the use of (69) imply that

$$
\begin{aligned}
&\int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s(t)|^2_{H^1(K)}\,\mathrm{d}t \\
&\le (C_A^-)^2 C_J^+ \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \\
&\le (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde{U}(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \\
&= (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \left(\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} |\nabla \tilde{U}(X,t)|^2\,\mathrm{d}X \right) \mathrm{d}t \\
&= (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \left(\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \underbrace{|\nabla(U(\mathcal{A}_t(X),t))|^2}_{=\left|B_t|_{\hat{K}}(\nabla U)(\mathcal{A}_t(X),t)\right|^2} \,\mathrm{d}X \right) \mathrm{d}t
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
&\le (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_K |\nabla U(x,t)|^2 J_K^{-1}\,\mathrm{d}x \right) \mathrm{d}t \\
&\le (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_K |\nabla U(t)|^2 \|B_t\|^2 J_K^{-1}\,\mathrm{d}x \right) \mathrm{d}t \\
&\le (C_A^-)^2 C_J^+ (C_J^-)^{-1} \tilde{c}^{(2)}_{CH} (C_A^+)^2 \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |U(t)|^2_{H^1(K)}\,\mathrm{d}t \\
&= C^{(a)}_{CH} \int_{I_m} |U(t)|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t,
\end{aligned}
$$

where $C^{(a)}_{CH} := (C_A^-)^2 C_J^+ (C_J^-)^{-1} \tilde{c}^{(2)}_{CH} (C_A^+)^2$.

Now we turn our attention to the term

$$
\int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2\,\mathrm{d}S \right) \mathrm{d}t.
$$

For simplicity we assume that $d = 2$. In Appendix we briefly describe the proof for $d = 3$. We will use estimate (6.162) from [20], which implies that

$$\begin{aligned}
&\int_{I_m} \left(\sum_{\hat\Gamma \in \mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat\Gamma)} \int_{\hat\Gamma} [\tilde{\mathcal{U}}_s]^2 \, \mathrm{d}S^{\hat\Gamma} \right) \mathrm{d}t \\
&\le \tilde{c}^{(3)}_{CH} \int_{I_m} \left(\sum_{\hat\Gamma \in \mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat\Gamma)} \int_{\hat\Gamma} [\tilde{U}]^2 \, \mathrm{d}S^{\hat\Gamma} \right) \mathrm{d}t.
\end{aligned} \tag{72}$$

(Here $\mathrm{d}S^{\hat\Gamma}$ denotes the element of the arc $\hat\Gamma$. Similarly we use the notation $\mathrm{d}S^{\Gamma}$.)

Now we consider the relation $\Gamma = \mathcal{A}_t(\hat\Gamma)$, $\hat\Gamma \in \mathcal{F}^I_{h,t_{m-1}}$, and introduce a parametrization of $\hat\Gamma$:

$$\hat\Gamma = \mathcal{B}^{\hat\Gamma}_{m-1}([0,1]) = \{X = \mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon); \upsilon \in [0,1]\}.$$

Then an element of $\hat\Gamma$ can be expressed as

$$\mathrm{d}S^{\hat\Gamma} = |(\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon)| \, \mathrm{d}\upsilon, \quad \upsilon \in [0,1].$$

These relations imply that

$$\begin{aligned}
\Gamma &= \{x = \mathcal{A}_t(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon)); \upsilon \in [0,1]\} \\
\mathrm{d}S^{\Gamma} &= \left| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon))(\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon) \right| \mathrm{d}\upsilon, \quad \upsilon \in [0,1].
\end{aligned}$$

The term $(\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon)$ is a tangent vector to $\hat\Gamma$ at the point $\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of

$$\frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon))(\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon)$$

are identical from the sides of both elements $K^{(L)}_{\hat\Gamma}$ and $K^{(R)}_{\hat\Gamma}$ adjacent to $\hat\Gamma$. Then we can use the above relations, inequalities (65), (38), and write

$$\begin{aligned}
&\int_{\Gamma} \frac{1}{h(\Gamma)} [\mathcal{U}_s]^2 \mathrm{d}S^{\Gamma} \\
&= \int_0^1 \frac{1}{h(\Gamma)} [\mathcal{U}_s(\mathcal{A}_t(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon)))]^2 \left| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon))(\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon) \right| \mathrm{d}\upsilon \\
&\le \int_0^1 \frac{1}{h(\Gamma)} [\tilde{\mathcal{U}}_s(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon))]^2 \underbrace{\left\| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon)) \right\|}_{\le C^+_A} \left| (\mathcal{B}^{\hat\Gamma}_{m-1})'(\upsilon) \right| \mathrm{d}\upsilon \\
&\le C^+_A \int_{\hat\Gamma} \frac{C^{**}_{L6}}{h(\hat\Gamma)} [\tilde{\mathcal{U}}_s]^2 \mathrm{d}S^{\hat\Gamma}.
\end{aligned} \tag{73}$$

From (72) and (73) we get

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[\mathcal{U}_s]^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t \tag{74}$$
$$\le \tilde{c}^{(3)}_{CH}C^+_A C^{**}_{L6}\int_{I_m}\left(\sum_{\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}}\frac{c_W}{h(\hat\Gamma)}\int_{\hat\Gamma}[\tilde U]^2\,\mathrm{d}S^{\hat\Gamma}\right)\mathrm{d}t.$$

Further, for $\Gamma=\mathcal{A}_t(\hat\Gamma)$, where $\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}$, we consider the parametrization

$$\begin{aligned}
&\Gamma=\{x=\mathcal{B}^\Gamma_t(\upsilon);\upsilon\in[0,1]\},\\
&\hat\Gamma=\{X=\mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(\upsilon));\upsilon\in[0,1]\},\\
&\mathrm{d}S^{\hat\Gamma}=\left|\frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(\upsilon))(\mathcal{B}^\Gamma_t)'(\upsilon)\right|\mathrm{d}\upsilon.
\end{aligned}$$

Then, by (39),

$$\begin{aligned}
\int_{\hat\Gamma}[\tilde U]^2\,\mathrm{d}S^{\hat\Gamma} &= \int_0^1\underbrace{[\tilde U(\mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(\upsilon)))]^2}_{[U(\mathcal{B}^\Gamma_t(\upsilon))]^2}\left|\frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(\upsilon))(\mathcal{B}^\Gamma_t)'(\upsilon)\right|\mathrm{d}\upsilon\\
&\le \int_0^1[U(\mathcal{B}^\Gamma_t(\upsilon))]^2\underbrace{\left\|\frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(\upsilon))\right\|}_{\le C^-_A}\left|(\mathcal{B}^\Gamma_t)'(\upsilon)\right|\mathrm{d}\upsilon\\
&\le C^-_A\int_0^1[U(\mathcal{B}^\Gamma_t(\upsilon))]^2|(\mathcal{B}^\Gamma_t)'(\upsilon)|\,\mathrm{d}\upsilon\\
&= C^-_A\int_\Gamma[U]^2\,\mathrm{d}S^\Gamma.
\end{aligned}$$

Substituting back to (74) and using (65), we find that

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[\mathcal{U}_s]^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t \tag{75}$$
$$\le \tilde{c}^{(3)}_{CH}C^+_A C^{**}_{L6}(C^*_{L6})^{-1}C^-_A\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[U]^2\,\mathrm{d}S\right)\mathrm{d}t$$
$$= C^{(b)}_{CH}\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[U]^2\,\mathrm{d}S\right)\mathrm{d}t,$$

where $C^{(b)}_{CH}=\tilde{c}^{(3)}_{CH}C^+_A C^{**}_{L6}(C^*_{L6})^{-1}C^-_A$.

Similarly we can prove the inequality

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\frac{c_W}{h(\Gamma)}\int_\Gamma|\mathcal{U}_s|^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t \tag{76}$$

$$\leq C_{CH}^{(c)}\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\frac{c_W}{h(\Gamma)}\int_\Gamma|U|^2\,\mathrm{d}S\right)\mathrm{d}t.$$

Finally, (71), (75) and (76) imply (67) with $c_{CH}^{(2)}=\max\{C_{CH}^{(a)},C_{CH}^{(b)},C_{CH}^{(c)}\}$. □

3.4. **Proof of the unconditional stability**

Theorem 2. *There exists a constant $C_{T2}>0$ such that*

$$\|U_m^-\|_{\Omega_{t_m}}^2-\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2+\|\{U\}_{m-1}\|_{\Omega_{t_{m-1}}}^2+\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,t}^2\mathrm{d}t \tag{77}$$

$$\leq C_{T2}\left(\int_{I_m}\|g\|_{\Omega_t}^2\mathrm{d}t+\int_{I_m}\|u_D\|_{DGB,t}^2\mathrm{d}t+\int_{I_m}\|U\|_{\Omega_t}^2\mathrm{d}t\right).$$

Proof. From (46), by virtue of (54), (50), (51), (52), (55) and (53), after some manipulation we get

$$\|U_m^-\|_{\Omega_{t_m}}^2-\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2+\|\{U\}_{m-1}\|_{\Omega_{t_{m-1}}}^2$$
$$+\beta_0\left(1-\frac{1}{k_1}-\frac{1}{k_2}-\frac{1}{k_3}\right)\int_{I_m}\|U\|_{DG,t}^2\mathrm{d}t$$
$$\leq\int_{I_m}\|g\|_{\Omega_t}^2\mathrm{d}t+\beta_0(1+k_3)\int_{I_m}\|u_D\|_{DGB,t}^2\mathrm{d}t$$
$$+\left(c_z+1+\frac{c_d}{\beta_0}+2c_b\right)\int_{I_m}\|U\|_{\Omega_t}^2\mathrm{d}t.$$

Hence, choosing $k_1=k_2=k_3=6$, we get (77) with $C_{T2}=\max\{1,7\beta_0,c_z+1+c_d/\beta_0+2c_b\}$. □

Theorem 3. *There exist constants $C_{T3}^*,C_{T3}^{**}>0$ such that for any $\delta_1>0$ we have*

$$\|U_m^-\|_{\Omega_{t_m}^2}+\|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2+\frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,t}^2\mathrm{d}t \tag{78}$$

$$\leq C_{T3}^*\int_{I_m}\|U\|_{\Omega_t}^2\mathrm{d}t+C_{T3}^{**}\int_{I_m}\left(\|g\|_{\Omega_t}^2+\|u_D\|_{DGB,t}^2\right)\mathrm{d}t$$
$$+\frac{2}{\delta_1}\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2+4\delta_1\|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2.$$

Proof. From (32), by virtue of (56), (50), (51), (52), (55) and (53), we get

$$\|U_m^-\|_{\Omega_{t_m}}^2+\|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2+\beta_0\left(1-\frac{1}{k_1}-\frac{1}{k_2}-\frac{1}{k_3}\right)\int_{I_m}\|U\|_{DG,t}^2\mathrm{d}t$$
$$\leq\int_{I_m}\|g\|_{\Omega_t}^2\mathrm{d}t+\beta_0(1+k_3)\int_{I_m}\|u_D\|_{DGB,t}^2\mathrm{d}t$$
$$+\left(1+c_z+2c_b+\frac{c_d}{\beta_0}\right)\int_{I_m}\|U\|_{\Omega_t}^2\mathrm{d}t+2\left(U_{m-1}^-,U_{m-1}^+\right)_{\Omega_{t_{m-1}}}.$$

Using Young's inequality for the term $2(U_{m-1}^-, U_{m-1}^+)$ and setting $k_1 = k_2 = k_3 = 6$, we get (78), where $C_{T3}^* = 1 + c_z + 2c_b + c_d/\beta_0$ and $C_{T3}^{**} = \max\{1, 7\beta_0\}$. □

We introduce the following notation:

$$\begin{aligned} t_{m-1+l/q} &= t_{m-1} + \tau_m \frac{l}{q}, \\ U_{m-1+l/q} &= U(t_{m-1+l/q}), \quad l = 0, \dots, q. \end{aligned}$$

Lemma 7. *There exist constants $L_q^*, M_q^* > 0$ such that for $m = 1, \dots, M$ we have*

$$\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \geq \frac{L_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \tag{79}$$

$$\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \leq \frac{M_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{80}$$

Proof. Using the equivalence of norms in the space of polynomials of degree $\leq q$, for $p(t) = \tilde{U}(X,t),\ t \in I_m$, and any fixed $X \in \Omega_{t_{m-1}}$, we have

$$\begin{aligned} \sum_{l=0}^{q} \tilde{U}^2\left(X, t_{m-1+l/q}\right) &\geq \frac{L_q}{\tau_m} \int_{I_m} \tilde{U}^2(X,t)\,\mathrm{d}t, \\ \tilde{U}^2\big(X, t_{m-1}^+\big) &\leq \frac{M_q}{\tau_m} \int_{I_m} \tilde{U}^2(X,t)\,\mathrm{d}t \end{aligned}$$

(Cf. [20], Section 6.2.3.2). integrating over $\Omega_{t_{m-1}}$ and using Fubini's theorem, we get

$$\begin{aligned} \sum_{l=0}^{q} \int_{\Omega_{t_{m-1}}} |\tilde{U}\left(X, t_{m-1+l/q}\right)|^2 \mathrm{d}X &\geq \frac{L_q}{\tau_m} \int_{\Omega_{t_{m-1}}} \left(\int_{I_m} |\tilde{U}(X,t)|^2 \mathrm{d}t \right) \mathrm{d}X \\ &= \frac{L_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t. \end{aligned}$$

Analogously we find that

$$\int_{\Omega_{t_{m-1}}} |\tilde{U}\big(X, t_{m-1}^+\big)|^2\, \mathrm{d}X \leq \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t.$$

Now the substitution $X = \mathcal{A}_t^{-1}(x)$, where $X \in \Omega_{t_{m-1}}$, $x \in \Omega_t$, relation $\tilde{U}(\mathcal{A}_t^{-1}(x), t) = U(x,t)$ and (37) imply that

$$\begin{aligned}
&\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \\
&\geq C_J^- \sum_{l=0}^{q} \int_{\Omega_{t_{m-1+l/q}}} |U(x, t_{m-1+l/q})|^2 J^{-1}(x, t_{m-1+l/q})\,\mathrm{d}x \\
&= C_J^- \sum_{l=0}^{q} \int_{\Omega_{t_{m-1}}} |\tilde{U}(X, t_{m-1+l/q})|^2 \mathrm{d}X \\
&\geq \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_t} |\tilde{U}(\mathcal{A}_t^{-1}(x), t)|^2 J^{-1}(x,t)\,\mathrm{d}x \right) \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{L_q}{\tau_m} (C_J^+)^{-1} C_J^- \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2\,\mathrm{d}x \right) \mathrm{d}t \\
&= \frac{L_q}{\tau_m} (C_J^+)^{-1} C_J^- \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t.
\end{aligned}$$

Hence, we get (79) with $L_q^* = L_q (C_J^+)^{-1} C_J^-$.

Further, since $x = \mathcal{A}_{t_{m-1}}(X) = X$ and, thus, $\tilde{U}(X, t_{m-1}^+) = U(x, t_{m-1}^+)$, using the substitution theorem and (37), we obtain

$$\begin{aligned}
\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} &= \int_{\Omega_{t_{m-1}}} |\tilde{U}\big(X, t_{m-1}^+\big)|^2 \mathrm{d}X \\
&\leq \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_t} |\tilde{U}(\mathcal{A}_t^{-1}, t)|^2 J^{-1}(x,t)\,\mathrm{d}x \right) \mathrm{d}t \\
&\leq \frac{M_q}{\tau_m} (C_J^-)^{-1} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2\,\mathrm{d}x \right) \mathrm{d}t \\
&= \frac{M_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,
\end{aligned}$$

where $M_q^* = M_q (C_J^-)^{-1}$. □

In what follows, because of simplicity, we use the notation $\tilde{U}' = \frac{\partial \tilde{U}}{\partial t}$ and do not write the arguments X and t in integrals.

Lemma 8. *There exists a constant $C_{L8} > 0$ such that*

$$\begin{aligned} &\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t + (\{U\}_{m-1}, \mathcal{U}_s(t_{m-1}^+))_{\Omega_{t_{m-1}}} \\ &\geq \frac{1}{2} \left(\|U(s-)\|_{\Omega_s}^2 + \|U(t_{m-1}^+)\|_{\Omega_{t_{m-1}}}^2 \right) \\ &\quad -C_{L8} \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t - (U_{m-1}^+, U_{m-1}^-)_{\Omega_{t_{m-1}}}. \end{aligned} \tag{81}$$

for any $s \in I_m,\ m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Proof. By virtue of the definition of the ALE derivative (9), the definitions of $\tilde{U}, \tilde{\mathcal{U}}_s, \mathcal{U}_s$, the fact that $\tilde{U}'$ is a polynomial of degree $\leq q - 1$ in time and the substitution theorem we can write

$$\begin{aligned} &\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t = \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} (1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &\quad + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} (D_t U, U)_{\Omega_t} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} (1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &\quad + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t. \end{aligned} \tag{82}$$

Now we estimate the second and third term on the right-hand side. We begin with the third term. The fact that J is constant on each $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and the substitution theorem imply that

$$\begin{aligned} &\left| \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| = \left| \sum_{\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} (J_{\hat{K}} - 1) \left(\int_{\hat{K}} \tilde{U}' \tilde{\mathcal{U}}_s \,\mathrm{d}X \right) \mathrm{d}t \right| \\ &\leq \sum_{\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t \in I_m} |J_{\hat{K}} - 1| \int_{I_m} \left(\int_{\hat{K}} |\tilde{U}' \tilde{\mathcal{U}}_s| \,\mathrm{d}X \right) \mathrm{d}t. \end{aligned}$$

Using the relation $J_{\hat{K}}(t_{m-1}) = 1$, we have

$$\max_{t \in I_m} |J_{\hat{K}} - 1| \leq \int_{t_{m-1}}^{t_m} |J_{\hat{K}}'| \,\mathrm{d}t \leq c_J \tau_m,$$

where $c_J > 0$ is a constant independent of h, τ_m, m. Then we find that

$$
\begin{aligned}
&\sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat K} - 1| \int_{I_m} \int_{\hat K} |\tilde U' \tilde{\mathcal{U}}_s| \,\mathrm{d}X \mathrm{d}t \\
&\le c_J \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \tau_m \int_{I_m} \left(\int_{\hat K} |\tilde U' \tilde{\mathcal{U}}_s| \,\mathrm{d}X \right) \mathrm{d}t \\
&= c_J \tau_m \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(\int_{I_m} |\tilde U' \tilde{\mathcal{U}}_s| \,\mathrm{d}t \right) \mathrm{d}X \\
&\le c_J \tau_m \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(\left(\int_{I_m} |\tilde U'|^2 \,\mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \,\mathrm{d}t \right)^{1/2} \right) \mathrm{d}X.
\end{aligned}
$$

Now we apply the inverse inequality in time: There exists a constant $\hat c_I$ such that

$$
\left(\int_{I_m} |\tilde U'(X,t)|^2 \,\mathrm{d}t \right)^{1/2} \le \frac{\hat c_I}{\tau_m} \left(\int_{I_m} |\tilde U(X,t)|^2 \,\mathrm{d}t \right)^{1/2} \tag{83}
$$

holds for every $X \in \Omega_{t_{m-1}}$, $\tau_m \in (0, \overline{\tau})$ and $m = 1, \dots, M$.

This inequality, Young's inequality, Fubini's theorem, (64), substitution theorem and (37) imply that

$$
\begin{aligned}
&\tau_m \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(\left(\int_{I_m} |\tilde U'|^2 \,\mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \,\mathrm{d}t \right)^{1/2} \right) \mathrm{d}X \\
&\le \hat c_I \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(\int_{I_m} |\tilde U|^2 \mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \mathrm{d}t \right)^{1/2} \mathrm{d}X \\
&\le \frac{\hat c_I}{2} \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(\int_{I_m} \left(|\tilde U|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}t \right) \mathrm{d}X \\
&= \frac{\hat c_I}{2} \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} \left(\int_{\hat K} \left(|\tilde U|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{\hat c_I}{2} \left(\int_{I_m} \|\tilde U\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \right) \\
&\le \frac{\hat c_I}{2} (1 + \tilde c^{(1)}_{CH}) \int_{I_m} \|\tilde U\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \frac{\hat c_I}{2} (1 + \tilde c^{(1)}_{CH}) \int_{I_m} \sum_{\hat K \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} \left(|\tilde U|^2 \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{\hat c_I}{2} (1 + \tilde c^{(1)}_{CH}) \int_{I_m} \left(\int_{\Omega_t} |U|^2 J^{-1} \mathrm{d}x \right) \mathrm{d}t \le c^* \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,
\end{aligned}
$$

where $c^* = (C_J^-)^{-1} \hat c_I (1 + \tilde c^{(1)}_{CH})/2$. Summarizing the obtained results, we see that we have proved the inequality

$$
\left| \int_{I_m} \left(\tilde U', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| \le c^* c_J \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{84}
$$

Similarly as above we can estimate the second term on the right-hand side of (82):

$$
\begin{aligned}
&\left|\int_{t_{m-1}}^{s}\left(\tilde{U}',\tilde{U}(1-J)\right)_{\Omega_{t_{m-1}}}\mathrm{d}t\right| \le \int_{I_m}\left|\tilde{U}',\tilde{U}(1-J)\right|_{\Omega_{t_{m-1}}}\mathrm{d}t\\
&\le \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}}\max_{t\in I_m}|1-J_{\hat{K}}|\int_{I_m}\int_{\hat{K}}|\tilde{U}'\tilde{U}|\,\mathrm{d}X\mathrm{d}t\\
&\le c_J\tau_m\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}}\int_{I_m}\int_{\hat{K}}|\tilde{U}'\tilde{U}|\,\mathrm{d}X\mathrm{d}t\\
&= c_J\tau_m\sum_{\hat{K}\in\mathcal{T}_{h,t_{m-1}}}\int_{\hat{K}}\left(\int_{I_m}|\tilde{U}'\tilde{U}|\mathrm{d}t\right)\mathrm{d}X\\
&\le c_J\tau_m\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}}\int_{\hat{K}}\left(\left(\int_{I_m}|\tilde{U}'|^2\,\mathrm{d}t\right)^{1/2}\left(\int_{I_m}|\tilde{U}|^2\,\mathrm{d}t\right)^{1/2}\right)\mathrm{d}X.
\end{aligned}
$$

Now the inverse inequality in time, Young's inequality, Fubini's theorem, (64) and (37) yield the inequality

$$
\left|\int_{t_{m-1}}^{s}\left(\tilde{U}',\tilde{U}(1-J)\right)_{\Omega_{t_{m-1}}}\mathrm{d}t\right| \le c_1\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t. \tag{85}
$$

with $c_1 = c_J(C_J^-)^{-1}\hat{c}_I/2$.

Finally, from (82), (84), (85) and analogy to (59), (63) putting $c_2 = c^*c_J + c_1$ we find that

$$
\begin{aligned}
&\int_{I_m}(D_tU,\mathcal{U}_s)_{\Omega_t}\mathrm{d}t + (\{U\}_{m-1},\mathcal{U}_s(t_{m-1}+))_{\Omega_{t_{m-1}}}\\
&\ge \int_{t_{m-1}}^{s}(D_tU,U)_{\Omega_t}\mathrm{d}t + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
&\quad -(U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}} - c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\
&= \frac{1}{2}\int_{t_{m-1}}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}U^2(x,t)\mathrm{d}x\right)\mathrm{d}t - \frac{1}{2}\int_{t_{m-1}}^{s}\left(U^2\mathrm{div}\,\boldsymbol{z}\right)_{\Omega_t}\mathrm{d}t\\
&\quad +\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}} - c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\
&= \frac{1}{2}\left(\|U(s-)\|^2_{\Omega_s} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) - \frac{c_z}{2}\int_{t_{m-1}}^{s}\|U\|_{\Omega_t}\mathrm{d}t\\
&\quad -c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t - (U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}},
\end{aligned}
$$

which implies (81) with $C_{L8} = c_z/2 + c_2$. □

In the following lemmas, for simplicity we use the notation $\mathcal{U}_l^*$ and $\tilde{\mathcal{U}}_l^*$ for the discrete characteristic functions to U and $\tilde{U}$, respectively at the time instant $t_{m-1+l/q}$.

Lemma 9. *There exists a constant $C_{L9} > 0$ such that*

$$
|a_h(U,\mathcal{U}_l^*,t) + \beta_0 J_h(U,\mathcal{U}_l^*,t)| \le C_{L9}\left(\|U\|^2_{DG,t} + \|\mathcal{U}_l^*\|^2_{DG,t} + \|u_D\|^2_{DGB,t}\right) \tag{86}
$$

for all $t, l \in I_m$, $m = 1, \dots, M$, $h \in (0, \overline{h})$.

Proof. From the definition of the forms a_h and J_h we immediately have

$$
\begin{aligned}
&a_h(U, \mathcal{U}_l^*, t) = \sum_{K \in \mathcal{T}_{h,t}} \int_K \beta(U) \nabla U \cdot \nabla \mathcal{U}_l^* \,\mathrm{d}x \\
&- \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \left(\langle \beta(U) \nabla U \rangle \cdot \mathbf{n}_\Gamma \, [\mathcal{U}_l^*] + \theta \, \langle \beta(U) \nabla \mathcal{U}_l^* \rangle \cdot \mathbf{n}_\Gamma \, [U] \right) \mathrm{d}S \\
&- \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \left(\beta(U) \nabla U \cdot \mathbf{n}_\Gamma \, \mathcal{U}_l^* + \theta \beta(U) \nabla \mathcal{U}_l^* \cdot \mathbf{n}_\Gamma \, U - \theta \beta(U) \nabla \mathcal{U}_l^* \cdot \mathbf{n}_\Gamma \, u_D \right) \mathrm{d}S, \\
&J_h(U, \mathcal{U}_l^*, t) = c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [U] \, [\mathcal{U}_l^*] \,\mathrm{d}S \\
&\qquad\qquad + c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma U \, \mathcal{U}_l^* \,\mathrm{d}S.
\end{aligned}
$$

Now, using the property of the function β, the Cauchy inequality and Young's inequality, we get

$$
\begin{aligned}
&|a_h(U, \mathcal{U}_l^*, t)| \le \beta_1 \sum_{K \in \mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2 \right) \mathrm{d}x \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W} \left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2 \right) + \frac{c_W}{h(\Gamma)} \, [\mathcal{U}_l^*]^2 \right) \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W} \left(|\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2 \right) + \frac{c_W}{h(\Gamma)} \, [U]^2 \right) \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W} |\nabla U|^2 + \frac{c_W}{h(\Gamma)} \, |\mathcal{U}_l^*|^2 \right) \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W} |\nabla \mathcal{U}_l^*|^2 + \frac{c_W}{h(\Gamma)} \, |U|^2 \right) \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*| \, |u_D| \,\mathrm{d}S.
\end{aligned} \tag{87}
$$

The last term can be estimated using Using Young's inequality and the relation $h(\Gamma) \le h_{K_\Gamma^{(L)}}$, for each $\varepsilon > 0$ we get

$$
\begin{aligned}
&\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*| \; |u_D| \;\mathrm{d}S \\
&\le \frac{\beta_1 \varepsilon}{2} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h(\Gamma)^{-1} \, |u_D|^2 \;\mathrm{d}S + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} \, |\nabla \mathcal{U}_l^*|^2 \;\mathrm{d}S \\
&\le \frac{\beta_1 \varepsilon}{2 c_W} J_h^B(u_D, u_D) + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_{\partial K_\Gamma^{(L)}} h_{K_\Gamma^{(L)}} \, |\nabla \mathcal{U}_l^*|^2 \;\mathrm{d}S.
\end{aligned}
$$

Now we express the first term on the right-hand side with the aid of the definition of the $\|\cdot\|_{DGB,t}$-norm and to the second term we apply the multiplicative trace inequality (42) and the inverse inequality (43). We get

$$\begin{aligned}&\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla \mathcal{U}^*_l|\ |u_D|\ \mathrm{d}S\\ &\le \frac{\beta_1\varepsilon}{2c_W}\|u_D\|^2_{DGB,t} + \frac{\beta_1}{2\varepsilon}c_M(c_I+1)\sum_{K\in\mathcal{T}_{h,t}}\|\nabla\mathcal{U}^*_l\|^2_{L^2(K)}.\end{aligned}$$

If we use the inequality $\sum_{K\in\mathcal{T}_{h,t}}\|\nabla\mathcal{U}^*_l\|^2_{L^2(K)} \le \|\mathcal{U}^*_l\|^2_{DG,t}$, which obviously follows from the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla \mathcal{U}^*_l|\ |u_D|\ \mathrm{d}S \le \frac{\beta_1\varepsilon}{2c_W}\|u_D\|^2_{DGB,t} + \frac{\beta_1}{2\varepsilon}c_M(c_I+1)\|\mathcal{U}^*_l\|^2_{DG,t}. \tag{88}$$

Setting $\varepsilon := \frac{\beta_1}{\beta_0}c_M(c_I+1)$ in (88) and substituting back to (87) we get

$$\begin{aligned}|a_h(U,\mathcal{U}^*_l,t)| &\le \beta_1 \sum_{K\in\mathcal{T}_{h,t}}\int_K \left(|\nabla U|^2 + |\nabla\mathcal{U}^*_l|^2\right)\mathrm{d}x\\
&+\beta_1\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma \frac{h(\Gamma)}{c_W}\left(|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2\right)\mathrm{d}S\\
&+\beta_1\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma \frac{h(\Gamma)}{c_W}|\nabla U|^2\,\mathrm{d}S\\
&+\beta_1\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma \frac{h(\Gamma)}{c_W}\left(|\nabla(\mathcal{U}^*_l)^{(L)}_\Gamma|^2 + |\nabla(\mathcal{U}^*_l)^{(R)}_\Gamma|^2\right)\mathrm{d}S\\
&+\beta_1\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma \frac{h(\Gamma)}{c_W}|\nabla \mathcal{U}^*_l|^2\,\mathrm{d}S + \frac{\beta_1^2}{2\beta_0 c_W}c_M(c_I+1)\|u_D\|^2_{DGB,t}\\
&+\frac{\beta_0}{2}\|\mathcal{U}^*_l\|^2_{DG,t} + \beta_1\,J_h(\mathcal{U}^*_l,\mathcal{U}^*_l,t) + \beta_1\,J_h(U,U,t).\end{aligned}$$

Using the inequality $h(\Gamma) \le h_K$ for $\Gamma \subset \partial K$, we have

$$
\begin{aligned}
|a_h(U,\mathcal{U}_l^*,t)| \le{}& \beta_1 \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \,\mathrm{d}x \\
&+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla U_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla U_\Gamma^{(R)}|^2\right) \,\mathrm{d}S \\
&+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla U|^2 \,\mathrm{d}S \\
&+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2\right) \,\mathrm{d}S \\
&+\frac{\beta_1}{c_W} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla \mathcal{U}_l^*|^2 \,\mathrm{d}S \\
&+\frac{\beta_1^2}{2\beta_0 c_W} c_M (c_I+1) \|u_D\|_{DGB,t}^2 + \frac{\beta_0}{2} \|\mathcal{U}_l^*\|_{DG,t}^2 \\
&+\beta_1\, J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \beta_1\, J_h(U,U,t)
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
\le{}& \beta_1 \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \,\mathrm{d}x \\
&+\frac{\beta_1}{c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \,\mathrm{d}S \\
&+\frac{\beta_1^2}{2\beta_0 c_W} c_M (c_I+1) \|u_D\|_{DGB,t}^2 + \frac{\beta_0}{2} \|\mathcal{U}_l^*\|_{DG,t}^2 \\
&+\beta_1\, J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \beta_1\, J_h(U,U,t).
\end{aligned}
$$

Now, applying the multiplicative inequality and the inverse inequality, we can estimate the term

$$
\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \,\mathrm{d}S
$$

as follows:

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}S \\
&= \sum_{K\in\mathcal{T}_{h,t}} h_K \left(\|\nabla U\|^2_{L^2(\partial K)} + \|\nabla \mathcal{U}_l^*\|^2_{L^2(\partial K)}\right) \\
&\le c_M \sum_{K\in\mathcal{T}_{h,t}} h_K (\|\nabla U\|_{L^2(K)} \underbrace{|\nabla U|_{H^1(K)}}_{\le c_I h_K^{-1}\|\nabla U\|_{L^2(K)}} + h_K^{-1}\|\nabla U\|^2_{L^2(K)}) \\
&+ c_M \sum_{K\in\mathcal{T}_{h,t}} h_K (\|\nabla \mathcal{U}_l^*\|_{L^2(K)} \underbrace{|\nabla \mathcal{U}_l^*|_{H^1(K)}}_{\le c_I h_K^{-1}\|\nabla \mathcal{U}_l^*\|_{L^2(K)}} + h_K^{-1}\|\nabla \mathcal{U}_l^*\|^2_{L^2(K)}) \\
&\le c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \left(\|\nabla U\|^2_{L^2(K)} + \|\nabla \mathcal{U}_l^*\|^2_{L^2(K)}\right) \\
&= c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \left(|U|^2_{H^1(\Omega)} + |\mathcal{U}_l^*|^2_{H^1(\Omega)}\right).
\end{aligned} \tag{90}$$

From (89), (90), the definition of the $\|\cdot\|_{DG,t}$-norm, using the inequality

$$J_{h,}(U,\mathcal{U}_l^*,t) \le J_h(U,U,t) + J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)$$

and putting $C_{L9} = \max\{\beta_0+\beta_1+\beta_1 c_M(c_I+1)/c_W, \beta_1^2 c_M(c_I+1)/(2\beta_0 c_W)\}$, we finally get

$$\begin{aligned}
&|a_h(U,\mathcal{U}_l^*,t) + \beta_0\, J_h(U,\mathcal{U}_l^*,t)| \le \left(\beta_1 + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |U|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\
&+ (\beta_0+\beta_1) J_h(U,U,t) + \left(\beta_1 + \frac{\beta_0}{2} + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |\mathcal{U}_l^*|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\
&+ (\beta_0+\beta_1) J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \frac{\beta_1^2}{2\beta_0 c_W} c_M(c_I+1)\|u_D\|^2_{DGB,t} \\
&\le C_{L9} \left(\|U\|^2_{DG,t} + \|\mathcal{U}_l^*\|^2_{DG,t} + \|u_D\|^2_{DGB,t}\right).
\end{aligned}$$

□

Lemma 10. *For each $k_1 > 0$ there exists a constant $c_b > 0$ such that for the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ we have the inequality*

$$\int_{I_m} |b_h(U,\mathcal{U}_l^*,t)| \mathrm{d}t \le \frac{\beta_0}{2k_1} \int_{I_m} \|\mathcal{U}_l^*\|^2_{DG,t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{91}$$

Proof. By (28),

$$\begin{aligned}
b_h(U,\mathcal{U}_l^*,t) = &\underbrace{-\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d f_s(U) \frac{\partial \mathcal{U}_l^*}{\partial x_s}\,\mathrm{d}x}_{:=\sigma_1} \\
&+ \underbrace{\sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(R)}, \mathbf{n}_\Gamma)\, [\mathcal{U}_l^*]_\Gamma\, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(L)}, \mathbf{n}_\Gamma)\, \mathcal{U}_l^*\, \mathrm{d}S.}_{:=\sigma_2}
\end{aligned} \tag{92}$$

Then from the Lipschitz-continuity of the functions f_s, $s = 1, \ldots, d$, with the modul $L_f > 0$, assumption that $f_s(0) = 0$ and the Cauchy inequality, we obtain

$$\begin{aligned} |\sigma_1| &\leq \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |f_s(U) - f_s(0)| \left| \frac{\partial \mathcal{U}_l^*}{\partial x_s} \right| \mathrm{d}x \\ &\leq L_f \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |U| \left| \frac{\partial \mathcal{U}_l^*}{\partial x_s} \right| \mathrm{d}x \leq L_f \sqrt{d}\, \|U\|_{\Omega_t} |\mathcal{U}_l^*|_{H^1(\Omega_t,\mathcal{T}_{h,t})}. \end{aligned} \tag{93}$$

Now we shall estimate σ_2. From the relation $f_s(0) = 0$, $s = 1, \ldots, d$, and the consistency property **(H2)** of the numerical flux H we have $H(0, 0, \mathbf{n}_\Gamma) = 0$. Then we can use the Lipschitz-continuity of H and get

$$\begin{aligned} |\sigma_2| &\leq L_H \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, [\mathcal{U}_l^*]\, \mathrm{d}S \\ &\quad + L_H \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(L)}|)\, |(\mathcal{U}_l^*)_\Gamma^{(L)}|\, \mathrm{d}S. \end{aligned}$$

Using the fact that $U_\Gamma^{(R)} = U_\Gamma^{(L)}$ for $\Gamma \in \mathcal{F}^B_{h,t}$, the Cauchy inequality and the relation $h(\Gamma) \leq h_K$, if $\Gamma \subset \partial K$, we obtain

$$\begin{aligned} &|\sigma_2| \\ &\leq \frac{L_H}{\sqrt{c_W}} \left(c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{[\mathcal{U}_l^*]^2}{h(\Gamma)} \mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{\left((\mathcal{U}_l^*)_\Gamma^{(L)} \right)^2}{h(\Gamma)} \mathrm{d}S \right)^{1/2} \\ &\quad \times \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h(\Gamma) \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)^2\, \mathrm{d}S \right)^{1/2} \\ &\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h(\Gamma) \int_\Gamma \left(|U_\Gamma^{(L)}|^2 + |U_\Gamma^{(R)}|^2 \right) \mathrm{d}S \right)^{1/2} \\ &\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \\ &\quad \times \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h_{K_\Gamma^{(L)}} \int_{\partial K_\Gamma^{(L)}\cap\Gamma} |U_\Gamma^{(L)}|^2 \mathrm{d}S + h_{K_\Gamma^{(R)}} \int_{\partial K_\Gamma^{(R)}\cap\Gamma} |U_\Gamma^{(R)}|^2\, \mathrm{d}S \right)^{1/2} \\ &\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |U|^2\, \mathrm{d}S \right)^{1/2} \\ &= \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2}. \end{aligned} \tag{94}$$

Substituting (93) and (94) into (92), using the Cauchy inequality and the definition of the $\|\cdot\|_{DG,t}$-norm, we find that

$$\begin{aligned} |b_h(U,\mathcal{U}_l^*,t)| &\leq L_f\sqrt{d}\,\|U\|_{\Omega_t}\,|\mathcal{U}_l^*|_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\ &\quad + \frac{L_H}{\sqrt{c_W}}\,J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)^{1/2}\left(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2} \end{aligned} \tag{95}$$

$$\begin{aligned} &\leq \left(L_f^2 d\,\|U\|^2_{\Omega_t} + \frac{L_H^2}{c_W}\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2} \\ &\quad\times\left(|\mathcal{U}_l^*|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} + J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)\right)^{1/2} \\ &\leq c\,\|\mathcal{U}_l^*\|_{DG,t}\left(\|U\|_{\Omega_t} + \left(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2}\right), \end{aligned}$$

where $c = \left(\max\{L_f^2\,d,\; L_H^2/c_W\}\right)^{1/2}$. Furthermore, the multiplicative trace inequality and the inverse inequality imply that

$$\begin{aligned} &\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)} \leq c_M \sum_{K\in\mathcal{T}_{h,t}} h_K\left(\|U\|_{L^2(K)}|U|_{H^1(K)} + h_K^{-1}\|U\|^2_{L^2(K)}\right) \\ &\leq c_M(c_I+1)\sum_{K\in\mathcal{T}_{h,t}} \|U\|^2_{L^2(K)} = c_M(c_I+1)\|U\|^2_{\Omega_t}. \end{aligned}$$

Hence, from this relation, (95) and Young's inequality we get

$$\begin{aligned} |b_h(U,\mathcal{U}_l^*,t)| &\leq c_1\,\|\mathcal{U}_l^*\|_{DG,t}\,\|U\|_{\Omega_t} \leq \frac{\beta_0}{2k_1}\|\mathcal{U}_l^*\|^2_{DG,t} + c_1^2\frac{k_1}{2\beta_0}\|U\|^2_{\Omega_t} \\ &= \frac{\beta_0}{2k_1}\,\|\mathcal{U}_l^*\|^2_{DG,t} + c_b\,\|U\|^2_{\Omega_t}, \end{aligned}$$

where $c_1 = c(1+\sqrt{c_M(c_I+1)}), k_1 > 0$ and $c_b = c_1^2 k_1/\beta_0$. Integrating over the interval I_m, we finally have (91). □

Lemma 11. *For each $k_2 > 0$ there exists a constant $c_d > 0$ such that the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ satisfy the inequality*

$$\int_{I_m} |d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t \leq \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_d}{2\beta_0}\int_{I_m}\|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t. \tag{96}$$

Proof. By (29), (40) and the Cauchy and Young's inequalities,

$$\begin{aligned}
&\int_{I_m} |d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t \le c_z \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d |\mathcal{U}_l^*||\frac{\partial U}{\partial x_s}|\,\mathrm{d}x\,\mathrm{d}t \\
&\le c_z \int_{I_m} \|\mathcal{U}_l^*\|_{\Omega_t} |U|_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t \\
&\le c_z \int_{I_m} \|\mathcal{U}_l^*\|_{\Omega_t} \|U\|_{DG,t}\,\mathrm{d}t \le \frac{\beta_0}{2k_2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_z^2 k_2}{2\beta_0}\int_{I_m} \|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t,
\end{aligned}$$

which is (96) with $c_d = c_z^2 k_2$. □

Lemma 12. *For the approximate solution U, the discrete characteristic function $\mathcal{U}_l^*$ and any $k_3 > 0$ we have*

$$\begin{aligned}
\int_{I_m} |l_h(\mathcal{U}_l^*,t)|\,\mathrm{d}t \le \frac{1}{2}\int_{I_m} \left(\|g\|^2_{\Omega_t} + \|\mathcal{U}_l^*\|^2_{\Omega_t}\right)\,\mathrm{d}t \\
+\frac{\beta_0 k_3}{2}\int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m} \|\mathcal{U}_l^*\|^2_{DG,t}\,\mathrm{d}t.
\end{aligned} \tag{97}$$

Proof. It follows from (30) that

$$|l_h(\mathcal{U}_l^*,t)| = |(g,\mathcal{U}_l^*) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\,\mathcal{U}_l^*\,\mathrm{d}S|.$$

After using the Cauchy inequality for the first term on the right-hand side and applying Young's inequality with $k_3 > 0$ to the second term, we find that

$$\begin{aligned}
&|(g,\mathcal{U}_l^*) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\,\mathcal{U}_l^*\,\mathrm{d}S| \\
&\le \frac{1}{2}(\|g\|^2_{\Omega_t} + \|\mathcal{U}_l^*\|^2_{\Omega_t}) + \frac{\beta_0 k_3}{2} \underbrace{c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma |u_D|^2\,\mathrm{d}S}_{=\|u_D\|^2_{DGB,t}} \\
&+\frac{\beta_0}{2k_3} \underbrace{c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma |\mathcal{U}_l^*|^2\,\mathrm{d}S}_{\le J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) \le \|\mathcal{U}_l^*\|^2_{DG,t}}.
\end{aligned}$$

Hence,

$$|l_h(\mathcal{U}_l^*,t)| \le \frac{1}{2}(\|g\|^2_{\Omega_t} + \|\mathcal{U}_l^*\|^2_{\Omega_t}) + \frac{\beta_0 k_3}{2}\|u_D\|^2_{DGB,t} + \frac{\beta_0}{2k_3}\|\mathcal{U}_l^*\|^2_{DG,t},$$

from which we get (97) by integrating both sides over the interval I_m. □

Theorem 4. *There exist constants $C_{T4}, C^*_{T4} > 0$ such that*

$$\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t \le C_{T4}\tau_m \left(\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t\right) \tag{98}$$

provided $0 < \tau_m < C^*_{T4}$.

Proof. For $q = 1$, the proof is contained in [5]. Let us assume that $q \geq 2$, $l \in \{1, \ldots, q-1\}$.
From the definition of the approximate solution (32)–(33) for $\varphi := \mathcal{U}^*_l$ we get

$$\int_{I_m} (D_t U, \mathcal{U}^*_l)_{\Omega_t}\, \mathrm{d}t + \big(\{U\}_{m-1}, \{\mathcal{U}^*_l\}^+_{m-1}\big)_{\Omega_{t_{m-1}}} \tag{99}$$
$$= \int_{I_m} (-a_h(U, \mathcal{U}^*_l, t) - \beta_0 J_h(U, \mathcal{U}^*_l, t) - b_h(U, \mathcal{U}^*_l, t))\, \mathrm{d}t$$
$$+ \int_{I_m} (-d_h(U, \mathcal{U}^*_l, t) + l_h(\mathcal{U}^*_l, t))\, \mathrm{d}t.$$

This relation and Lemma 8 imply that

$$\frac{1}{2}\left(\left\|U^-_{m-1+l/q}\right\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) \tag{100}$$
$$\leq \int_{I_m} |a_h(U, \mathcal{U}^*_l, t) + \beta_0 J_h(U, \mathcal{U}^*_l, t)|\, \mathrm{d}t + \int_{I_m} |b_h(U, \mathcal{U}^*_l, t)|\, \mathrm{d}t$$
$$+ \int_{I_m} |d_h(U, \mathcal{U}^*_l, t)|\, \mathrm{d}t + \int_{I_m} |l_h(\mathcal{U}^*_l, t)|\, \mathrm{d}t$$
$$+ \big(U^-_{m-1}, U^+_{m-1}\big)_{\Omega_{t_{m-1}}} + C_{L8} \int_{I_m} \|U\|^2_{\Omega_t}\, \mathrm{d}t \equiv \mathrm{RHS}.$$

Now we need to estimate the right-hand side of (100) from above. Using (86), (91), (96),(97) with $k_1 = k_2 = k_3 = 1$, (81) and Young's inequality with any $\delta_2 > 0$, we get

$$\begin{aligned}
\mathrm{RHS} \;\leq\; & C_{L9} \int_{I_m} \big(\|U\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{DG,t} + \|u_D\|^2_{DGB,t}\big)\, \mathrm{d}t \\
& + \frac{\beta_0}{2} \int_{I_m} \|\mathcal{U}^*_l\|^2_{DG,t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t \\
& + \frac{c_d}{2\beta_0} \int_{I_m} \|\mathcal{U}^*_l\|^2_{\Omega_t} \mathrm{d}t + \frac{1}{2} \int_{I_m} \big(\|g\|^2_{\Omega_t} + \|\mathcal{U}^*_l\|^2_{\Omega_t}\big)\, \mathrm{d}t \\
& + \frac{\beta_0}{2} \int_{I_m} \|u_D\|^2_{DGB,t} \mathrm{d}t + \frac{\beta_0}{2} \int_{I_m} \|\mathcal{U}^*_l\|^2_{DG,t} \\
& + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + C_{L8} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{RHS} \\
&\leq c_1 \int_{I_m} \big(\|U\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{\Omega_t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\big)\, \mathrm{d}t \\
&\quad + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}},
\end{aligned}$$

where $c_1 = \max\{C_{L9} + \beta_0 + c_d/(2\beta_0) + 1/2, c_b + C_{L8}\}$. Now we apply Theorem 1 on the continuity of the discrete characteristic function:

$$\int_{I_m} \|\mathcal{U}_l^*\|^2_{\Omega_t} \mathrm{d}t \le c^{(1)}_{\mathrm{CH}} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \quad \int_{I_m} \|\mathcal{U}_l^*\|^2_{DG,t} \mathrm{d}t \le C^{(2)}_{\mathrm{CH}} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t.$$

Hence,

$$\begin{aligned} \text{RHS} \quad \le \quad & c_2 \int_{I_m} \left(\|U\|^2_{DG,t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t \\ & + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}, \end{aligned}$$

with $c_2 = c_1 \max\{1 + c^{(1)}_{CH}, 1 + c^{(2)}_{CH}\}$. Then it follows from (100) that

$$\begin{aligned} & \frac{1}{2}\left(\|U^-_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) \\ & \le c_2 \int_{I_m} \left(\|U\|^2_{DG,t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} \\ & \quad + \delta_2 \|U^+_{m-1}\|_{\Omega_{t_{m-1}}}. \end{aligned} \tag{101}$$

Further, multiplying (101) by $\frac{\beta_0}{4c_2(q-1)}$, summing over $l = 1, \dots, q-1$ and adding to (78), we find that

$$\begin{aligned} & \|U^-_m\|^2_{\Omega_{t_m}} + \frac{\beta_0}{8c_2(q-1)} \sum_{l=1}^{q-1} \|U\|^2_{\Omega_{t_{m-1+l/q}}} + \left(\frac{\beta_0}{8c_2} + 1\right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \\ & \quad + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t \\ & \le \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t + \left(\frac{\beta_0}{4} + C^*_{T3}\right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \\ & \quad + \left(\frac{\beta_0}{4} + C^{**}_{T3}\right) \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t \\ & \quad + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1\right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}. \end{aligned}$$

Setting $c_3 := \min\left\{\frac{\beta_0}{8c_2(q-1)}, \frac{\beta_0}{8c_2} + 1\right\}$ and rearranging, we get

$$\begin{aligned} & c_3 \underbrace{\left(\|U^-_m\|_{\Omega^2_{t_m}} + \sum_{l=1}^{q-1} \|U^2_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)}_{=\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}}} \\ & + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t \end{aligned}$$

$$\leq \left(\frac{\beta_0}{4} + C_{T3}^*\right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \left(\frac{\beta_0}{4} + C_{T3}^{**}\right) \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t$$
$$+ \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1\right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}.$$

It follows from inequalities (79) and (80) that

$$\frac{c_3 L_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t$$
$$\leq \left(\frac{\beta_0\delta_2 M_q^*}{4c_2\tau_m} + \frac{4\delta_1 M_q^*}{\tau_m} + \frac{\beta_0}{4} + C_{T3}^*\right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$$
$$+ \left(\frac{\beta_0}{4} + C_{T3}^{**}\right) \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DG,t}\right) \mathrm{d}t$$
$$+ \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.$$

Setting $\delta_1 = \frac{c_3 L_q^*}{16 M_q^*}$, $\delta_2 = \frac{c_3 c_2 L_q^*}{\beta_0 M_q^*}$, $c_4 := \frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}$, $c_5 := \frac{\beta_0}{4} + C_{T3}^{**}$ we get

$$\left(\frac{c_3 L_q^*}{2\tau_m} - \frac{\beta_0}{4} - C_{T3}^*\right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t \tag{102}$$
$$\leq c_5 \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t + c_4 \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.$$

If the condition $0 < \tau_m \leq C_{T4}^* := \frac{c_3 L_q^*}{4(\frac{\beta_0}{4} + C_{T3}^*)}$ is satisfied, then $\frac{\beta_0}{4} + C_{T3}^* \geq \frac{c_3 L_q^*}{4\tau_m}$ and from (102) we obtain the estimate

$$\frac{c_3 L_q^*}{4\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t$$
$$\leq c_5 \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right) \mathrm{d}t + c_4 \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}},$$

which implies (98). □

The stability analysis will be finished by the application of the following auxiliary lemma.

Lemma 13. *(Discrete Gronwall inequality) Let x_m, a_m, b_m and y_m, where $m = 1, 2, \ldots$, be non-negative sequences and let the sequence a_m be nondecreasing. Then, if*

$$x_0 + y_0 \leq a_0,$$
$$x_m + y_m \leq a_m + \sum_{j=0}^{m-1} b_j x_j \quad \textit{for} \quad m \geq 1,$$

we have

$$x_m + y_m \leq a_m \prod_{j=0}^{m-1} (1 + b_j) \quad \textit{for} \quad m \geq 0.$$

The proof can be carried out by induction.

Now, if (98) is substituted into (77), an inequality is obtained, which is a basis of the proof of our main result about the stability:

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\le (C_{T2} + C_{T4}\,\tau_m)\int_{I_m} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\,\mathrm{d}t + C_{T2}C_{T4}\,\tau_m\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}. \end{aligned} \tag{103}$$

Theorem 5. *Let $0 < \tau_m \le C^*_{T4}$ for $m = 1, \ldots, M$. Then there exists a constant $C_{T5} > 0$ such that*

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\le C_{T5}\left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t\right), \quad m = 1, \ldots, M,\ h \in (0, \overline{h}), \end{aligned} \tag{104}$$

where $R_{t,j} = (C_{T2} + C_{T4}\,\tau_j)\,(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})$ for $t \in I_j$.

Proof. Writing j instead of m in (103),we obtain

$$\begin{aligned} &\|U_j\|^2_{\Omega_{t_j}} - \|U_{j-1}^-\|_{\Omega_{t_{j-1}}} + \|\{U\}_{j-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\le \int_{I_j} R_{t,j}\,\mathrm{d}t + C_{T2}C_{T4}\,\tau_j\|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}}. \end{aligned}$$

Let $m \ge 1$. The summation over all $j = 1, \ldots, m$ yields the inequality

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\le \|U_0^-\|^2_{\Omega_0} + C_{T2}C_{T4}\sum_{j=0}^m \tau_{j+1}\|U_j^-\|^2_{\Omega_{t_j}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t. \end{aligned}$$

The use of the discrete of Gronwall inequality with setting

$$\begin{aligned} x_0 &= a_0 = \|U_0^-\|^2_{\Omega_{t_0}}, \quad c_0 = 0, \\ x_m &= \|U_m^-\|^2_{\Omega_{t_m}}, \\ y_m &= \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t, \\ a_m &= \|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t, \\ b_j &= C_{T2}C_{T4}\,\tau_{j+1}, \quad j = 0, 1, \ldots, m, \end{aligned}$$

yield

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\le \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_j R_{t,j}\,\mathrm{d}t \right) \prod_{j=0}^{m-1} \big(1 + C_{T2}C_{T4}\,\tau_{j+1}\big). \end{aligned} \tag{105}$$

Finally (105) and the inequality $1+\sigma < \exp(\sigma)$ valid for any $\sigma > 0$ immediately yield (104) with the constant $C_{T5} := \exp(C_{T2}C_{T4}T)$. □

4. Conclusion

This paper is devoted to the stability analysis of the space-time discontinuous Galerkin method (STDGM) applied to the numerical solution of a initial-boundary value problem for a nonlinear convection-diffusion equation in a time-dependent domain. The problem is formulated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method. In the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The space discretization uses piecewise polynomial approximations of degree $\le p$ with an integer $p \ge 1$. For the discontinuous Galerkin discretization in time we use polynomials of degree $\le q$ with $q \ge 2$. (If $q = 0$, then we get the backward Euler time discretization and the case $q = 1$ was analyzed in [5].) Here the situation is much more complicated and a special technique based on the ALE-generalization of the concept of the discrete characteristic function has been applied. This approach combined with a number of various estimates results in the proof of unconditional stability of the method. The obtained results represent a theoretical support of the ALE-STDGM developed in [16] for the numerical solution of compressible Navier-Stokes equations in time-dependent domains and interaction of compressible flow with elastic structures. Further step will be the application of derived results to the analysis of error estimates of the ALE-STDGM in time-dependent domains.

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5. Appendix: proof of estimates (75) and (76) from the proof of Theorem 1 in the 3D case (by Z. Vlasáková)

We introduce a parametrization of $\hat{\Gamma}$. Let Δ^2 be a reference simplex in $\mathbb{R}^2$ (with one vertex being the origin and all of the other vertices have only one non-zero coordinate equal to 1). Now

$$\begin{aligned}
&\Gamma = \mathcal{A}_t(\hat{\Gamma}), \quad \hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}},\\
&\hat{\Gamma} = \mathcal{B}^{\hat{\Gamma}}_{m-1}(\Delta^2) = \{X = \mathcal{B}^{\hat{\Gamma}}_{m-1}(v);\, v \in \Delta^2\},\\
&\mathrm{d}S^{\hat{\Gamma}} = \left\| \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2, \quad v \in \Delta^2,\\
&\Gamma = \{x = \mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v));\, v \in \Delta^2\},\\
&\mathrm{d}S^{\Gamma} = \left\| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2,\\
&\quad v \in \Delta^2.
\end{aligned}$$

By the symbol $\times$ we denote the vector product. The terms $\frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^i}(v)$ are tangent vectors to $\hat{\Gamma}$ at the point $\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of $\frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^i}(v)$ are identical from the sides of both elements $\hat{K}^{\hat{\Gamma}}_L$ and $\hat{K}^{\hat{\Gamma}}_R$ adjacent to $\hat{\Gamma}$.

Then we can write

$$\begin{aligned}
&\int_\Gamma \frac{1}{h(\Gamma)} [U_s]^2 \mathrm{d}S^\Gamma \qquad (106)\\
&= \int_{\Delta^2} \frac{1}{h(\Gamma)} [U_s(\mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)))]^2\\
&\qquad \left\| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2\\
&\le \int_{\Delta^2} \frac{1}{h(\Gamma)} [\tilde{U}_s(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))]^2 \left\| \frac{d\mathcal{A}_t}{dX}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \right\|^2\\
&\qquad \left\| \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2\\
&\le (C^+_A)^2 \int_{\hat{\Gamma}} \frac{C^{**}_{L6}}{h(\hat{\Gamma})} [\tilde{U}_s]^2 \mathrm{d}S^{\hat{\Gamma}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [U_s]^2 \mathrm{d}S^\Gamma \right) \mathrm{d}t \qquad (107)\\
&\le \tilde{c}^{(3)}_{CH} (C^+_A)^2 C^{**}_{L6} \int_{I_m} \left(\sum_{\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat{\Gamma})} \int_{\hat{\Gamma}} [\tilde{U}]^2 \mathrm{d}S^{\hat{\Gamma}} \right) \mathrm{d}t.
\end{aligned}$$

Further for $\Gamma = \mathcal{A}_t(\hat{\Gamma})$, $\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}$, we consider the parametrization

$$
\begin{aligned}
&\Gamma = \{x = \mathcal{B}^\Gamma_t(v);\, v \in \Delta^2\},\\
&\hat{\Gamma} = \{X = \mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(v));\, v \in \Delta^2\},\\
&\mathrm{d}S^\Gamma = \left\| \frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}v, \quad v \in \Delta^2\\
&\mathrm{d}S^{\hat{\Gamma}} = \left\| \frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(v)) \frac{\partial \mathcal{B}^\Gamma_t}{\partial x^1}(v) \times \frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(v)) \frac{\partial \mathcal{B}^\Gamma_t}{\partial x^2}(v) \right\| \mathrm{d}v, \quad v \in \Delta^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{\hat{\Gamma}} [\tilde{U}]^2 \mathrm{d}S^{\hat{\Gamma}} \qquad (108)\\
&= \int_{\Delta^2} [\tilde{U}(\mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(v)))]^2\\
&\qquad \left\| \frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(v)) \frac{\partial \mathcal{B}^\Gamma_t}{\partial x^1}(v) \times \frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(v)) \frac{\partial \mathcal{B}^\Gamma_t}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2\\
&\le \int_{\Delta^2} [U(\mathcal{B}^\Gamma_t(v))]^2 \left\| \frac{d\mathcal{A}_t^{-1}}{dx}(\mathcal{B}^\Gamma_t(v)) \right\|^2 \left\| \frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2\\
&\le (C_A^-)^2 \int_{\Delta^2} [U]^2 \mathrm{d}S^\Gamma.
\end{aligned}
$$

Together we get

$$
\begin{aligned}
&\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [U_s]^2 \mathrm{d}S^\Gamma \right) \mathrm{d}t \qquad (109)\\
&\le \tilde{c}^{(3)}_{CH} (C_A^+)^2 C^{**}_{L6} (C^*_{L6})^{-1} (C_A^-)^2 \int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [U]^2 \mathrm{d}S^\Gamma \right) \mathrm{d}t,
\end{aligned}
$$

which is the 3D version of (75). Similarly we proof (76) in the 3D case.

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