Algebraic description of the finite Stieltjes moment problem*

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Abstract

The Stieltjes problem of moments seeks for a nondecreasing positive distribution function $\mu(\lambda)$ on the semi-axis $[0, +\infty)$ so that its moments match a given infinite sequence of positive real numbers $m_0, m_1, \ldots$. In his seminal paper *Investigations on continued fractions* published in 1894 Stieltjes gave a complete solution including the conditions for the existence and uniqueness in relation to his main goal, the convergence theory of continued fractions.

One can also reformulate the Stieltjes problem of moments as looking for *a sequence of positive distribution functions* $\mu^{(1)}(\lambda), \mu^{(2)}(\lambda), \ldots$, where the nth distribution function has n points of increase and $m_0, m_1, \ldots, m_{2n-1}$ represent its (first) $2n$ moments, i.e., as the sequence of the *finite Stieltjes moment problems.* This view can be linked to iterative solution of (large) linear algebraic systems. Providing that $m_0, m_1, \ldots$ *are* moments of some linear, self-adjoint and coercive operator $\mathcal{A}$ on a Hilbert space with respect to a given vector f, the finite Stieltjes moment problems determine the iterations of the conjugate gradient method applied for solving $\mathcal{A}u = f$, and vice versa. Here the existence and uniqueness is guaranteed by the properties of the operator $\mathcal{A}$ (reformulation for finite sequences, matrices and finite vectors is obvious).

This fundamental link raises a question on how the solution of the finite Stieltjes moment problem can be described purely algebraically. This has motivated the presented exposition built upon ideas published previously by several authors. Since the description uses matrices of moments, it is not intended for numerical computations.

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1. Introduction

Krylov subspace methods belong among important tools for solving large systems of linear algebraic equations arising from many applications, and they are counted among

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the most important algorithmic discoveries of the 20th century. Hestenes and Stiefel pointed out in their seminal paper published in 1952 [22] that the conjugate gradient method (CG) "is related to the theory of orthogonal polynomials and to continued fraction expansions" and they developed the connection in Sections 14–18 of their paper, with references to the monographs on orthogonal polynomials by Szegö [42] and on analytic theory of continued fractions by Wall [48]. This line of thought has however been largely overshadowed by the primarily algorithmic description of CG and of Krylov subspace methods in most of the literature. It has been rarely mentioned that what we now consider the state-of-the-art computational methods for solving large scale problems is in fact very closely related to the discoveries in classical analysis and approximation theory of the 19th century that culminates in several ways in the works of Thomas Jan Stieltjes. From the other side, the modern analytic works on continued fractions, orthogonal polynomials, quadrature and approximations of functions very rarely mention that there is a very closely related area of computational mathematics with widespread practical applications and that understanding of the related methods and algorithms is built upon the same mathematical principles (for more on this relationship we refer to [29]; see also the text below.)

Motivated by the preceding reasoning, the presented text will focus on the description of the problem of moments with two goals. First, it would like to pay a tribute to Stieltjes and briefly recall his analytic solution embedded in the theory of convergence of continued fractions. Second, it will reformulate the moment problem purely algebraically and describe its solution using the Cholesky factorization of the associated Hankel matrices with pointing out the connection to CG. We hope that this historical essay, which will also point out to a certainly very incomplete list of cornerstone publications, can be of interest to readers working in related but different areas.

The problem of moments appeared in mathematics (with the notion of moments inspired from mechanics) in the second half of the 19th century with the works of Chebyshev, Markov, Christoffel, Heine, Stieltjes and others. They were related to several very close lines of thoughts in the fields that are now unfortunately considered in our fragmented and overspecialized world distinct and far from each other. While the main motivation of Chebyshev and Markov was obtaining limiting results in probability theory, Stieltjes was primarily interested in the question of convergence of continued fractions, i.e., in approximation of analytic functions. The term *moment problem* was used for the first time by Stieltjes in his seminal paper *Investigations on continued fractions* [40] published in 1894, the year of his death at the age of 38 (please notice that throughout our text we refer to the English translation published by Springer in 1993). This paper has influenced development of a large area of mathematics, pure and applied, as well as development of various methods used in computational sciences.

To give an example, the paper by Stieltjes also introduced (what is now called) the Riemann-Stieltjes integral, which influenced in a substantial way development of the spectral theory of self adjoint operators in Hilbert spaces in the works of Hilbert, F. Riesz, Stone, Wintner, von Neumann, Hellinger, Toeplitz and many others. These results now represent a classical part of functional analysis and operator theory, but they also formed mathematical foundations of quantum mechanics. The integral representation of operators, moment problem, method of moments and the closely related method of continued fractions represent important tools in mathematical and theoretical physics. The operator theory view to the problem of moments, its connections and impact are beautifully documented, e.g., in the classical monograph on linear operators by Dunford and Schwartz published in three parts of total 2591 pages within the years 1958 [11] covering general theory, 1963 [12] covering spectral theory of self-adjoint operators in Hilbert spaces and 1971 [13] devoted to the so called spectral operators. In relation to the moment problem and the impact of Stieltjes, an interested reader may enjoy, in particular, [12], Notes and Remarks to Chapter X, pp. 926–936, Section 8 *Moment Theorems* of Chapter XII, pp. 1250–1256, and Notes and Remarks to the same chapter, pp. 1268–1277. The Hamburger moment problem using infinite dimensional Jacobi matrices is investigated in [41, Chapter X, § 4, in particular pp. 606–614]; see also [50, Chapter 6, § 114 and § 115, pp. 238–242].

In [12, Chapter XII, Section 8] the Stieltjes moment problem results (and the related Hamburger moment problem results) are proved using the spectral theory of self adjoint operators on Hilbert spaces. In relation to this approach it is interesting to recall contributions of Akhiezer (and the associated results of Krein), which are referred to in [12], as well as the contributions of Vorobyev, which are not referred to there and which remain, up to now, almost unknown in the operator theory literature. The seminal paper by Akhiezer [1] published in Russian in 1941 used for solving the problem of moments the link with infinite Jacobi matrices. The concept has later been extended and beautifully exposed in the monograph [2, see, in particular, Chapter 4], with the original Russian version published in 1961. Another beautiful monograph by Gantmacher and Krein [15], with the first Russian edition published in 1941, does not deal with the problem of moments, but it presents a remarkable mechanical interpretation of the results published by Stieltjes in [40]; see in particular the comprehensive summary due to Krein in [15, Supplement II, pp. 283–297].

The paper by Vorobyev [46] published in 1954 contains in Section 1 a concise description of the so called *restricted problem of moments in the Hilbert space* with the solution based on the restriction of operators to the finite dimensional subspace and the subsequent spectral representation. The focus of the paper, as well as of the monograph [47], published originally in Russian in 1958, is on *solving problems in application areas.* This is also reflected in the title of [47] with the key words *method of moments*, which is used since then in many publications throughout theoretical physics, computational sciences and engineering. Vorobyev also emphasized the direct link between the work of Stieltjes and his results presented in [40], the results of Chebyshev and Markov, and the ideas behind Krylov subspace methods for solving linear equations and approximating eigenvalues. He referred to the works of A. N. Krylov, Lanczos, Hestenes and Stiefel, which gave birth to Krylov subspace methods that are, as mentioned above, of primary importance in many large scale computations; see also [8]. Vorobyev also referred to the works of Karush [24] and Stesin [39] on extension of the given ideas to problems with compact self-adjoint operators. In the paper on CG [22], Hestenes and Stiefel made the link to Gauss quadrature and they used the Riemann-Stieltjes integral representation of operators. This link and the seminal discoveries of C. C. Paige (see, e.g., [34]) has later inspired the ground breaking results of Greenbaum on the behavior of the Lanczos method and CG in finite precision arithmetic [21]; see also [33], [29, Section 5.9]. Vorobyev realized that CG and the Lanczos method for approximating eigenvalues are mathematically equivalent to the method of moments using operator formulation, and he derived the CG and the Lanczos method from this formulation. In this way he completed the link between the Stieltjes (Hamburger) moment problem (in his terminology "the scalar problem of moments") with the operator moment problem, and between the classical analytic investigations and applied computations used in solving algebraic problems in sciences and engineerings.

It is worth noticing that Akhiezer and Vorobyev did not refer to the each other work. This can perhaps be partially attributed to the separation of the pure and applied mathematics as different disciplines. Such separation neither existed at the time of Euler, nor at the times of Gauss, Jacobi, Chebyshev, Markov and Stieltjes. It has arised only in the 20th century, and it unfortunately remains present despite warnings of many distinguished mathematicians, working also in other areas such as theoretical physics, including Lanczos. The presented text wishes to contribute (in a rather modest way) towards emphasizing the links by presenting two different views to proving existence and uniqueness of solution to the (finite) moment problem.

Section 2 will recall, with giving detailed references, the moment problem related results in the Stieltjes paper [40], which can be useful, in complement to the existing broader surveys, in particular for nonspecialist readers. We believe that the work of Stieltjes deserves wider recognition. Many references to this work in the classical monograph on orthogonal polynomials by Szegö [42], including the paper [40], can serve as a strong supporting argument. On the other hand, the very interesting proceedings [28] showing the widespread use of moments in mathematics (and behind) surprisingly does not contain a quote to the Stieltjes paper [40]. Section 2 of the presented paper pays a tribute to the work of Stieltjes. Section 3 describes the algebraic solution of the finite Stieltjes moment problem. Since our exposition is built upon results scattered in literature, this might be useful for those who like to see the relationships between approaches for solving the same problem using different mathematical thoughts.

1.1. Brief comments on literature

In the following we will briefly comment on a (certainly rather incomplete) list of monographs, surveys and articles which may be of interest to readers who are not specialists in the area. The survey [27] by Kjeldsen represents a highly recommendable reading to all interested in the history of the moment problem and of the contribution by Stieltjes. The quotes to the correspondence between Stieltjes and Hermite is of particular interest; it offers an insight into the process that led Stieltjes to formulating and solving the problem (see pp. 21–35). The second part of [27] (pp. 35–43) explains the generalization to the Hamburger moment problem and how the generalizations became independent of the theory of continued fractions, moving towards the field of complex function theory (with the work of Nevanlinna) and to the field of functional analysis (with the work of M. Riesz). The beautiful survey [45] by Van Assche describes the work of Stieltjes on continued fractions and on the moment problem (Sections 1 and 2), including some related topics, namely the electrostatic interpretation of the roots of orthogonal polynomials (Section 3), the Markov-Stieltjes inequalities and the Gauss quadrature (Section 4), and some special orthogonal polynomials (Section 5). Throughout the survey Van Assche referred to many developments that were deeply influenced by the contribution of Stieltjes. Further clarifications and insightful comments can be found in [44], an addendum to [45] by Valent and Van Assche.

Shohat and Tamarkin gave a theoretical presentation of the moment problem in their classical monograph [38], with Section 1 of the Introduction (pp. vii–xi) presenting a brief historical review of the problem. Chapter I starts from a more general case of a multiple sequence of real numbers and then restricts to the case of the Hamburger and the Stieltjes moment problems in Section 2 (pp. 4–6) and Section 6 (pp. 19–21). Chapter II thoroughly investigates the Hamburger moment problem, in particular explaining the connection with orthogonal polynomials and with continued fractions; note that Sections 24-25 (pp. 72–76) are devoted to the Stieltjes moment problem. Chapter III presents various modifications of the moment problem and Chapter IV connection with quadrature formulas. The book [3] by Akhiezer and Krein is composed of six articles dealing with several specialized questions regarding the moment problem. The first article presents the basic results and it is subdivided into three chapters. In particular, the solution of the finite Hamburger moment problem can be found in Chapter I (see Theorem 3, p. 8), while the solution of the infinite case is given in Chapter II (see Theorem 7, pp. 51–52). Articles II-IV treat the moment problem in the light of functional analysis and Articles V and VI apply the derived results to the study of a special class of functions. The monograph [2] by Akhiezer mentioned above discusses in the first chapter properties of the infinite Jacobi matrices and of the associated orthogonal polynomials. In particular, Section 4, Chapter 1 (pp. 20–24) is devoted to the connections with quadrature formulas and continued fractions. Chapter 2 gives the conditions for the existence of the solution of the Hamburger moment problem (Section 1, pp. 29–34). The chapter also discusses the connection with completeness of some related spaces of functions (Section 2–3, pp. 34–49), relationship with certain analytic functions (Section 4–5, pp. 49–67), and interpretation of the moment problem as a problem of continuation of a positive functional (Section 6, pp. 68–79). Stieltjes' results on continued fractions can be found in the Appendix (pp. 232–242). The remaining chapters are devoted to the connection with interpolation problems in the theory of functions (Chapter 3); with the spectral theory of operators (Chapter 4); with the trigonometric moment problem and its connection with the integral representation of some specific functions (Chapter 5).

A geometrical point of view on the moment problem is offered by the monograph [23] by Karlin and Shapley that thoroughly investigates the moment spaces using convex sets and distributions, which represents an interesting complement to the approaches linked with continued fractions. The monumental historical monograph [7] and the monograph [6] by Brezinski about formal (general) orthogonal polynomials and Padé-type approximations give, besides mathematical descriptions, an incredible amount of well-sorted historical information. In [6] the Hamburger and the Stieltjes moment problems are presented as particular cases in Section 2.10 (pp. 115–125). The connection with continued fractions is given in Section 3.2 (pp. 152–159). In [7] the historical description of Stieltjes' work on the convergence of continued fractions can be found in Section 5.2.4 (pp. 227–235), and the more general case connected with the Hamburger moment problem is presented in Section 6.3 (pp. 284–291).

The book on Krylov subspace methods [29] can serve as a reference for readers interested in the connections between the Stieltjes moment problem (Section 3.1, pp. 73–76), the model reduction and the Gauss quadrature (Section 3.2, pp. 76–88), orthogonal polynomials and continued fractions (Section 3.3, pp. 89–108), Jacobi matrices (Section 3.4, pp. 108–136), the Lanczos algorithm and CG (Section 3.5, pp. 136–142), and many other related topics. In particular, various useful information can be found in the historical comments in Section 2.5.7 (pp. 64–69), Sections 3.3.5 and 3.3.6 (pp. 104–108), Section 3.4.3 (pp. 130–136), Remark 3.5.1 (pp. 139–140) and Section 4.9 (pp. 222–226).

Among the extensive literature on orthogonal polynomials and continued fractions we refer the reader to the classical monograph on orthogonal polynomials [42] by Szegö, the books on continued fractions [48] by Wall and [30] by Lorentzen and Waadeland. The monograph [9] by Bultheel and Van Barel emphasizes interconnections between analytic and algebraic descriptions and it covers rational approximations, orthogonal polynomials, related matrix theory, as well as the connection to linear dynamical systems and signal processing. The monograph [26] by Khrushchev focuses on the historical development and provides references to very many original sources, including the contributions of Euler, Chebyshev and Markov. It is also worth pointing out the paper [20] by Gragg on the matrix interpretation of continued fraction algorithm where many links between different views can be found. Pointers to many references can also be found in the papers [36, 37] focusing the more general context of Gauss quadratures for linear functionals and their connection with formal orthogonal polynomials, complex Jacobi matrices, and the non-Hermitian Lanczos algorithm.

2. The Stieltjes moment problem

Given a finite sequence of $2n$ positive real numbers $m_0, m_1, \ldots, m_{2n-1}$, the *finite Stieltjes problem of moments* addressed in this paper looks for a positive solution $\omega_\ell^{(n)} > 0$, $\ell = 1, \ldots, n$, and $0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$ of the system of $2n$ equations

$$\sum_{\ell=1}^{n} \omega_\ell^{(n)} \left\{ \lambda_\ell^{(n)} \right\}^j = m_j, \quad j = 0, 1, \ldots, 2n-1. \tag{2.1}$$

In the 1894 paper [40] Stieltjes considered the case of an infinite sequence of positive real numbers $m_0, m_1, \ldots$ and he proposed the problem of finding a positive mass distribution with mass $\omega_\ell > 0$ concentrated at the distance $\lambda_\ell > 0$ from the origin so that

$$\sum_{\ell=1}^{\infty} \omega_\ell \left\{ \lambda_\ell \right\}^j = m_j, \quad j = 0, 1, \ldots;$$

see [40, Section 24, pp. 648–650] (whenever we point to particular pages in [40], we always refer to the English translation published by Springer in 1993). Moreover, in [40, Sections 37–38, pp. 665–669] Stieltjes introduced what is known today as the *Riemann-Stieltjes integral*, and he reformulated the moment problem as the problem of finding a nondecreasing positive distribution function $\mu(\lambda)$ so that the associated Riemann-Stieltjes integral satisfies

$$\int_0^\infty \lambda^j \, \mathrm{d}\mu(\lambda) = m_j, \qquad \text{for} \qquad j = 0, 1, \ldots\,; \tag{2.2}$$

see [40, Section 48, pp. 685–686, Sections 51–53, pp. 688–695]. If such distribution function exists, then the real numbers $m_0, m_1, \ldots$ are known as its *moments*. If $m_0 = 1$, the moment problem is normalized and the total mass $\sum_{\ell=1}^{\infty} \omega_\ell$ is equal to one. Stieltjes did not consider $m_0 = 1$ and we follow his setting here. The modification for $m_0 = 1$ is obvious.

Inspired by the presentation of the moment problem history by Kjeldsen [27], we briefly recall main ideas behind the solution of Stieltjes. It was embedded in investigation of convergence of continued fractions with positive coefficients $a_1, a_2, \ldots$

$$S(\lambda) = \cfrac{1}{-a_1\lambda + \cfrac{1}{a_2 + \cfrac{1}{-a_3\lambda + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}}} \tag{2.3}$$

(Stieltjes used the variable $z = -\lambda$). The $2n$th convergent $S_{2n}(\lambda)$ of $S(\lambda)$

$$S_{2n}(\lambda) = \cfrac{1}{-a_1\lambda + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{2n-2} + \cfrac{1}{-a_{2n-1}\lambda + \cfrac{1}{a_{2n}}}}}} = \frac{p_{2n}(\lambda)}{q_{2n}(\lambda)} \tag{2.4}$$

is a rational function where the numerator $p_{2n}(\lambda)$ has degree $n-1$ and the denominator $q_{2n}(\lambda)$ has degree n. Analogously, the $(2n+1)$st convergent $S_{2n+1}(\lambda) = p_{2n+1}(\lambda)/q_{2n+1}(\lambda)$ is the rational function obtained by truncating $S(\lambda)$ after the term $-a_{2n+1}\lambda$, with $p_{2n+1}(\lambda)$ of degree n and $q_{2n+1}(\lambda)$ of degree $n+1$; see [40, Section 2, p. 616]. We will now recall how Stieltjes defined the distribution function associated with the given continued fraction $S(\lambda)$. Then we turn into his solution of the moment problem.

Considering the decomposition into partial fraction

$$S_{2n}(\lambda) = \frac{p_{2n}(\lambda)}{q_{2n}(\lambda)} = \sum_{\ell=1}^{n} \frac{\omega_\ell^{(n)}}{\lambda_\ell^{(n)} - \lambda}, \tag{2.5}$$

the first $2n$ coefficients $m_0, m_1, \ldots, m_{2n-1}$ of the formal power series expansion

$$S_{2n}(\lambda) = -\frac{m_0}{\lambda} - \frac{m_1}{\lambda^2} - \cdots - \frac{m_{2n-1}}{\lambda^{2n}} - \cdots,$$

can be expressed as

$$m_j = \sum_{\ell=1}^{n} \omega_\ell^{(n)} \left\{\lambda_\ell^{(n)}\right\}^j, \quad j = 0, 1, \ldots, 2n-1;$$

see [40, Section 8, p. 625]. Moreover, the roots $0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$ of $q_{2n}(\lambda)$ are positive and distinct, and the coefficients $\omega_1^{(n)}, \omega_2^{(n)}, \ldots, \omega_n^{(n)}$ are positive; see [40, Section 3, pp. 617–618]. A similar result holds for the first $2n+1$ coefficients $m_0, m_1, \ldots, m_{2n}$ of the formal power series expansion of $S_{2n+1}(\lambda)$, which can be expressed as

$$m_j = \sum_{\ell=0}^{n} \nu_\ell^{(n)} \left\{\theta_\ell^{(n)}\right\}^j, \quad j = 0, 1, \ldots, 2n,$$

with $\nu_0^{(n)}, \nu_1^{(n)}, \dots, \nu_n^{(n)}$ positive coefficients and $0 = \theta_0^{(n)} < \theta_1^{(n)} < \cdots < \theta_n^{(n)}$; see [40, Section 3, pp. 617–618, and Section 8, p. 625].

Summarizing, given the continued fraction (2.3), the convergent $S_{2n}(\lambda)$ determines the nondecreasing positive distribution function $\mu^{(n)}(\lambda)$ with n points of increase (see [40, Section 36, p. 665])

$$\mu^{(n)}(\lambda) = \begin{cases} 0 & \text{if } \lambda < \lambda_1^{(n)} \\ \sum_{\ell=1}^{k} \omega_\ell^{(n)} & \text{if } \lambda_k^{(n)} \le \lambda < \lambda_{k+1}^{(n)}, \qquad k = 1, \dots, n-1, \\ \sum_{\ell=1}^{n} \omega_\ell^{(n)} & \text{if } \lambda_n^{(n)} \le \lambda \end{cases} \tag{2.6}$$

where $\omega_k^{(n)} > 0$ is the size of the jump at the node $\lambda_k^{(n)} > 0$, such that

$$\int_0^\infty \lambda^j \, \mathrm{d}\mu^{(n)}(\lambda) = \sum_{\ell=1}^{n} \omega_\ell^{(n)} \left\{ \lambda_\ell^{(n)} \right\}^j = m_j, \quad j = 0, 1, \dots, 2n-1. \tag{2.7}$$

Similarly, $S_{2n+1}(\lambda)$ determines the distribution function $\widetilde{\mu}^{(n)}(\lambda)$ with the $n+1$ points of increase (in increasing order) $\theta_0^{(n)} = 0, \theta_1^{(n)}, \dots, \theta_n^{(n)}$ and the size of the jumps $\nu_0^{(n)}, \nu_1^{(n)}, \dots, \nu_n^{(n)}$.

In [40, Section 44, p. 677] Stieltjes defined the following distribution functions that can be expressed in nowadays terminology as (see [27, p. 31])

$$\begin{aligned} \mu(\lambda) &:= \frac{1}{2}\left(\limsup_{n\to\infty} \mu^{(n)}(\lambda) + \liminf_{n\to\infty} \mu^{(n)}(\lambda) \right), \\ \widetilde{\mu}(\lambda) &:= \frac{1}{2}\left(\limsup_{n\to\infty} \widetilde{\mu}^{(n)}(\lambda) + \liminf_{n\to\infty} \widetilde{\mu}^{(n)}(\lambda) \right), \end{aligned}$$

which satisfy

$$m_j = \int_0^\infty \lambda^j \, \mathrm{d}\mu(\lambda) = \int_0^\infty \lambda^j \, \mathrm{d}\widetilde{\mu}(\lambda) \quad \text{for} \quad j = 0, 1, \dots;$$

see [40, Section 48, p. 685]. Given the continued fraction $S(\lambda)$ with positive coefficients, the distribution functions $\mu(\lambda)$ and $\widetilde{\mu}(\lambda)$ exist but they may not be equal.

Up to now, the primary information was the continued fraction (2.3). Now we return back to the problem of moments with the sequence of positive real numbers $m_0, m_1, \dots$ being the primary given data. Consider the Hankel matrices composed by the sequence $m_0, m_1, \dots$

$$H_j^{(k)} = \begin{bmatrix} m_k & m_{k+1} & \dots & m_{k+j} \\ m_{k+1} & m_{k+2} & \dots & m_{k+j+1} \\ \vdots & \vdots & & \vdots \\ m_{k+j} & m_{k+j+1} & \dots & m_{k+2j} \end{bmatrix}, \tag{2.8}$$

and their determinants $\Delta_j^{(k)}$ (with $\Delta_j = \Delta_j^{(0)}$ and $H_j = H_j^{(0)}$ for simplicity of notation). The coefficients of the continued fraction $S(\lambda)$ can then be expressed in terms of the given data $m_0, m_1, \dots$ by the formulas

$$a_{2j-1} = \frac{(\Delta_{j-2}^{(1)})^2}{\Delta_{j-2}\Delta_{j-1}} \quad \text{and} \quad a_{2j} = \frac{(\Delta_{j-1})^2}{\Delta_{j-1}^{(1)}\Delta_{j-2}^{(1)}}, \quad j = 1, 2, \dots, \tag{2.9}$$

with $\Delta_{-1} = \Delta_{-1}^{(1)} = 1$; see [40, Section 11, Equation (7), p. 630]. Therefore, given a sequence of positive real numbers

$$m_0, m_1, m_2, \ldots,$$

with the Hankel determinants

$$\Delta_j > 0 \text{ and } \Delta_j^{(1)} > 0, \quad \text{for} \quad j = 0, 1, \ldots, \tag{2.10}$$

there exists a continued fraction (2.3) with positive coefficients $a_1, a_2, \ldots$ determining the distribution functions $\mu(\lambda)$ and $\widetilde{\mu}(\lambda)$ solving the infinite Stieltjes moment problem. We see that H_{n-1}, $H_{n-1}^{(1)}$ being positive definite for all $n = 1, 2, \ldots$ is *sufficient* for the existence of the solutions $\mu(\lambda)$ and $\widetilde{\mu}(\lambda)$ to the infinite moment problem (2.2); see [40, Section 51, pp. 688–690].

The assumption of H_{n-1}, $H_{n-1}^{(1)}$ being positive definite, $n = 1, 2, \ldots$, is also necessary. Indeed, in [40, Section 8, p. 625] Stieltjes showed that equality (2.7) implies the Hankel matrix $H_j^{(k)}$ to be positive definite for every $k + 2j \leq 2n - 1$. In particular, H_{n-1} and $H_{n-1}^{(1)}$ are positive definite. Therefore, (2.10) is the necessary and sufficient condition for the existence of a solution of the infinite Stieltjes moment problem (2.2); see [40, Section 24, p. 649].

Finally, we consider the question as to whether the constructed solution is unique. In the derivation of the previous results Stieltjes distinguished two cases. If

$$\sum_{n=1}^{\infty} a_n < +\infty \quad (\textit{indeterminate case}),$$

then $S(\lambda)$ does not converge, while each of the subsequences $S_{2n}(\lambda)$ and $S_{2n+1}(\lambda)$ converges to a different analytic function in $\mathbb{C} \setminus \mathbb{R}_+$, where $\mathbb{R}_+$ is the positive real line, i.e.,

$$\begin{aligned} \lim_{n\to\infty} S_{2n}(\lambda) &= -\int_0^{\infty} \frac{1}{\lambda - \xi}\, \mathrm{d}\mu(\xi), \\ \lim_{n\to\infty} S_{2n+1}(\lambda) &= -\int_0^{\infty} \frac{1}{\lambda - \xi}\, \mathrm{d}\widetilde{\mu}(\xi). \end{aligned}$$

Hence $\mu(\lambda)$ and $\widetilde{\mu}(\lambda)$ are two of the infinitely many solutions of the infinite moment problem (2.2); see [40, Section 24, pp. 649–650, Sections 51–53, pp. 688–694]. Moreover, for any solution $\psi(\lambda)$ of the moment problem (2.2) we get the bounds

$$-\int_0^{\infty} \frac{1}{\lambda - \xi}\, \mathrm{d}\mu(\xi) \leq -\int_0^{\infty} \frac{1}{\lambda - \xi}\, \mathrm{d}\psi(\xi) \leq -\int_0^{\infty} \frac{1}{\lambda - \xi}\, \mathrm{d}\widetilde{\mu}(\xi);$$

see [40, Section 52, pp. 690–692]. If, on the other hand,

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \quad (\textit{determinate case}),$$

then the continued fraction $S(\lambda)$ converges to an analytic function in $\mathbb{C} \setminus \mathbb{R}_+$ which can be expressed as

$$S(\lambda) = \lim_{n\to\infty} S_{2n}(\lambda) = \lim_{n\to\infty} S_{2n+1}(\lambda) = -\int_0^\infty \frac{1}{\lambda-\xi}\, \mathrm{d}\mu(\xi) = -\int_0^\infty \frac{1}{\lambda-\xi}\, \mathrm{d}\widetilde{\mu}(\xi)$$

and $\mu(\lambda) = \widetilde{\mu}(\lambda)$ is the unique solution of the moment problem (2.2); see [40, Section 54, pp. 694–695].

As remarked by Stieltjes on p. 689 of [40], Chebyshev, Heine, and Darboux studied continued fractions of the kind

$$F(\lambda) \;=\; \cfrac{\gamma_0}{\alpha_1 - \lambda - \cfrac{\gamma_1}{\alpha_2 - \lambda - \cfrac{\gamma_2}{\alpha_3 - \lambda - \cfrac{\gamma_3}{\cdots}}}}. \tag{2.11}$$

In [40, Introduction, pp. 609–613], Stieltjes described that under certain conditions[1] on the coefficients α_j and γ_j the form (2.11) is equivalent to the form (2.3). Moreover, if the two continued fractions are equivalent, then the nth convergent $F_n(\lambda)$ of (2.11) is equal to the $2n$th convergent $S_{2n}(\lambda)$ of (2.3). As shown by Stieltjes, $F_n(\lambda) = S_{2n}(\lambda)$ converges to an analytic function on $\mathbb{C} \setminus \mathbb{R}_+$. Convergence of $S(\lambda)$ cannot be investigated using only its even convergents and this has probably contributed to the fact that it had not been taken into consideration before Stieltjes work.

Obviously, the previous development also gives solution to the finite moment problem (2.1). Consider a finite sequence of $2n$ positive real numbers $m_0, m_1, \ldots, m_{2n-1}$. If H_{n-1} and $H_{n-1}^{(1)}$ are positive definite, then there exists a continued fraction $S(\lambda)$ of the kind (2.3) such that the $2n$th convergent $S_{2n}(\lambda)$ can be developed into the power series around infinity

$$S_{2n}(\lambda) = \frac{p_{2n}(\lambda)}{q_{2n}(\lambda)} = -\frac{m_0}{\lambda} - \frac{m_1}{\lambda^2} - \cdots - \frac{m_{2n-1}}{\lambda^{2n}} - \ldots,$$

where the coefficients a_i of $S(\lambda)$ are given for $i = 1, \ldots, 2n$ by the equations (2.9) and by any chosen sequence of positive real numbers for $i = 2n+1, 2n+2, \ldots$. The distribution function (2.6) then gives solution to the finite Stieltjes moment problem (2.1). Vice versa, using the results in [40, Section 8, p. 625], existence of a solution of the finite moment problem (2.1) implies that the matrices H_{n-1} and $H_{n-1}^{(1)}$ are positive definite. In general, there exist infinitely many distribution functions having moments $m_0, m_1, \ldots, m_{2n-1}$, consider, e.g., $\mu^{(j+1)}(\lambda)$ and $\widetilde{\mu}^{(j)}(\lambda)$ for $j \geq n$; see also [23, Section 21]. However, since the decomposition into the partial fraction (2.5) is unique, $\mu^{(n)}(\lambda)$ is the unique distribution function with n points of increase having moments $m_0, m_1, \ldots, m_{2n-1}$, i.e., the unique solution to the finite Stieltjes moment problem (2.1). Summarizing, we have the following theorem.

[1] Such conditions are satisfied, e.g., for continued fractions associated with CG applied to a linear system with a Hermitian positive definite matrix, respectively with a linear self-adjoint and coercive operator on a Hilbert space; see [22, Section 18] and [29, Section 3.3.2], respectively [31, Sections 5.1, 5.2 and Chapter 11].

Theorem 2.1 (Classical solution of the finite Stieltjes moment problem). *Consider $2n$ positive real numbers $m_0, m_1, \ldots, m_{2n-1}$. The system of $2n$ equations*

$$\sum_{\ell=1}^{n} \omega_\ell^{(n)} \left\{ \lambda_\ell^{(n)} \right\}^j = m_j, \quad j = 0, 1, \ldots, 2n-1$$

has the positive solution $\omega_\ell^{(n)} > 0$, $\ell = 1, \ldots, n$, $0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$ if and only if the Hankel matrices H_{n-1} and $H_{n-1}^{(1)}$ composed of $m_0, m_1, \ldots, m_{2n-1}$ (see (2.8)) are positive definite. The solution is unique and it is given by the poles $\lambda_\ell^{(n)}$ and the weights $\omega_\ell^{(n)}$ obtained by the decomposition of the rational function $p_{2n}(\lambda)/q_{2n}(\lambda)$ into the partial fraction

$$\frac{p_{2n}(\lambda)}{q_{2n}(\lambda)} = \sum_{\ell=1}^{n} \frac{\omega_\ell^{(n)}}{\lambda_\ell^{(n)} - \lambda},$$

where $S_{2n}(\lambda) = p_{2n}(\lambda)/q_{2n}(\lambda)$ is the finite continued fraction given by (2.4) whose $2n$ positive coefficients $a_1, a_2, \ldots, a_{2n}$ are given by

$$a_{2j-1} = \frac{(\Delta_{j-2}^{(1)})^2}{\Delta_{j-2}\Delta_{j-1}} \quad \text{and} \quad a_{2j} = \frac{(\Delta_{j-1})^2}{\Delta_{j-1}^{(1)}\Delta_{j-2}^{(1)}}, \quad j = 1, \ldots n,$$

with Δ_j and $\Delta_j^{(1)}$ the determinants of H_j and $H_{j-1}^{(1)}$, $\Delta_{-1} = \Delta_{-1}^{(1)} = 1$.

The following part of the text will derive the necessary and sufficient conditions for the existence and uniqueness and it will present a straightforward algebraic solution to the finite Stieltjes moment problem by means of the Cholesky decomposition of the Hankel matrix H_{n-1}.

3. Algebraic solution of the finite Stieltjes moment problem

Consider first a specific variant of the moment problem associated with a linear, self-adjoint and coercive operator $\mathcal{A}$ on a Hilbert space (or, analogously, with a finite Hermitian positive definite matrix). Given a nonzero vector f, the solution of the linear equation

$$\mathcal{A}x = f \tag{3.1}$$

always exists and it is unique. Forming a sequence of moments

$$m_j = (f, \mathcal{A}^j f)_H, \quad j = 0, 1, \ldots, \tag{3.2}$$

where $(\cdot, \cdot)_H$ denotes the given Hilbert space inner product, we can consider a sequence of the finite Stieltjes moment problems determined by $m_0, m_1, \ldots, m_{2n-1}$, $n = 1, 2, \ldots$. They are associated with the spectral distribution function determined by the spectral decomposition of $\mathcal{A}$ and the spectral projection of f; see, e.g., [31, Section 5.2 and the references given there] or [29, Chapter 3]. Each finite Stieltjes moment problem determined by the data $m_0, m_1, \ldots, m_{2n-1}$, $n = 1, 2, \ldots$, has the unique solution given by the eigenvalues and the normalized first components of the eigenvectors of the Jacobi matrices J_n from the Lanczos recurrence

$$\mathcal{A}\, Q_n = Q_n J_n + \beta_n q_{n+1} e_n^*, \quad n = 1, 2, \ldots, \tag{3.3}$$

where Q_n is the (formal) matrix containing as its columns the orthonormal basis of the nth Krylov subspace $\{q_1, \mathcal{A}\, q_1, \ldots, \mathcal{A}^{n-1}\, q_1\}$, $q_1 = f/(f, f)_H^{1/2}$.

In this way, any finite Stieltjes moment problem defined by the first $2n$ moments $m_0, m_1, \ldots, m_{2n-1}$ given by (3.2) is uniquely solved by the first n steps of the Lanczos method applied to $\mathcal{A}, q_1$, or by CG applied to (3.1) (here we consider, with no loss of generality, a zero initial approximation).

Unlike in [31] and [29], the following text (as well as the whole paper) considers the moment problem without an underlying assumption that the data $m_0, m_1, \ldots, m_{2n-1}$ are determined as moments (3.2), i.e., are determined by some distribution function that is going to be approximated. Given any sequence of positive real numbers $m_0, m_1, \ldots, m_{2n-1}$, we have to therefore primarily address the questions of existence and uniqueness. We will present an algebraic construction of the solution of the Stieltjes moment problem (2.1) that does not use the concept of continued fractions and combines Cholesky factorization of the matrix H_{n-1} with the three-term recurrences for orthogonal polynomials and the associated Lanczos vectors.

Consider the symmetric bilinear form $(\cdot, \cdot) : \mathcal{P}_{n-1} \times \mathcal{P}_{n-1} \to \mathbb{R}$, which is defined on the space of polynomials of degree at most $n - 1$ with real coefficients $\mathcal{P}_{n-1}$, by the prescribed $2n$ positive real numbers

$$(\lambda^i, \lambda^j) = m_{i+j} \quad \text{for} \quad 0 \leq i + j \leq 2n - 1. \tag{3.4}$$

Notice that the bilinear form is also defined for $(\lambda^n, \lambda^j) = (\lambda^j, \lambda^n) = m_{n+j}$ for $j = 0, \ldots, n - 1$. If the Hankel matrix H_{n-1} composed of the numbers $m_0, m_1, \ldots, m_{2n-2}$ (see (2.8)) is positive definite, i.e., if $\Delta_j > 0$ for $j = 0, \ldots, n - 1$, then by the Cholesky factorization we get the unique lower triangular matrix L_{n-1} with positive elements on the diagonal so that

$$H_{n-1} = L_{n-1} L_{n-1}^*, \tag{3.5}$$

with I_n the identity matrix of dimension n (L^* denotes the Hermitian transpose of L; the matrices used here are real and we use this notation for an ordinary transpose). Consider the inverse of the Cholesky factor

$$L_{n-1}^{-1} = \begin{bmatrix} \xi_{1,1} & & \\ \vdots & \ddots & \\ \xi_{n,1} & \cdots & \xi_{n,n} \end{bmatrix}$$

and the polynomials $\varphi_0(\lambda), \varphi_1(\lambda), \ldots, \varphi_{n-1}(\lambda)$ defined by the rows $1, 2, \ldots, n$ of L_{n-1}^{-1} respectively,

$$\varphi_i(\lambda) = \sum_{j=0}^{i} \xi_{i+1,j+1} \lambda^j, \quad \text{for} \quad i = 0, \ldots, n - 1; \tag{3.6}$$

see [18, Section 4] and [32]. Then (whenever appropriate we skip the argument λ for simplicity of notation)

$$L_{n-1}^{-1}H_{n-1}L_{n-1}^{-*} = \begin{bmatrix} (\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) & \cdots & (\varphi_0,\varphi_{n-1}) \\ (\varphi_1,\varphi_0) & (\varphi_1,\varphi_1) & \cdots & (\varphi_1,\varphi_{n-1}) \\ \vdots & \vdots & & \vdots \\ (\varphi_{n-1},\varphi_0) & (\varphi_{n-1},\varphi_1) & \cdots & (\varphi_{n-1},\varphi_{n-1}) \end{bmatrix} = I_n,$$

i.e.,

$$(\varphi_i,\varphi_j) = \delta_{ij}, \qquad \text{for} \qquad i,j = 0,\ldots,n-1,$$

where δ_{ij} denotes the Kronecker delta. With H_{n-1} positive definite, any polynomial $\varphi(\lambda) \neq 0$ in $\mathcal{P}_{n-1}$ can be written as $\varphi(\lambda) = \sum_{\ell=0}^{n-1} \eta_\ell \varphi_\ell(\lambda)$, with polynomials $\varphi_j(\lambda)$ defined in (3.6). Hence,

$$(\varphi,\varphi) = \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-1} \eta_\ell \eta_k (\varphi_\ell,\varphi_k) = \sum_{\ell=0}^{n-1} \eta_\ell^2 > 0.$$

Consequently, the bilinear form $(\cdot,\cdot)$ defined by (3.4) is an *inner product* on $\mathcal{P}_{n-1}$ and the polynomials $\varphi_0,\varphi_1,\ldots,\varphi_{n-1}$ are *orthonormal polynomials* with respect to this inner product. It remains to show that the inner product (3.4) is given as

$$(\varphi,\psi) = \sum_{\ell=1}^{n} \omega_\ell^{(n)}\, \varphi\Big(\lambda_\ell^{(n)}\Big)\, \psi\Big(\lambda_\ell^{(n)}\Big) \tag{3.7}$$

with $\omega_\ell^{(n)} > 0$, $\ell = 1,\ldots,n$, and $0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$, which will provide, as proved below, the solution to the finite Stieltjes moment problem (2.1).

The orthonormal polynomials defined by (3.4)

$$\varphi_0(\lambda) = \frac{1}{\sqrt{m_0}},\ \varphi_1(\lambda),\ \ldots,\ \varphi_{n-1}(\lambda)$$

satisfy the three-term recurrence

$$\beta_j\varphi_j(\lambda) = (\lambda - \alpha_{j-1})\varphi_{j-1}(\lambda) - \beta_{j-1}\varphi_{j-2}(\lambda), \qquad j = 1,2,\ldots,n-1, \tag{3.8}$$

where $\beta_0 = 0$, $\varphi_{-1}(\lambda) = 0$, $\varphi_0(\lambda) = 1/\sqrt{m_0}$ and the coefficients α_{j-1}, β_j are given by

$$\alpha_{j-1} = (\lambda\varphi_{j-1},\varphi_{j-1}),\ \ \beta_j = \|(\lambda-\alpha_{j-1})\varphi_{j-1} - \beta_{j-1}\varphi_{j-2}\|, \tag{3.9}$$

with the norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$; see, e.g., [10, Chapter I, Section 4], [29, Section 3.3.1]. Consider now for $j = n$ the polynomial

$$\widehat{\varphi}_n(\lambda) = (\lambda - \alpha_{n-1})\varphi_{n-1}(\lambda) - \beta_{n-1}\varphi_{n-2}(\lambda), \tag{3.10}$$

where $\alpha_{n-1} = (\lambda\varphi_{n-1},\varphi_{n-1})$ is well-defined due to (3.4). Clearly,

$$(\widehat{\varphi}_n,\varphi_j) = 0, \quad j = 0,\ldots,n-1$$

by construction (please notice that here $(\cdot,\cdot)$ represents only the bilinear form). We will denote

$$\psi_n(\lambda) = \left(\lambda - \lambda_1^{(n)}\right) \cdots \left(\lambda - \lambda_n^{(n)}\right) \tag{3.11}$$

the monic counterpart of $\widehat{\varphi}_n(\lambda)$; see [29, relations (3.2.11)–(3.2.12), pp. 80–81]. We will show that (3.11) indeed defines the positive distinct nodes needed for (2.1).

The recurrences (3.8)–(3.10) can be written in the compact form as (see, e.g., [49, Section 2.4], [29, Section 3.3.1])

$$\lambda \Phi_n(\lambda) = J_n \Phi_n(\lambda) + \widehat{\varphi}_n(\lambda) e_n, \tag{3.12}$$

where $\Phi_n(\lambda) = [\varphi_0(\lambda), \varphi_1(\lambda), \ldots, \varphi_{n-1}(\lambda)]^T$, $e_n = [0, \ldots, 0, 1]^T$ is the nth vector of the Euclidean basis, and J_n is the nth Jacobi matrix

$$J_n = \begin{bmatrix} \alpha_0 & \beta_1 & & \\ \beta_1 & \alpha_1 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_{n-1} \end{bmatrix}, \quad \beta_j > 0, \quad j = 1, \ldots, n-1.$$

Since $\widehat{\varphi}_n(\lambda_\ell^{(n)}) = 0$, (3.12) immediately gives

$$J_n \, \Phi_n\!\left(\lambda_\ell^{(n)}\right) = \lambda_\ell^{(n)} \, \Phi_n\!\left(\lambda_\ell^{(n)}\right), \quad \ell = 1, \ldots, n$$

and therefore $\lambda_\ell^{(n)}$, $\ell = 1, \ldots, n$, represent the eigenvalues of J_n. Notice that $\Phi_n(\lambda) \neq 0$ for every λ since $\varphi_0(\lambda) = 1/\sqrt{m_0} \neq 0$.

Let in addition to H_{n-1}, also the symmetric matrix (see (2.8))

$$H_{n-1}^{(1)} = \begin{bmatrix} (\lambda, 1) & (\lambda, \lambda) & \ldots & (\lambda, \lambda^{n-1}) \\ (\lambda^2, 1) & (\lambda^2, \lambda) & \ldots & (\lambda^2, \lambda^{n-1}) \\ \vdots & \vdots & & \vdots \\ (\lambda^n, 1) & (\lambda^n, \lambda) & \ldots & (\lambda^n, \lambda^{n-1}) \end{bmatrix}$$

be positive definite. Since an easy manipulation gives (see [25, Theorem 2])

$$L_{n-1}^{-1} H_{n-1}^{(1)} L_{n-1}^{-*} = \begin{bmatrix} (\lambda\varphi_0, \varphi_0) & (\lambda\varphi_0, \varphi_1) & \ldots & (\lambda\varphi_0, \varphi_{n-1}) \\ (\lambda\varphi_1, \varphi_0) & (\lambda\varphi_1, \varphi_1) & \ldots & (\lambda\varphi_1, \varphi_{n-1}) \\ \vdots & \vdots & & \vdots \\ (\lambda\varphi_{n-1}, \varphi_0) & (\lambda\varphi_{n-1}, \varphi_1) & \ldots & (\lambda\varphi_{n-1}, \varphi_{n-1}) \end{bmatrix} = J_n, \tag{3.13}$$

where L_{n-1} is defined by (3.5), the symmetric matrix J_n is positive definite if and only if $H_{n-1}^{(1)}$ is positive definite. The eigenvalues of the Jacobi matrix J_n are distinct (see, e.g., [29, p. 115], [35, Lemma 7.7.1]), and therefore we can with no loss of generality in ordering the eigenvalues write

$$0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}.$$

Consider the spectral decomposition

$$J_n = Z_n \, \Lambda \, Z_n^*, \quad \Lambda = \mathrm{diag}\Big(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}\Big)\,,$$

where $Z_n = [z_1^{(n)}, \ldots, z_n^{(n)}]$ is the matrix of the associated normalized eigenvectors of J_n as its columns. The spectral decomposition rewritten as

$$\Lambda \, Z_n^* = Z_n^* \, J_n \tag{3.14}$$

represents the Lanczos process with the diagonal matrix Λ and the starting vector given by $v_1 = Z_n^* e_1$, $v_1^* v_1 = 1$, i.e., composed of the first elements of the normalized eigenvectors of the matrix J_n. The vectors $v_j = Z_n^* e_j$, $j = 1, \ldots, n$ are then given in terms of polynomials in the diagonal matrix Λ as

$$v_j = p_{j-1}(\Lambda)\, v_1, \quad j = 1, \ldots, n, \quad p_0(\lambda) = 1.$$

From the orthonormality of the vectors $v_i^* v_j = \delta_{ij}$, $i, j = 1, \ldots, n$, we immediately get the orthonormality of the polynomials $p_0(\lambda), p_1(\lambda), \ldots, p_{n-1}(\lambda)$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{P}}$ on the space of polynomials $\mathcal{P}_{n-1}$

$$(p_i, p_j)_{\mathcal{P}} = \sum_{\ell=1}^{n} \Big(e_1^* z_\ell^{(n)}\Big)^2 \, p_i\Big(\lambda_\ell^{(n)}\Big) \, p_j\Big(\lambda_\ell^{(n)}\Big)\,.$$

Moreover, the polynomials $p_0(\lambda) = 1, p_1(\lambda), \ldots, p_{n-1}(\lambda)$ must due to (3.14) satisfy the same three-term recurrence as the polynomials $\varphi_0(\lambda), \varphi_1(\lambda), \ldots, \varphi_{n-1}(\lambda)$; see (3.8). Therefore

$$\varphi_j(\lambda) = \frac{1}{\sqrt{m_0}} \, p_j(\lambda), \quad j = 0, \ldots, n-1$$

and the inner product $(\cdot,\cdot)$ defined by (3.4) is equivalently given by[2]

$$(\varphi, \psi) = \sum_{\ell=1}^{n} \frac{(e_1^* z_\ell^{(n)})^2}{m_0} \, \varphi\Big(\lambda_\ell^{(n)}\Big) \, \psi\Big(\lambda_\ell^{(n)}\Big)\,, \tag{3.15}$$

which determines the nodes in (3.7) as expected and gives the weights

$$\omega_\ell^{(n)} = \frac{(e_1^* z_\ell^{(n)})^2}{m_0}, \quad \ell = 1, \ldots, n.$$

Rewriting the equality (3.13) as

$$H_{n-1}^{(1)} = L_{n-1} \begin{bmatrix} (\lambda\varphi_0, \varphi_0) & (\lambda\varphi_0, \varphi_1) & \ldots & (\lambda\varphi_0, \varphi_{n-1}) \\ (\lambda\varphi_1, \varphi_0) & (\lambda\varphi_1, \varphi_1) & \ldots & (\lambda\varphi_1, \varphi_{n-1}) \\ \vdots & \vdots & & \vdots \\ (\lambda\varphi_{n-1}, \varphi_0) & (\lambda\varphi_{n-1}, \varphi_1) & \ldots & (\lambda\varphi_{n-1}, \varphi_{n-1}) \end{bmatrix} L_{n-1}^*,$$

[2] In the context of calculating the nodes and the weights of the Gauss quadrature, $\lambda_\ell^{(n)}$ and $\omega_\ell^{(n)}$ are given as the eigenvalues and the squared first elements of the associated eigenvectors of J_n in [19, Section III] and [18, Section 2]. The proofs are different.

proves that the constructed inner product (3.15) indeed solves the finite Stieltjes moment problem (2.1). Uniqueness follows from the construction.

It remains to show that the assumptions on H_{n-1}, $H_{n-1}^{(1)}$ being positive definite, which were used in the construction of the inner product (3.15), are also necessary. Given the inner product (3.7) solving the finite Stieltjes moment problem (2.1), consider the sequence of orthonormal polynomials $\varphi_0(\lambda), \varphi_1(\lambda), \ldots, \varphi_{n-1}(\lambda)$, and the associated positive definite Jacobi matrix J_n. Since for any nonzero vector v

$$v^* H_{n-1} v = (\varphi_v, \varphi_v) > 0 \quad \text{for some } \varphi_v \in \mathcal{P}_{n-1},\ \varphi_v \neq 0,$$

H_{n-1} must be positive definite. Using the Cholesky decomposition (3.5) and the relationships (3.13) between $H_{n-1}^{(1)}$ and J_n, $H_{n-1}^{(1)}$ must be positive definite as well. Summarizing, we have proved the following theorem, where the first part with the necessary and sufficient conditions for the existence of the solution is identical to the first part of Theorem 2.1. Its proof is, as shown by the construction above, substantially different.

Theorem 3.1 (Algebraic solution of the finite Stieltjes moment problem). *Consider $2n$ positive real numbers $m_0, m_1, \ldots, m_{2n-1}$. The system of $2n$ equations*

$$\sum_{\ell=1}^{n} \omega_\ell^{(n)} \left\{ \lambda_\ell^{(n)} \right\}^j = m_j, \quad j = 0, 1, \ldots, 2n-1$$

has the positive solution $\omega_\ell^{(n)} > 0$, $\ell = 1, \ldots, n$, $0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$ if and only if the Hankel matrices H_{n-1} and $H_{n-1}^{(1)}$ composed of $m_0, m_1, \ldots, m_{2n-1}$ (see (2.8)) are positive definite. The solution is unique and it is given by the eigenvalues $\lambda_\ell^{(n)}$ and the rescaled first components of the associated normalized eigenvectors $z_\ell^{(n)}$ of the Jacobi matrix J_n,

$$J_n z_\ell^{(n)} = \lambda_\ell^{(n)} z_\ell^{(n)}, \quad \omega_\ell^{(n)} = \frac{(e_1^* z_\ell^{(n)})^2}{m_0}, \quad \ell = 1, 2, \ldots, n,$$

where J_n results from the Cholesky factorization of H_{n-1} and the subsequent simple manipulations

$$H_{n-1} = L_{n-1} L_{n-1}^*, \quad J_n = L_{n-1}^{-1} H_{n-1}^{(1)} L_{n-1}^{-*}.$$

Remark 3.2. *It should be understood that the presented results and the summary formulated as Theorem 3.1 on the algebraic solution of the Stieltjes moment problem are by no means meant as a suggestion for constructing a practically usable computational algorithm. Because of the notorious ill-conditioning of the moment matrices (see, e.g., [43], [14], [4], and [5]) the factorization of the explicitly formed moment matrices are numerically unfeasible. For pointers to practically usable algorithms for computation of the nodes and weights of the Gauss quadrature, which represents a related but different problem, extended discussions and/or references to the relevant literature we refer the interested reader to, e.g., [16, Chapter 2], [33, Section 3], [17, Chapter 5], and [29, Section 3.6].*

4. Conclusion

The finite Stieltjes moment problem (2.1) has a unique solution if and only if the Hankel matrices H_{n-1} and $H_{n-1}^{(1)}$ (2.8) are positive definite. Using the approach of Stieltjes embedded in the theory of continued fractions, the solution can be expressed through the finite continued fraction (2.4) whose coefficients are given by (2.9); see Theorem 2.1 where we have summarized the results contained in [40]. We remark that the equations (2.9) link the positive definiteness of H_{n-1} and $H_{n-1}^{(1)}$ with the construction of the continued fraction and, as a consequence, it leads to the solution presented in Theorem 2.1. In a more direct purely algebraic approach summarized in Theorem 3.1, the positive definiteness of H_{n-1} allows to use its Cholesky decomposition and the positive definiteness of $H_{n-1}^{(1)}$ completes the argument by using the spectral decomposition of the associated (positive definite) Jacobi matrix J_n.

Finally, the Jacobi matrix J_n in Theorem 3.1 reveals the link between the Stieltjes moment problem, the Gauss quadrature, the Lanczos method for self-adjoint eigenvalue problems, and CG for solving equations with linear, self-adjoint, and coercive operators. An interested reader can find more information on these relationships (and many references to an extensive existing literature) in the monographs [29, in particular Chapter 3] and [31, Chapters 5 and 11].

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