LARGE DATA EXISTENCE THEORY FOR THREE-DIMENSIONAL UNSTEADY FLOWS OF RATE-TYPE VISCOELASTIC FLUIDS WITH STRESS DIFFUSION

MICHAL BATHORY, MIROSLAV BULÍČEK, AND JOSEF MÁLEK

Abstract. We prove that there exists a weak solution to a system governing an unsteady flow of a viscoelastic fluid in three dimensions, for arbitrarily large time interval and data. The fluid is described by the incompressible Navier-Stokes equations for the velocity $\boldsymbol{v}$, coupled with a diffusive variant of a combination of the Oldroyd-B and the Giesekus models for a tensor $\mathbb{B}$. By a proper choice of the constitutive relations for the Helmholtz free energy (which, however, is non-standard in the current literature despite the fact that this choice is well motivated from the point of view o physics) and for the energy dissipation, we are able to prove that $\mathbb{B}$ enjoys the same regularity as $\boldsymbol{v}$ in the classical three-dimensional Navier-Stokes equations. This enables us to handle any kind of objective derivative of $\mathbb{B}$, thus obtaining existence results for the class of diffusive Johnson-Segalman models as well. Moreover, using a suitable approximation scheme, we are able to show that $\mathbb{B}$ remains positive definite if the initial datum was a positive definite matrix (in a pointwise sense). We also show how the model we are considering can be derived from basic balance equations and thermodynamical principles in a natural way.

1. Introduction

We aim to establish a global-in-time and large-data existence theory, within the context of weak solutions, to a class of homogeneous incompressible rate-type viscoelastic fluid flowing in a closed three-dimensional container. The studied class of models can be seen as the Navier-Stokes system coupled with a viscoelastic rate-type fluid model that shares the properties of both Oldroyd-B and Giesekus models and is completed with a diffusion term. Such models are frequently encountered in the theory of non-Newtonian fluid mechanics, see [12, 11] and further references cited in [11].

In order to precisely formulate the problems investigated in this study, we start introducing notation. For a bounded domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary $\partial\Omega$ and a time interval of the length $T > 0$, we set $Q := (0,T) \times \Omega$ for a time-space cylinder and $\Sigma := (0,T) \times \partial\Omega$ for the evolving boundary. The symbol $\boldsymbol{n}$ denotes the outward unit normal vector on $\partial\Omega$, and for any vector $\mathbf{z}$, the vector $\mathbf{z}_\tau$ denotes the projection of a vector to a tangent plane on $\partial\Omega$, i.e., $\mathbf{z}_\tau := \mathbf{z} - (\mathbf{z} \cdot \boldsymbol{n})\boldsymbol{n}$. Then,

1991 *Mathematics Subject Classification.* Primary 35Q35, 76A05, 76A10.

Key words and phrases. viscoelasticity, Oldroyd-B, Johnson-Segalman, large-data and long-time existence, weak solution.

Michal Bathory was supported by the project No. 1652119 financed by the Charles University Grant Agency (GAUK). Miroslav Bulíček and Josef Málek acknowledge the support of the project No. 18-12719S financed by Czech science foundation (GAČR). Miroslav Bulíček and Josef Málek are members of the Jindřich Nečas center for mathematical modelling.

for a given density of the external body forces $\boldsymbol{f} : Q \to \mathbb{R}^3$, a given initial velocity $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^3$ and a given initial extra stress tensor $\mathbb{B}_0 : \Omega \to \mathbb{R}^{3\times 3}_{>0}$ (here $\mathbb{R}^{3\times 3}_{>0}$ denotes the set of positively definite symmetric (3×3)-matrices), we look for a vector field $\boldsymbol{v} : Q \to \mathbb{R}^3$, a scalar field $p : Q \to \mathbb{R}$ and a positive definite matrix field $\mathbb{B} : Q \to \mathbb{R}^{3\times 3}_{>0}$ solving the following system in Q:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} = -\nabla p + 2\mu a \operatorname{div}((1-\gamma)(\mathbb{B} - \mathbb{I}) + \gamma(\mathbb{B}^2 - \mathbb{B})) + \boldsymbol{f}, \tag{1.2}$$

$$\begin{aligned} \partial_t \mathbb{B} + (\boldsymbol{v} \cdot \nabla)\mathbb{B} + \delta_1(\mathbb{B} - \mathbb{I}) + \delta_2(\mathbb{B}^2 - \mathbb{B}) - \lambda \Delta \mathbb{B} \\ = \tfrac{a+1}{2}(\nabla \boldsymbol{v} \mathbb{B} + (\nabla \boldsymbol{v} \mathbb{B})^T) + \tfrac{a-1}{2}(\mathbb{B} \nabla \boldsymbol{v} + (\mathbb{B} \nabla \boldsymbol{v})^T), \end{aligned} \tag{1.3}$$

and being completed by the following boundary conditions on Σ:

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= 0, \\ -\sigma \boldsymbol{v}_\tau &= ((2\nu \mathbb{D}\boldsymbol{v} + 2a(1-\gamma)(\mathbb{B} - \mathbb{I}) + 2a\gamma(\mathbb{B}^2 - \mathbb{B}))\boldsymbol{n})_\tau, \\ (\boldsymbol{n} \cdot \nabla)\mathbb{B} &= \mathbb{O}, \qquad \text{(here } \mathbb{O} \text{ stands for zero } 3 \times 3\text{-matrix)} \end{aligned} \tag{1.4}$$

and by the initial conditions in Ω:

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \tag{1.5}$$

$$\mathbb{B}(0) = \mathbb{B}_0. \tag{1.6}$$

The parameters $\gamma \in (0, 1)$, $\nu, \lambda, \sigma > 0$, $\delta_1, \delta_2 \geq 0$ and $a \in \mathbb{R}$ are given numbers.

The main result of this study can be stated as follows.

Theorem. *Let $\boldsymbol{v}_0$ and $\mathbb{B}_0$ be such that the initial total energy is bounded. Then for sufficiently regular $\boldsymbol{f}$, there exists global-in-time weak solution to* (1.1)–(1.6)*.*

Although the above theorem is stated vaguely, we would like to emphasize that we are going to establish the **long-time** existence of weak solution for **large data** and for **three-dimensional** flows. A more precise and rigorous version of the above result including the correct function spaces and the properly defined weak formulation is stated in Theorem 2.

We complete the introductory part by providing physical background relevant to the studied problem and by recalling the earlier results relevant to the problem (1.1)–(1.6) analyzed here.

1.1. **Mathematical and thermodynamical background.** The system (1.1)–(1.4) can be rewritten into a more concise form once one recognizes some physical quantities. First of all, let

$$\mathbb{D}\boldsymbol{v} = \tfrac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) \quad \text{and} \quad \mathbb{W}\boldsymbol{v} = \tfrac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T)$$

denote the symmetric and antisymmetric parts of the velocity gradient $\nabla \boldsymbol{v}$, respectively. Then, looking at the equation (1.2), we see that (1.2) is obtained from a general form of the balance of linear momentum, namely

$$\varrho \overset{\bullet}{\boldsymbol{v}} = \operatorname{div} \mathbb{T} + \varrho \boldsymbol{f}, \tag{1.7}$$

once we set the density $\varrho = 1$ and require that the Cauchy stress tensor $\mathbb{T}$ has the form

$$\mathbb{T} = -p\mathbb{I} + 2\nu \mathbb{D}\boldsymbol{v} + 2a\mu((1-\gamma)(\mathbb{B} - \mathbb{I}) + \gamma(\mathbb{B}^2 - \mathbb{B})). \tag{1.8}$$

In (1.7), $\overset{\bullet}{\boldsymbol{v}}$ stands for the material time derivative of $\boldsymbol{v}$, i.e., $\overset{\bullet}{\boldsymbol{v}} = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. Setting similarly the material time derivative of a tensor $\mathbb{B}$ by $\overset{\bullet}{\mathbb{B}}$, i.e.,

$$\overset{\bullet}{\mathbb{B}} = \partial_t \mathbb{B} + (\boldsymbol{v} \cdot \nabla)\mathbb{B},$$

we can recognize the presence of a general objective derivative in (1.3). Namely, defining

$$\overset{\square}{\mathbb{B}} = \overset{\bullet}{\mathbb{B}} - a(\mathbb{D}\boldsymbol{v}\mathbb{B} + \mathbb{B}\mathbb{D}\boldsymbol{v}) - (\mathbb{W}\boldsymbol{v}\mathbb{B} - \mathbb{B}\mathbb{W}\boldsymbol{v}),$$

we can rewrite the system (1.1)–(1.3) into a more familiar form as

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1.9}$$

$$\overset{\bullet}{\boldsymbol{v}} = \operatorname{div} \mathbb{T} + \boldsymbol{f} \tag{1.10}$$

$$\overset{\square}{\mathbb{B}} + \delta_1(\mathbb{B} - \mathbb{I}) + \delta_2(\mathbb{B}^2 - \mathbb{B}) = \lambda \Delta \mathbb{B}, \tag{1.11}$$

which is supposed to hold true in Q and which is completed by the initial conditions (1.5), (1.6) fulfilled in Ω and by the boundary conditions (1.4) on Σ that take the form:

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \tag{1.12}$$

$$(\mathbb{T}\boldsymbol{n})_\tau = -\sigma \boldsymbol{v}_\tau, \tag{1.13}$$

$$(\boldsymbol{n} \cdot \nabla)\mathbb{B} = \mathbb{O}. \tag{1.14}$$

We provide several comments regarding the equations above, both in the bulk and on the boundary. The Navier slip boundary condition (1.13) is considered here just for simplicity; note that for smooth domains, namely if $\Omega \in \mathcal{C}^{1,1}$, we can introduce the pressure p as an integrable function, e.g. by using an additional layer of approximation as in [4], see also [9, 8] or [2] how to treat the pressure in evolutionary models with the Navier boundary conditions. Nevertheless, since we shall always deal with formulation without the pressure, see the statement of Theorem 2, we can also treat the Dirichlet boundary condition as well, or a very general kind of implicitly given boundary condition see e.g., [22, 5, 6] or [2]. The Neumann boundary condition for $\mathbb{B}$ is here considered just for simplicity and without any specific physical meaning.

Next aspect, which makes the above system more complicated than the Navier-Stokes equation is the form of the Cauchy stress tensor $\mathbb{T}$ as in (1.8). The term $-p\mathbb{I} + 2\nu\mathbb{D}\boldsymbol{v}$ corresponds to the standard Newtonian fluid with a constant kinematic viscosity ν. The next part of the Cauchy stress which depends linearly on $\mathbb{B}$ appears in all the viscoelastic rate-type fluid models - see, e.g., [18, (7.20b), (8.20e)], [14, (6.43e)] or [11, (43a)]. On the other hand, the addition of the term $2a\mu\gamma(\mathbb{B}^2 - \mathbb{B})$ is, to our best knowledge, considered here for the first time. The fact that we require that γ is positive (and strictly less than 1) plays a key role in the analysis of the problem, as will be shown below.

The quantity $\mathbb{B}$ takes into account the elastic responses of the fluid and the equation (1.11) describes its evolution in the current configuration (Eulerian description), just as the velocity $\boldsymbol{v}$. It is frequent to call the tensor $\mu(\mathbb{B} - \mathbb{I})$ the extra stress or conformation tensor and to denote it by $\boldsymbol{\tau}$. More importantly, since the material derivative of $\mathbb{B}$ is not objective, it must be "corrected" and this is the reason, why in (1.11) the derivative $\overset{\square}{\mathbb{B}}$ appears. The parameter a in the definition of $\overset{\square}{\mathbb{B}}$

determines the type of the objective derivative. The case $a = 1$ leads to the upper convected Oldroyd derivative, that has favorable physical properties and that leads to the clear interpretation of $\mathbb{B}$ within the thermodynamical framework developed in [24], see also [25, 20, 21, 19]. Next, the case $a = 0$ leads to the corrotational Jaumann-Zaremba derivative and this is the only case for which the analysis is much simpler than in other cases. Furthermore, if $a \in [-1, 1]$, one obtains the whole class of Gordon-Schowalter derivatives. However, it turns out that the physical properties of these derivatives are irrelevant for the analysis presented below (except the case $a = 0$), therefore we may take any $a \in \mathbb{R}$. For $a = 1$ and $\lambda = 0$ we distinguis two cases: if $\delta_1 > 0$ and $\delta_2 = 0$ we obtain the classical Oldroyd-B model while if $\delta_1 = 0$ and $\delta_2 > 0$ we get the Giesekus model. Next, by considering $a \in [-1, 1]$, we obtain the class of Johnson-Segalman models. If we further let $\lambda > 0$, we are introducing a diffusive variants of the previous models. It has been observed that including the diffusion term in (1.11) is physically reasonable, see, e.g., [12] or [11] and references therein. However, up to now, it has been unknown what precise form should the diffusion term take and also whether it actually helps in the analysis of the model. Our main result provides a partial answer to this question, namely: for $\gamma \in (0, 1)$ and with the diffusion term being of the form $\Delta\mathbb{B}$ (or more generally, the linear second order operator), the global existence of a weak solution is available.

The reader familiar with the equations describing flows of the standard Oldroyd-B viscoleastic rate-type fluid can identify two deviations in the set of equations (1.9)–(1.11) studied hereafter. We provide a few comments on these differences.

The first deviation concerns the incorporation of the stress diffusion, i.e. the term $-\Delta\mathbb{B}$, into the equations. Following the pioneering work of [12] it is clear that the quantity related to $|\nabla\mathbb{B}|^2$ has to be added into the list of underlying dissipating mechanisms. On the other hand, the precise form in which the stress diffusion should appear depends on the choice of a thermodynamical approach and specific assumptions. In fact, using the thermodynamical concepts as in [18] or [11], one can derive models, where the stress diffusion term takes the form $-\mathbb{B}\Delta\mathbb{B} - \Delta\mathbb{B}\mathbb{B}$, $-\mathbb{B}^{\frac{1}{2}}\Delta\mathbb{B}\mathbb{B}^{\frac{1}{2}}$ etc, however, we would prefer to have $-\Delta\mathbb{B}$ simply because it coincides with the form proposed by [12], and, from the perspective of PDE analysis, one prefers to deal with stress diffusion that leads to a linear operator. Viscoelastic models with a stress diffusion, but without the term $\mathbb{B}^2$ in the stress tensor are derived, e.g., in [18] and [11] even in the temperature dependent case. Here, we will briefly explain the approach in a simplified isothermal setting (sufficient for the purpose of this study), referring to the mentioned works for the derivation in a complete thermal setting and for more details.

The second deviation from usual viscoelastic models consists in the presence of the term $(\mathbb{B}^2 - \mathbb{B})$ in the Cauchy stress tensor, see (1.8). This term arises if we slightly modify energy storage mechanism and apply thermodynamic approach as developed in [18]. In what follows, we shall give the clear interpretation and thermodynamic derivation of our model.

First, we postulate the constitutive equation for the Helmholtz free energy in the form

$$\psi(\mathbb{B}) := \mu((1-\gamma)(\operatorname{tr}\mathbb{B} - 3 - \ln\det\mathbb{B}) + \tfrac{1}{2}\gamma|\mathbb{B} - \mathbb{I}|^2), \tag{1.15}$$

where $\mu > 0$ and $\gamma \in [0, 1]$ is a kind of parameter interpolating between two forms of the energy. The choice $\gamma = 0$ would lead to a standard Oldroyd-B diffusive model. To our best knowledge, the case $\gamma > 0$ was not considered before in literature. The term $\frac{1}{2}\gamma|\mathbb{B}-\mathbb{I}|^2$, which is newly included in ψ is obviously convex with the minimum at $\mathbb{B}=\mathbb{I}$ and depends just on $\operatorname{tr}\mathbb{B}$ and on $\operatorname{tr}(\mathbb{B}\mathbb{B})$, i.e., on invariants of $\mathbb{B}$, therefore it does not violate any of the basic principles of continuum physics. Moreover, such an addition does not affect the first three terms in the asymptotic expansion of ψ near $\mathbb{I}$, on the logarithmic scale. To see this, let $\mathbb{H}$ denote the Hencky logarithmic tensor satisfying $e^{\mathbb{H}}=\mathbb{B}$ (which exists due to the positive definiteness of $\mathbb{B}$). Using Jacobi's identity, we compute that

$$\operatorname{tr}\mathbb{B}-3-\ln\det\mathbb{B}=\operatorname{tr}(e^{\mathbb{H}}-\mathbb{I}-\mathbb{H})=\operatorname{tr}(\tfrac{1}{2}\mathbb{H}^2+O(\mathbb{H}^3)).$$

On the other hand, we easily get

$$\tfrac{1}{2}|\mathbb{B}-\mathbb{I}|^2=\tfrac{1}{2}\operatorname{tr}(e^{2\mathbb{H}}-2e^{\mathbb{H}}+\mathbb{I})=\operatorname{tr}(\tfrac{1}{2}\mathbb{H}^2+O(\mathbb{H}^3)),$$

hence we also have

$$(1-\gamma)(\operatorname{tr}\mathbb{B}-3-\ln\det\mathbb{B})+\tfrac{1}{2}\gamma|\mathbb{B}-\mathbb{I}|^2=\operatorname{tr}(\tfrac{1}{2}\mathbb{H}^2+O(\mathbb{H}^3))$$

and we see that for $\mathbb{B}$ being close to identity, the form of ψ is almost independent of the choice of parameter γ and the second part of ψ in (1.15) can be just understood as a correction for large values of $\mathbb{B}$.

Next, we show how the constitutive equation for $\mathbb{T}$ (see (1.8)) appears naturally if we start with the choice of the Helmholtz free energy (1.15) and require that the form of the equation for $\mathbb{B}$ is given by (1.11). For the derivation, we followed the approach developed in [18] that stems from the balance equations and requires the knowledge how the material stores the energy, but we simplify it by considering that the density is constant (in fact we set for simplicity $\varrho=1$ and hence $\operatorname{div}\boldsymbol{v}=0$) and the flows are isothermal, i.e. the temperature θ is constant as well. Under these assumptions the balance equations of continuum physics (for linear and angular momenta, energy and for formulation of the second law of thermodynamics) take the form

$$\begin{aligned}\overset{\bullet}{\boldsymbol{v}}&=\operatorname{div}\mathbb{T},\quad \mathbb{T}=\mathbb{T}^T,\\ \overset{\bullet}{e}&=\mathbb{T}\cdot\mathbb{D}\boldsymbol{v}-\operatorname{div}\boldsymbol{j}_e,\\ \overset{\bullet}{\eta}&=\xi-\operatorname{div}\boldsymbol{j}_\eta\qquad\text{with } \xi\geq 0,\end{aligned}$$

where e is the (specific) internal energy, η is the entropy, ξ is the rate of entropy production, $\mathbb{T}$ is the Cauchy stress tensor and the quantities $\boldsymbol{j}_e$, $\boldsymbol{j}_\eta$ represent the internal and the entropy fluxes, respectively. Since the quantities ψ, e, θ and η are related thorugh the thermodynamical identity

$$e=\psi+\theta\eta,$$

we can easily deduce from above identities that

$$\theta\xi=\theta\overset{\bullet}{\eta}+\operatorname{div}(\theta\boldsymbol{j}_\eta)=\mathbb{T}\cdot\mathbb{D}\boldsymbol{v}-\operatorname{div}(\boldsymbol{j}_e-\theta\boldsymbol{j}_\eta)-\overset{\bullet}{\psi}. \tag{1.16}$$

To evaluate the last term, we rewrite (1.11) as

$$-\overset{\bullet}{\mathbb{B}}=-\lambda\Delta\mathbb{B}-a(\mathbb{D}\boldsymbol{v}\mathbb{B}+\mathbb{B}\mathbb{D}\boldsymbol{v})-(\mathbb{W}\boldsymbol{v}\mathbb{B}-\mathbb{B}\mathbb{W}\boldsymbol{v})+\delta_1(\mathbb{B}-\mathbb{I})+\delta_2(\mathbb{B}^2-\mathbb{B}). \tag{1.17}$$

Next, it follows from (1.15) that

$$\frac{\partial\psi(\mathbb{B})}{\partial\mathbb{B}}=\mathbb{J},$$

where $\mathbb{J}$ is defined as

$$\mathbb{J} := \mu(1-\gamma)(\mathbb{I}-\mathbb{B}^{-1}) + \mu\gamma(\mathbb{B}-\mathbb{I}).$$

Consequently, taking the scalar product of (1.17) with $\mathbb{J}$ we observe that (since $\mathbb{B}\mathbb{J} = \mathbb{J}\mathbb{B}$, the term with $\mathbb{W}\boldsymbol{v}$ vanishes)

$$\begin{aligned} -\overset{\bullet}{\psi} &= -\lambda\Delta\mathbb{B}\cdot\mathbb{J} - a(\mathbb{D}\boldsymbol{v}\mathbb{B}+\mathbb{B}\mathbb{D}\boldsymbol{v})\cdot\mathbb{J} - (\mathbb{W}\boldsymbol{v}\mathbb{B}-\mathbb{B}\mathbb{W}\boldsymbol{v})\cdot\mathbb{J} \\ &\quad + \delta_1(\mathbb{B}-\mathbb{I})\cdot\mathbb{J} + \delta_2(\mathbb{B}^2-\mathbb{B})\cdot\mathbb{J} \\ &= -\lambda\operatorname{div}(\nabla\psi(\mathbb{B})) - a(\mathbb{D}\boldsymbol{v}\mathbb{B}+\mathbb{B}\mathbb{D}\boldsymbol{v})\cdot\mathbb{J} \\ &\quad + \delta_1(\mathbb{B}-\mathbb{I})\cdot\mathbb{J} + \delta_2(\mathbb{B}^2-\mathbb{B})\cdot\mathbb{J} + \lambda\nabla\mathbb{B}\cdot\nabla\mathbb{J}. \end{aligned} \tag{1.18}$$

To evaluate the terms on the last line, we use the symmetry and the positive definiteness of the matrix $\mathbb{B}$ to obtain

$$\begin{aligned} (\mathbb{B}-\mathbb{I})\cdot\mathbb{J} &= \mu(1-\gamma)|\mathbb{B}^{\frac12}-\mathbb{B}^{-\frac12}|^2 + \mu\gamma|\mathbb{B}-\mathbb{I}|^2, \\ (\mathbb{B}^2-\mathbb{B})\cdot\mathbb{J} &= \mu(1-\gamma)|\mathbb{B}-\mathbb{I}|^2 + \mu\gamma|\mathbb{B}^{\frac32}-\mathbb{B}^{\frac12}|^2, \\ \nabla\mathbb{B}\cdot\nabla\mathbb{J} &= \mu\gamma|\nabla\mathbb{B}|^2 - \mu(1-\gamma)\nabla\mathbb{B}\cdot\nabla\mathbb{B}^{-1} \\ &= \mu\gamma|\nabla\mathbb{B}|^2 + \mu(1-\gamma)\nabla\mathbb{B}\cdot\mathbb{B}^{-1}\nabla\mathbb{B}\mathbb{B}^{-1} \\ &= \mu\gamma|\nabla\mathbb{B}|^2 + \mu(1-\gamma)|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}|^2. \end{aligned} \tag{1.19}$$

Similarly, we obtain

$$a(\mathbb{B}\mathbb{D}\boldsymbol{v}+\mathbb{D}\boldsymbol{v}\mathbb{B})\cdot\mathbb{J} = \left[2\mu a((1-\gamma)(\mathbb{B}-\mathbb{I}) + \gamma(\mathbb{B}^2-\mathbb{B}))\right]\cdot\mathbb{D}\boldsymbol{v}. \tag{1.20}$$

Thus, using (1.18)–(1.20) in (1.16), we conclude that

$$\begin{aligned} \theta\xi &= -\operatorname{div}(\lambda\nabla\psi(\mathbb{B}) + \boldsymbol{j}_e - \theta\boldsymbol{j}_\eta) \\ &\quad + \left[\mathbb{T} - 2a\mu((1-\gamma)(\mathbb{B}-\mathbb{I}) + \gamma(\mathbb{B}^2-\mathbb{B}))\right]\cdot\mathbb{D}\boldsymbol{v} \\ &\quad + \mu\lambda(\gamma|\nabla\mathbb{B}|^2 + (1-\gamma)|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}|^2) \\ &\quad + \mu\left((1-\gamma)\delta_1|\mathbb{B}^{\frac12}-\mathbb{B}^{-\frac12}|^2 + \gamma\delta_2|\mathbb{B}^{\frac32}-\mathbb{B}^{\frac12}|^2\right) \\ &\quad + \mu\left(((1-\gamma)\delta_2+\gamma\delta_1)|\mathbb{B}-\mathbb{I}|^2\right). \end{aligned} \tag{1.21}$$

Hence, assuming that the fluxes fulfill

$$\lambda\nabla\psi(\mathbb{B}) + \boldsymbol{j}_e - \theta\boldsymbol{j}_\eta = 0, \tag{1.22}$$

and setting (compare with (1.8))

$$\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}\boldsymbol{v} + 2a\mu((1-\gamma)(\mathbb{B}-\mathbb{I}) + \gamma(\mathbb{B}^2-\mathbb{B})),$$

the identity (1.21) reduces to (noticing that $-p\mathbb{I}\cdot\mathbb{D}\boldsymbol{v} = -p\operatorname{div}\boldsymbol{v} = 0$)

$$\begin{aligned} \theta\xi &= \mu\lambda(\gamma|\nabla\mathbb{B}|^2 + (1-\gamma)|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}|^2) + 2\nu|\mathbb{D}\boldsymbol{v}|^2 \\ &\quad + \mu\left((1-\gamma)\delta_1|\mathbb{B}^{\frac12}-\mathbb{B}^{-\frac12}|^2 + \gamma\delta_2|\mathbb{B}^{\frac32}-\mathbb{B}^{\frac12}|^2\right) \\ &\quad + \mu\left(((1-\gamma)\delta_2+\gamma\delta_1)|\mathbb{B}-\mathbb{I}|^2\right), \end{aligned} \tag{1.23}$$

which gives the nonnegative rate of the entropy production. Moreover, we have seen how the form of the Cauchy stress tensor $\mathbb{T}$ in (1.8) is dictated by the second line in (1.21). Furthermore, we can also see in (1.23) how the choice of the free energy (1.15) affects the entropy production due to the presence of the diffusive term $\Delta\mathbb{B}$ in (1.3).

1.2. **Concept of the weak solution and energy (in)equality.** In order to introduce the proper concept of a weak solution, we first derive the basic energy estimates based on the observation from the previous section. First, taking the scalar product of (1.10) and $\boldsymbol{v}$ we deduce the kinetic energy identity

$$\frac{1}{2}\partial_t|\boldsymbol{v}|^2 + \frac{1}{2}\operatorname{div}(|\boldsymbol{v}|^2\boldsymbol{v}) - \operatorname{div}(\mathbb{T}\boldsymbol{v}) + \mathbb{T}\cdot\mathbb{D}\boldsymbol{v} = \boldsymbol{f}\cdot\boldsymbol{v}$$

and replacing the term $\mathbb{T}\cdot\mathbb{D}\boldsymbol{v}$ from the equation (1.16), and using then also (1.22) and (1.23), we finally obtain

$$\begin{aligned}&\partial_t(\psi + \tfrac{1}{2}|\boldsymbol{v}|^2) + \operatorname{div}((\psi + \tfrac{1}{2}|\boldsymbol{v}|^2)\boldsymbol{v}) - \operatorname{div}(\mathbb{T}\boldsymbol{v} + \lambda\nabla\psi(\mathbb{B})) + 2\nu|\mathbb{D}\boldsymbol{v}|^2\\ &\quad + \mu\lambda(\gamma|\nabla\mathbb{B}|^2 + (1-\gamma)|\mathbb{B}^{-\frac{1}{2}}\nabla\mathbb{B}\mathbb{B}^{-\frac{1}{2}}|^2)\\ &\quad + \mu\Big((1-\gamma)\delta_1|\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}|^2 + \gamma\delta_2|\mathbb{B}^{\frac{3}{2}} - \mathbb{B}^{\frac{1}{2}}|^2 + ((1-\gamma)\delta_2 + \gamma\delta_1)|\mathbb{B}-\mathbb{I}|^2\Big)\\ &\qquad = \boldsymbol{f}\cdot\boldsymbol{v}.\end{aligned}$$

Integrating the above identity over Ω, using the integration by parts and the boundary conditions (1.12)–(1.14), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}|\boldsymbol{v}|^2 + \psi(\mathbb{B}) + \sigma\int_{\partial\Omega}|\boldsymbol{v}|^2 + 2\nu\int_\Omega|\mathbb{D}\boldsymbol{v}|^2\\ &\quad + \mu\lambda\int_\Omega \gamma|\nabla\mathbb{B}|^2 + (1-\gamma)|\mathbb{B}^{-\frac{1}{2}}\nabla\mathbb{B}\mathbb{B}^{-\frac{1}{2}}|^2\\ &\quad + \mu\int_\Omega(1-\gamma)\delta_1|\mathbb{B}^{\frac{1}{2}} - \mathbb{B}^{-\frac{1}{2}}|^2 + \gamma\delta_2|\mathbb{B}^{\frac{3}{2}} - \mathbb{B}^{\frac{1}{2}}|^2\\ &\quad + \mu\int_\Omega((1-\gamma)\delta_2 + \gamma\delta_1)|\mathbb{B}-\mathbb{I}|^2 = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}.\end{aligned}\tag{1.24}$$

The identity (1.24) evokes the proper choice of the function spaces for the solution $(\boldsymbol{v},\mathbb{B})$ and the form of the (weak) formulation of the solution to (1.1)–(1.6).

Definition 1. Let $T>0$ and $\Omega\subset\mathbb{R}^3$ be a Lipschitz domain. Suppose that $\gamma\in(0,1)$, $\nu,\sigma,\lambda>0$, $\delta_1,\delta_2\geq 0$, $a\in\mathbb{R}$, and $\boldsymbol{f}\in L^2(0,T;W^{-1,2}_{\boldsymbol{n},\mathrm{div}})$, $\boldsymbol{v}_0\in L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)$. Furthermore, let $\mathbb{B}_0\in L^2(\Omega)$ be such that

$$-\int_\Omega \ln\det\mathbb{B}_0 < \infty.$$

We say that a couple $(\boldsymbol{v},\mathbb{B}) : Q\to\mathbb{R}^3\times\mathbb{R}^{3\times 3}_{>0}$ is a weak solution to (1.1)–(1.6) if $\mathbb{B}=\mathbb{B}^T$ and the following holds:

$$\begin{aligned}&\boldsymbol{v}\in L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}})\cap L^\infty(0,T;L^2(\Omega)),\quad \partial_t\boldsymbol{v}\in L^{\frac{4}{3}}(0,T;W^{-1,2}_{\boldsymbol{n},\mathrm{div}}),\\ &\mathbb{B}\in L^2(0,T;W^{1,2}(\Omega))\cap L^\infty(0,T;L^2(\Omega)),\quad \partial_t\mathbb{B}\in L^{\frac{4}{3}}(0,T;W^{-1,2}(\Omega));\end{aligned}$$

For all $\boldsymbol{\varphi}\in L^4(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}})$ we have

$$\begin{aligned}&\int_0^T\langle\partial_t\boldsymbol{v},\boldsymbol{\varphi}\rangle + \int_Q(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{\varphi} + \sigma\int_0^T\int_{\partial\Omega}\mathcal{T}\boldsymbol{v}\cdot\mathcal{T}\boldsymbol{\varphi}\\ &\quad = -\int_Q(2\nu\mathbb{D}\boldsymbol{v} + 2a\mu((1-\gamma)(\mathbb{B}-\mathbb{I}) + \gamma(\mathbb{B}^2-\mathbb{B})))\cdot\nabla\boldsymbol{\varphi} + \int_0^T\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle;\end{aligned}\tag{1.25}$$

For all $\mathbb{A} \in L^4(0,T;W^{1,2}(\Omega))$, $\mathbb{A} = \mathbb{A}^T$, we have

$$\begin{aligned}\int_0^T \langle \partial_t \mathbb{B}, \mathbb{A}\rangle + \int_Q ((\boldsymbol{v}\cdot\nabla)\mathbb{B} + 2\mathbb{B}\mathbb{W}\boldsymbol{v} - 2a\,\mathbb{B}\mathbb{D}\boldsymbol{v})\cdot\mathbb{A} \\ + \int_Q (\delta_1(\mathbb{B}-\mathbb{I}) + \delta_2(\mathbb{B}^2 - \mathbb{B}))\cdot\mathbb{A} + \lambda\nabla\mathbb{B}\cdot\nabla\mathbb{A} = 0;\end{aligned} \tag{1.26}$$

The initial conditions are satisfied in the following sense

$$\lim_{t\to 0_+} (\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{B}(t) - \mathbb{B}(0)\|_2) = 0. \tag{1.27}$$

Moreover, we say that the solution satisfies the energy inequality if for every $t \in (0,T)$:

$$\begin{aligned}&\int_\Omega \left(\frac{|\boldsymbol{v}(t)|^2}{2} + \psi(\mathbb{B}(t))\right) + \int_0^t 2\nu\|\mathbb{D}\boldsymbol{v}\|_2^2 + \sigma\|\mathcal{T}\boldsymbol{v}\|_{2,\partial\Omega}^2 \\ &\quad + \mu\lambda\int_0^t (1-\gamma)\left\|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}\right\|_2^2 + \gamma\|\nabla\mathbb{B}\|_2^2 \\ &\quad + \mu\int_0^t (1-\gamma)\delta_1\left\|\mathbb{B}^{\frac12} - \mathbb{B}^{-\frac12}\right\|_2^2 + \gamma\delta_2\left\|\mathbb{B}^{\frac32} - \mathbb{B}^{\frac12}\right\|_2^2 \\ &\quad + \mu\int_0^t (\gamma\delta_1 + (1-\gamma)\delta_2)\|\mathbb{B}-\mathbb{I}\|_2^2 \\ &\leq \int_\Omega \left(\frac{|\boldsymbol{v}_0|^2}{2} + \psi(\mathbb{B}_0)\right) + \int_0^t \langle \boldsymbol{f}, \boldsymbol{v}\rangle.\end{aligned} \tag{1.28}$$

In the above definition we used the notation that is also used through the whole paper. By $L^p(\Omega)$ and $W^{n,p}(\Omega)$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, we denote the usual Lebesgue and Sobolev space, with their usual norms denoted as $\|\cdot\|_p$ and $\|\cdot\|_{n,p}$, respectively. The trace operator that maps $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$, for certain q, $q \geq 1$, will be denoted by $\mathcal{T}$. Further, we set $W^{-1,p'}(\Omega) = (W^{1,p}(\Omega))^*$, where $p' = p/(p-1)$. We shall use the same notation for the function spaces of scalar-, vector-, or tensor-valued functions, but we will distinguish the functions themselves using different fonts such as a for scalers, $\boldsymbol{a}$ for vectors and $\mathbb{A}$ for tensors. Also, we do not specify the meaning of the duality pairing $\langle\cdot,\cdot\rangle$, assuming it is clear from the context. Moreover, for certain subspaces of vector valued functions, we shall use the following notation:

$$\begin{aligned}C^\infty_{\boldsymbol{n}} &= \{\boldsymbol{w} : \Omega \to \mathbb{R}^3 : \boldsymbol{w} \text{ infinitely differentiable}, \ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ C^\infty_{\boldsymbol{n},\operatorname{div}} &= \{\boldsymbol{w} \in C^\infty_{\boldsymbol{n}} : \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega\}, \\ L^2_{\boldsymbol{n},\operatorname{div}} &= \overline{C^\infty_{\boldsymbol{n},\operatorname{div}}}^{\|\cdot\|_2}, \quad W^{1,2}_{\boldsymbol{n},\operatorname{div}} = \overline{C^\infty_{\boldsymbol{n},\operatorname{div}}}^{\|\cdot\|_{1,2}}, \quad W^{3,2}_{\boldsymbol{n},\operatorname{div}} = \overline{C^\infty_{\boldsymbol{n},\operatorname{div}}}^{\|\cdot\|_{3,2}}, \\ W^{-1,2}_{\boldsymbol{n},\operatorname{div}} &= (W^{1,2}_{\boldsymbol{n},\operatorname{div}})^*, \quad W^{-3,2}_{\boldsymbol{n},\operatorname{div}} = (W^{3,2}_{\boldsymbol{n},\operatorname{div}})^*.\end{aligned}$$

Occasionally, we shall denote the standard scalar products in $L^2(\Omega)$ and $L^2(\partial\Omega)$ as $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\partial\Omega}$, respectively. The Bochner spaces of mappings from $(0,T)$ to a Banach space X will be denoted as $L^p(0,T;X)$ with the norm $\|\cdot\|_{L^p(0,T;X)} = (\int_0^T \|\cdot\|_X^p)^{\frac1p}$. If $X = L^q(\Omega)$, or $X = W^{k,q}(\Omega)$, we will write just $\|\cdot\|_{L^pL^q}$, or $\|\cdot\|_{L^pW^{k,q}}$, respectively. The symbol $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ denotes the set of symmetric 3×3 real matrices. Furthermore, by $\mathbb{R}^{3\times3}_{>0}$ we denote the subset of $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ which consists of positive definite matrices, i.e., those which satisfy

$$\mathbb{A}\boldsymbol{z}\cdot\boldsymbol{z}>0 \quad \text{for all } \boldsymbol{z}\in\mathbb{R}^3\setminus\{0\}.$$

1.3. **The main result.** The key result of the paper is the following

Theorem 2. *Let all assumptions of Definition 1 be satisfied. Then there exists a weak solution satisfying the energy inequality.*

Let us briefly explain the main difficulties connected with the analysis of the system (1.9)–(1.13) and our ideas how to solve them. In the standard models where $\gamma=0$, to get an a priori estimate for $\mathbb{B}$, the right test function to take in (1.11) is $\mathbb{I}-\mathbb{B}^{-1}$. Then, using (1.9) and (1.10) tested by $\boldsymbol{v}$, one can eliminate the problematic terms like $\mathbb{B}\cdot\mathbb{D}\boldsymbol{v}$ coming from the objective derivative. However, the non-negative quantity to be controlled which comes from the diffusion term turns out to be just $|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}|^2$ and this provides little to no information. In particular, the terms $\nabla\boldsymbol{v}\mathbb{B}$ appearing in (1.11) are going to be just integrable and it is unclear if one can show a strong convergence of $\mathbb{B}$. Instead, one would like to test also by $\mathbb{B}$ to get control over $|\nabla\mathbb{B}|^2$. But this is not possible, since the term $\nabla\boldsymbol{v}\mathbb{B}\cdot\mathbb{B}$ which is created cannot be estimated without some serious simplifications (such as boundedness of $\nabla\boldsymbol{v}$, two or one dimensional setting or small data). Quite remarkably, this problem is solved just by adding $\gamma(\mathbb{B}-\mathbb{I})^2$ into the constitutive form for ψ. More precisely, considering $\gamma\in(0,1)$ we observe that the right test function in (1.11) is in fact $(1-\gamma)(\mathbb{I}-\mathbb{B}^{-1})+\gamma(\mathbb{B}-\mathbb{I})$. Indeed, the terms from the objective derivative cancel again due to the presence of $\gamma(\mathbb{B}^2-\mathbb{B})$ in $\mathbb{T}$. But now, we also get $\gamma|\nabla\mathbb{B}|^2$ under control, which is much better information than in the case $\gamma=0$ and it will imply compactness of all the terms appearing in (1.10) and (1.11). We have seen above that such a modification of ψ, and consequently of $\mathbb{T}$, is not ad-hoc and that it lies on solid physical grounds.

The second and also the last major difficulty which we will encounter is how one can justify testing of (1.11) by $\mathbb{B}^{-1}$ on the approximate (discrete level), where $\mathbb{B}^{-1}$ might not even exist. This we overcome by designing a delicate approximation scheme, which takes into account the smallest eigenvalue of $\mathbb{B}$, and also by noting that testing (1.11) only by $\mathbb{B}$ yields sufficient a priori estimates for the initial limit passage (in the Galerkin approximation of $\mathbb{B}$).

Up to now, there have been no results on global existence of weak solutions to Oldroyd-B models in three dimensions, including either the standard, or diffusive variants. The closest result so far is probably [23, Theorem 4.1], however there it is assumed that $\delta_2>0$ and $\lambda=0$ (Giesekus model), whereas we treat also the case $\delta_2=0$, but with $\lambda>0$ (diffusive Oldroyd-B or Giesekus model). Moreover, in [23], only the weak sequential stability of some hypothetical approximations is proved. We, on the other hand, provide the complete existence proof, including the construction of approximate solutions (which, in viscoelasticity, is generally a non-trivial task). In the article [16], Lions and Masmoudi prove the global existence in three dimensions, but only for $a=0$ (corrotational case), which is known to be much easier. The local in time existence of regular solutions for the non-diffusive variants of the models above ($\lambda=0$) is done in the pioneering work [13, Theorem 2.4.]. There, also the global existence for small data is shown. In two dimensions, the problem is solved in [10] in the case $\lambda>0$, $\delta_1>0$, $\delta_2=0$ (diffusive Oldroyd-B model). There are also global large data existence results in three dimensions for a slightly different class of diffusive rate-type viscoelastic models, but under the simplifying assumption that $\mathbb{B} = b\mathbb{I}$ - see [7] and [3]. This assumption, however, turns (1.11) into a much simpler scalar equation. Moreover, note that if $\mathbb{B} = b\mathbb{I}$, then the equations (1.10) and (1.11) decouple (which is not the case in [7] and [3] since there the considered constitutive relation for $\mathbb{T}$ is more complicated than here). Finally, in [1] (see also [15]), the global existence of a weak solution is shown for certain regularized Oldroyd-B model (including a cut-off or nonlinear $p-$Laplace operator in the diffusive term in $\mathbb{B}$). Thus, one might argue that since the case $\gamma > 0$ could be also seen as a regularization of the original model, we are just proving an existence of a solution to another regularization. However, this argument is not, in our opinion, correct for several reasons. First of all, the "regularization" $\gamma > 0$ does not touch the equation (1.11) at all. Second, it is not obvious why the nonlinear term $\gamma(\mathbb{B} - \mathbb{I})^2$ should have any regularization effect. And, perhaps most importantly, we already showed in Section 1.1 that the model with $\gamma > 0$ is physically well sounded and worth of studying on its own.

Since the topic is quite new and unexplored, we decided, for brevity and clarity of presentation to consider only the isothermal case. However, we believe that the framework and ideas presented here are robust enough to provide an existence analysis also for the full thermodynamical model if the evolution of the internal energy is described correctly. This is the subject of our forthcoming study.

Remark 3. Finally, we end this section by several concluding remarks on possible extensions, but we do not provide their proofs in this paper.

(i) The theorem holds also in arbitrary dimension $d > 3$ (in $d \le 2$, it is known), however with worse function spaces for the time derivatives and better for the test functions. Indeed, the only dimension-specific argument in the proof below is in the derivation of interpolation inequalities, which are then used to estimate $\partial_t \boldsymbol{v}$ and $\partial_t \mathbb{B}$. Moreover, all the non-linear terms in (1.25), (1.26) are integrable for arbitrary d if the test functions are smooth. In addition, if $d = 2$, then we can prove the existence of a weak solution satisfying even the energy equality, i.e., (1.28) holds with the equality sign.

(ii) When Ω has $C^{1,1}$ boundary, then, in addition, there exists a pressure $p \in L^{\frac{5}{3}}(Q)$, which appears in (1.2). Then, the test functions in (1.25) need not be divergence free if we include the term $\int_\Omega p \operatorname{div} \boldsymbol{\varphi}$ in (1.25). This follows in a standard way, using the Helmholtz decomposition of $\boldsymbol{v}$ (see, e.g., [2] for details).

(iii) It is possible to replace (1.12), (1.13) by the no-slip boundary condition $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$. Then, we only need to change all the spaces $W^{1,2}_{\boldsymbol{n}}$ to $W^{1,2}_0$ and so on. However, then it seems that the pressure p can be only obtained as a distribution (see [2]).

2. Proof of Theorem 2

The general scheme of the proof of Theorem 2 is the following: In order to invert the matrix $\mathbb{B}$ and to avoid problems with low integrability in the convected derivative, we introduce the special cut-off function

$$\rho_\varepsilon(\mathbb{A}) := \frac{\max\{0, \Lambda(\mathbb{A}) - \varepsilon\}}{\Lambda(\mathbb{A})(1 + \varepsilon|\mathbb{A}|^3)} \quad \text{for } \mathbb{A} \in \mathbb{R}^{3\times3}_{\mathrm{sym}},$$

where $\Lambda(\mathbb{A})$ denotes a minimal eigenvalue of $\mathbb{A}$ (whose spectrum is real due to the symmetry of $\mathbb{A}$) and thus is a continuous function of $\mathbb{A}$. Note that for any positively definite matrix $\mathbb{A}$ there holds $\rho_\varepsilon(\mathbb{A}) \to 1$ as $\varepsilon \to 0_+$. In order to shorten all formulae, we also denote

$$
\begin{aligned}
S(\mathbb{A}) &= (1-\gamma)(\mathbb{A}-\mathbb{I}) + \gamma(\mathbb{A}^2 - \mathbb{A}) && \text{for } \mathbb{A} \in \mathbb{R}^{3\times 3},\\
R(\mathbb{A}) &= \delta_1(\mathbb{A}-\mathbb{I}) + \delta_2(\mathbb{A}^2 - \mathbb{A}) && \text{for } \mathbb{A} \in \mathbb{R}^{3\times 3}.
\end{aligned}
$$

We construct a solution by an approximative scheme with approximative parameters k, l and ε, where $k, l \in \mathbb{N}$ correspond to the Galerkin approximation for $\boldsymbol{v}$ and $\mathbb{B}$, respectively, and ε corresponds to the presence of the cut-off function $\rho_\varepsilon(\mathbb{B})$ in certain terms. Then we pass to the limit in the approximative parameters in the following order. First, we let $l \to \infty$, which corresponds to the limit in the equation for $\mathbb{B}$. This allows us to prove certain minimum principle for $\mathbb{B}$ and also to obtain information about $\mathbb{B}^{-1}$. Next, we let $\varepsilon \to 0_+$ in order to remove the truncation function and finally we let $k \to \infty$, which corresponds to the limiting procedure in the equation for the velocity $\boldsymbol{v}$.

Also to simplify the presentation, we assume here that $\lambda = \mu = \nu = \sigma \equiv 1$ and refer to Section 1.1 for detail computation for general parameters.

2.1. **Galerkin approximation.** Following e.g., [17, Appendix A.4], we know that there exists a basis $\{\boldsymbol{w}_i\}_{i=1}^\infty$ of $W^{3,2}_{\boldsymbol{n},\mathrm{div}}$, which is orthonormal in $L^2(\Omega)$ and orthogonal in $W^{3,2}_{\boldsymbol{n},\mathrm{div}}$. Moreover, the projection $P_k : L^2(\Omega) \to \mathrm{span}\{\boldsymbol{w}_i\}_{i=1}^k$, defined as[1]

$$
P_k \boldsymbol{\varphi} = \sum_{i=1}^k (\boldsymbol{\varphi}, \boldsymbol{w}_i)\boldsymbol{w}_i, \quad \boldsymbol{\varphi} \in L^2(\Omega),
$$

is continuous in $L^2(\Omega)$ and also in $W^{3,2}_{\boldsymbol{n},\mathrm{div}}$ independently of k, i.e.,

$$
\|P_k\boldsymbol{\varphi}\|_2 \le C\|\boldsymbol{\varphi}\|_2 \qquad \|P_k\boldsymbol{\varphi}\|_{W^{3,2}_{\boldsymbol{n},\mathrm{div}}} \le C\|\boldsymbol{\varphi}\|_{W^{3,2}_{\boldsymbol{n},\mathrm{div}}}
$$

for all $\boldsymbol{\varphi} \in W^{3,2}_{\boldsymbol{n},\mathrm{div}}$, where the constant C is independent of k. Furthermore, thanks to the standard embedding, we also have that $W^{3,2}_{\boldsymbol{n},\mathrm{div}} \hookrightarrow W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Similarly, we construct the basis $\{\mathbb{W}_j\}_{j=1}^\infty$ of $W^{1,2}(\Omega)$, which is again L^2-orthonormal, $W^{1,2}$-orthogonal and the projection

$$
Q_l\mathbb{A} = \sum_{j=1}^l (\mathbb{A}, \mathbb{W}_j)\mathbb{W}_j, \quad \mathbb{A} \in L^2(\Omega),
$$

is continuous in $L^2(\Omega)$ and in $W^{1,2}(\Omega)$ independently of l.

Then for fixed $k, l \in \mathbb{N}$ and $\varepsilon > 0$, we look for the functions $\boldsymbol{v}^{k,l}_\varepsilon, \mathbb{B}^{k,l}_\varepsilon$ being of the form

$$
\boldsymbol{v}^{k,l}_\varepsilon(t,x) = \sum_{i=1}^k c_i^{k,l,\varepsilon}(t)\boldsymbol{w}_i(x) \quad \text{and} \quad \mathbb{B}^{k,l}_\varepsilon(t,x) = \sum_{j=1}^l d_j^{k,l,\varepsilon}(t)\mathbb{W}_j(x),
$$

where $c_i^{k,l,\varepsilon}, d_j^{k,l,\varepsilon}$, $i = 1, \dots, k$, $j = 1, \dots, l$, are unknown functions of time, and we require that $\boldsymbol{v}^{k,l}_\varepsilon, \mathbb{B}^{k,l}_\varepsilon$ (and consequently the functions $c_i^{k,l,\varepsilon}(t)$ and $d_j^{k,l,\varepsilon}(t)$) satisfy

[1]We recall here the definition $(a,b) := \int_\Omega ab$.

the following system of $(k+l)$ ordinary differential equations in time interval $(0,T)$:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_\varepsilon^{k,l},\boldsymbol{w}_i) + ((\boldsymbol{v}_\varepsilon^{k,l}\cdot\nabla)\boldsymbol{v}_\varepsilon^{k,l},\boldsymbol{w}_i) + 2(\mathbb{D}\boldsymbol{v}_\varepsilon^{k,l},\nabla\boldsymbol{w}_i) + (\mathcal{T}\boldsymbol{v}_\varepsilon^{k,l},\mathcal{T}\boldsymbol{w}_i)_{\partial\Omega} \\
&= -2a(\rho_\varepsilon(\mathbb{B}_\varepsilon^{k,l})S(\mathbb{B}_\varepsilon^{k,l}),\nabla\boldsymbol{w}_i) + \langle\boldsymbol{f},\boldsymbol{w}_i\rangle \text{ for } i=1,\ldots,k,
\end{aligned}\tag{2.1}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}_\varepsilon^{k,l},\mathbb{W}_j) + ((\boldsymbol{v}_\varepsilon^{k,l}\cdot\nabla)\mathbb{B}_\varepsilon^{k,l},\mathbb{W}_j) + (\rho_\varepsilon(\mathbb{B}_\varepsilon^{k,l})R(\mathbb{B}_\varepsilon^{k,l}),\mathbb{W}_j) \\
&= -(\nabla\mathbb{B}_\varepsilon^{k,l},\nabla\mathbb{W}_j) + 2(\rho_\varepsilon(\mathbb{B}_\varepsilon^{k,l})\mathbb{B}_\varepsilon^{k,l}(a\mathbb{D}\boldsymbol{v}_\varepsilon^{k,l} - \mathbb{W}\boldsymbol{v}_\varepsilon^{k,l}),\mathbb{W}_j) \text{ for } i=1,\ldots,l.
\end{aligned}\tag{2.2}
$$

Due to the L^2-orthonormality of the bases $\{\boldsymbol{w}_i\}_{i=1}^\infty$ and $\{\mathbb{W}_j\}_{j=1}^\infty$, the system (2.1)–(2.2) can be rewritten as a nonlinear system of ordinary differential equations for $c_i^{k,l,\varepsilon}$ and $d_j^{k,l,\varepsilon}$, where $i=1,\ldots,k$ and $j=1,\ldots,l$, and we equip this system with the initial conditions as follows

$$
c_i^{k,l,\varepsilon}(0) = (\boldsymbol{v}_0,\boldsymbol{w}_i) \qquad \text{and} \qquad d_j^{k,l,\varepsilon}(0) = (\mathbb{B}_0^\varepsilon,\mathbb{W}_j). \tag{2.3}
$$

Here, $\mathbb{B}_0^\varepsilon$ is defined as

$$
\mathbb{B}_0^\varepsilon(x) = \begin{cases} \mathbb{B}_0(x) & \text{if } \Lambda(\mathbb{B}_0(x)) > \varepsilon, \\ \mathbb{I} & \text{elsewhere.} \end{cases}
$$

Since $\mathbb{B}_0(x)\in\mathbb{R}^{3\times3}_{>0}$ for almost all $x\in\Omega$, we have that $\Lambda(\mathbb{B}_0(x))>0$ for almost all $x\in\Omega$. Consequently, using the fact $\mathbb{B}_0\in L^2(\Omega)$, we obtain

$$
\|\mathbb{B}_0^\varepsilon - \mathbb{B}_0\|_2^2 = \int_{\Lambda(\mathbb{B}_0)\le\varepsilon} |\mathbb{I}-\mathbb{B}_0|^2 \to 0
$$

as $\varepsilon\to0_+$. Note also that the initial conditions (2.3) can equivalently be written as $\boldsymbol{v}_l(0)=P_k\boldsymbol{v}_0$ and $\mathbb{B}_l(0)=Q_l\mathbb{B}_0^\varepsilon$.

For the system (2.1)–(2.3), the Carathéodory's theorem can be applied and therefore there exists $T^*>0$ and absolutely continuous functions $c_i^{k,l,\varepsilon}$, $d_j^{k,l,\varepsilon}$ satisfying (2.1)–(2.2) almost everywhere in $(0,T^*)$ with the initial conditions (2.3). If T^* is the maximal time, for which the solution exists and $T^*<T$, then at least one of the functions $c_i^{k,l,\varepsilon}$, $d_j^{k,l,\varepsilon}$ must blow up as $t\to T^*_-$. But using the estimate from the next section (see (2.8) valid for all $t\in(0,T^*)$), this will never happen. Thus, we can set $T^*=T$.

2.2. **Limit** $l\to\infty$**.** Since the first limit passage in the paper is $l\to\infty$, we just use the simplified notation and denote the approximative solution, constructed in the previous section, by $(\boldsymbol{v}_l,\mathbb{B}_l) := (\boldsymbol{v}_\varepsilon^{k,l},\mathbb{B}_\varepsilon^{k,l})$. Furthermore, we also simplify the cut-off term and write

$$
r_\varepsilon^l := \rho_\varepsilon(\mathbb{B}_\varepsilon^{k,l}).
$$

We start this part by proving estimates independent of l. Since $\mathbb{B}_l(t)$ and $\boldsymbol{v}_l(t)$ belong for almost all t to the linear hull of $\{\mathbb{W}_j\}_{j=1}^l$ and $\{\boldsymbol{w}_i\}_{i=1}^k$, respectively, we can use $\boldsymbol{v}_l$ instead of $\boldsymbol{w}_i$ in (2.1) and $\mathbb{B}_l$ instead of $\mathbb{W}_j$ in (2.2) to deduce,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{B}_l\|_2^2 + \|\nabla\mathbb{B}_l\|_2^2 = 2a(r_\varepsilon^l\mathbb{B}_l\mathbb{D}\boldsymbol{v}_l,\mathbb{B}_l) - (r_\varepsilon^l R(\mathbb{B}_l),\mathbb{B}_l), \tag{2.4}
$$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_l\|_2^2 + 2\|\mathbb{D}\boldsymbol{v}_l\|_2^2 + \|\mathcal{T}\boldsymbol{v}_l\|_{2,\partial\Omega}^2 = -2a(r_\varepsilon S(\mathbb{B}_l),\mathbb{D}\boldsymbol{v}_l) + \langle\boldsymbol{f},\boldsymbol{v}_l\rangle, \tag{2.5}
$$

where we used the integration by parts formula and the facts that $\operatorname{div} \boldsymbol{v}_l = 0$ and $\mathcal{T}\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Next, it follows from the definition of r^l_ε and the functions R and S that

$$r^l_\varepsilon(|S(\mathbb{B}_l)| + |R(\mathbb{B}_l)||\mathbb{B}_l| + |\mathbb{B}_l|^2) \leq C\frac{1+|\mathbb{B}_l|^3}{1+\varepsilon|\mathbb{B}_l|^3} \leq C(\varepsilon). \tag{2.6}$$

Hence, summing (2.4) and (2.5) and using the estimate (2.6) to bound the term on the right hand side, we obtain with the help of Hölder's, Young's and Korn's inequalities that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}_l\|_2^2 + \|\mathbb{B}_l\|_2^2\right) + \|\mathbb{D}\boldsymbol{v}_l\|_2^2 + \|\mathcal{T}\boldsymbol{v}_l\|_{2,\partial\Omega}^2 + \|\nabla\mathbb{B}_l\|_2^2 \\ \leq C(\varepsilon) + C\|\boldsymbol{f}\|^2_{W^{-1,2}_{\boldsymbol{n},\mathrm{div}}}.\end{aligned}$$

Integration with resect to time then leads to the following bound

$$\begin{aligned}&\sup_{t\in(0,T)}\left(\|\boldsymbol{v}_l\|_2^2 + \|\mathbb{B}_l\|_2^2\right) + \int_0^T (\|\mathbb{D}\boldsymbol{v}_l\|_2^2 + \|\mathcal{T}\boldsymbol{v}_l\|_{2,\partial\Omega}^2 + \|\nabla\mathbb{B}_l\|_2^2) \\ &\qquad \leq C(\varepsilon) + \|P_k\boldsymbol{v}_0\|_2^2 + \|Q_l\mathbb{B}_0^\varepsilon\|_2^2 + C\int_0^T \|\boldsymbol{f}\|^2_{W^{-1,2}_{\boldsymbol{n},\mathrm{div}}} \leq C(\varepsilon),\end{aligned} \tag{2.7}$$

where the last inequality follows from the continuity of the projections P_k and Q_l and from the assumption on initial data, namely that

$$\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{B}_0\|_2^2 + \|\ln\det\mathbb{B}_0\|_1 + C\int_0^T \|\boldsymbol{f}\|^2_{W^{-1,2}_{\boldsymbol{n},\mathrm{div}}} < \infty.$$

In (2.6) and (2.7) the notation $C(\varepsilon)$ emphasizes that the constant C depends on ε and we keep this notation also in what follows.

Next, we focus also on the estimate for time derivatives. First, it follows from L^2-orthonormality of the bases and the estimate (2.7) that

$$\sum_{i=1}^k c_i(t)^2 + \sum_{j=1}^l d_j(t)^2 \leq C(\varepsilon). \tag{2.8}$$

Then, since $\boldsymbol{v}_l$ is a linear combination of $\{\boldsymbol{w}_i\}_{i=1}^k \subset W^{1,\infty}(\Omega)$, we can estimate

$$\|\boldsymbol{v}_l\|_{L^\infty W^{1,\infty}} \leq \operatorname*{ess\,sup}_{t\in(0,T)} \sum_{i=1}^k |c_i(t)|\|\boldsymbol{w}_i\|_{1,\infty} \leq C(\varepsilon, k). \tag{2.9}$$

Moreover, we can read from (2.1) that

$$\|\partial_t\boldsymbol{v}\|_{L^\infty W^{1,\infty}} \leq C(\varepsilon, k). \tag{2.10}$$

Finally, it follows from (2.2) and (2.7) that

$$\|\partial_t\mathbb{B}_k\|_{L^2 W^{-1,2}} \leq C(\varepsilon, k). \tag{2.11}$$

Using (2.7), (2.9) (2.11), (2.10) and Banach-Alaoglu's theorem, we can find subsequences (which we do not relabel) and corresponding weak limits, such that

$$\boldsymbol{v}_l \rightharpoonup \boldsymbol{v}_\varepsilon \qquad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{2.12}$$

$$\boldsymbol{v}_l \overset{*}{\rightharpoonup} \boldsymbol{v}_\varepsilon \qquad \text{weakly}^* \text{ in } L^\infty(0,T;W^{1,\infty}(\Omega)),$$

$$\partial_t\boldsymbol{v}_l \overset{*}{\rightharpoonup} \partial_t\boldsymbol{v}_\varepsilon \qquad \text{weakly}^* \text{ in } L^\infty(0,T;W^{1,\infty}(\Omega)), \tag{2.13}$$

$$\mathcal{T}\boldsymbol{v}_l \rightharpoonup \mathcal{T}\boldsymbol{v}_\varepsilon \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)),$$

$$\mathbb{B}_l \rightharpoonup \mathbb{B}_\varepsilon \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{2.14}$$

$$\partial_t\mathbb{B}_l \rightharpoonup \partial_t\mathbb{B}_\varepsilon \qquad \text{weakly in } L^2(0,T;W^{-1,2}(\Omega)), \tag{2.15}$$

as $l \to \infty$. Moreover, it follows from (2.12), (2.13), (2.14), (2.15) and from Aubin-Lions' lemma that for some further subsequences, we have

$$\boldsymbol{v}_l \to \boldsymbol{v}_\varepsilon \qquad \text{strongly in } L^2(Q),$$

$$\mathbb{B}_l \to \mathbb{B}_\varepsilon \qquad \text{strongly in } L^2(Q),$$

$$r^l_\varepsilon \to r_\varepsilon := \rho_\varepsilon(\mathbb{B}_\varepsilon) \qquad \text{a.e. in } Q. \tag{2.16}$$

Using the convergence results (2.12)–(2.16), it is rather standard to let $l \to \infty$ in (2.1)–(2.2) and deduce that

$$\begin{aligned}(\partial_t\boldsymbol{v}_\varepsilon, \boldsymbol{w}_i) + (\boldsymbol{v}_\varepsilon\cdot\nabla)\boldsymbol{v}_\varepsilon, \boldsymbol{w}_i) + 2(\mathbb{D}\boldsymbol{v}_\varepsilon, \nabla\boldsymbol{w}_i)\\ = -(\mathcal{T}\boldsymbol{v}_\varepsilon, \mathcal{T}\boldsymbol{w}_i)_{\partial\Omega} - 2a(r_\varepsilon S(\mathbb{B}_\varepsilon), \phi\nabla\boldsymbol{w}_i) + \langle\boldsymbol{f}, \boldsymbol{w}_i\rangle\end{aligned} \tag{2.17}$$

for all $i = 1,\ldots,k$ and all $t \in (0,T)$ and

$$\begin{aligned}\langle\partial_t\mathbb{B}_\varepsilon, \mathbb{A}\rangle + ((\boldsymbol{v}_\varepsilon\cdot\nabla)\mathbb{B}_\varepsilon, \mathbb{A}) + (\nabla\mathbb{B}_\varepsilon, \nabla\mathbb{A})\\ = 2(r_\varepsilon\mathbb{B}_\varepsilon(a\mathbb{D}\boldsymbol{v}_\varepsilon - \mathbb{W}\boldsymbol{v}_\varepsilon), \mathbb{A}) - (r_\varepsilon R(\mathbb{B}_\varepsilon), \mathbb{A})\end{aligned} \tag{2.18}$$

for all $\mathbb{A} \in W^{1,2}(\Omega)$. Moreover, it follows from (2.14) and (2.15) that $\mathbb{B}_\varepsilon \in \mathcal{C}(0,T;L^2)$ and it is classical to show that $\mathbb{B}_\varepsilon(0) = \mathbb{B}^\varepsilon_0$ and $\boldsymbol{v}_\varepsilon(0) = P_k\boldsymbol{v}_0$.

2.3. **Limit** $\varepsilon \to 0$**.** In this part we consider the solution $(\boldsymbol{v}_\varepsilon, \mathbb{B}_\varepsilon)$ constructed in preceding section for $\varepsilon \in (0,1)$ and we let $\varepsilon \to 0_+$. To do so, we first derive estimates that are uniform with respect to ε. However, in this step we must test the equation for $\mathbb{B}_\varepsilon$ by $\mathbb{J}$, which contains also $\mathbb{B}_\varepsilon^{-1}$. Hence, we need to justify that $\mathbb{B}_\varepsilon^{-1}$ is a proper test function.

2.3.1. *Estimates for the inverse matrix - still* ε *dependent.* First, we prove that $\Lambda(\mathbb{B}_\varepsilon) \geq \varepsilon$. Hence, let $\boldsymbol{z} \in \mathbb{R}^3$ be arbitrary and set

$$\mathbb{A} = (\mathbb{B}_\varepsilon\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\,(\boldsymbol{z}\otimes\boldsymbol{z}) \tag{2.19}$$

in (2.18), where $(\boldsymbol{z}\otimes\boldsymbol{z})_{ij} := z_iz_j$, and the integrate the result with respect to time $t \in (0,\tau)$ with some fixed $\tau \in (0,T)$. Note that $\mathbb{A}$ clearly satisfies $\mathbb{A} \in L^2(0,T;W^{1,2}(\Omega))$ and therefore it can be used in (2.18). Next, we evaluate all terms in (2.18). For the time derivative, we have

$$\begin{aligned}\int_0^\tau \langle\partial_t\mathbb{B}_\varepsilon, \mathbb{A}\rangle &= \int_0^\tau \Big\langle\partial_t(\mathbb{B}_\varepsilon\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2), (\mathbb{B}_\varepsilon\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\Big\rangle\\ &= \frac{1}{2}\big\|(\mathbb{B}_\varepsilon(\tau)\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\big\|_2^2 - \frac{1}{2}\big\|(\mathbb{B}^\varepsilon_0\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\big\|_2^2\\ &= \frac{1}{2}\big\|(\mathbb{B}_\varepsilon(\tau)\boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\big\|_2^2,\end{aligned} \tag{2.20}$$

where for the last equality used the definition of $\mathbb{B}_0^\varepsilon$. Furthermore, using (2.19) again, we obtain

$$\begin{aligned}\int_Q \nabla\mathbb{B}_\varepsilon \cdot \nabla\mathbb{A} &= \int_0^\tau \int_\Omega \nabla(\mathbb{B}_\varepsilon - \varepsilon\mathbb{I}) \cdot \nabla((\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\,(\boldsymbol{z}\otimes\boldsymbol{z})) \\ &= \int_0^\tau \left\|\nabla(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\right\|_2^2 \geq 0\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}\int_Q (\boldsymbol{v}_\varepsilon\cdot\nabla)\mathbb{B}_\varepsilon \cdot \mathbb{A} &= \int_0^\tau \int_\Omega \boldsymbol{v}_\varepsilon \cdot \nabla(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_- \\ &= \frac{1}{2}\int_0^\tau \int_\Omega \boldsymbol{v}_\varepsilon \cdot \nabla((\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-^2 \\ &= -\frac{1}{2}\int_0^\tau \int_\Omega ((\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-^2 \operatorname{div}\boldsymbol{v}_\varepsilon = 0,\end{aligned} \tag{2.22}$$

where we used integration by parts and the fact that $\operatorname{div}\boldsymbol{v}_\varepsilon = 0$. Furthermore, since

$$\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} \geq \Lambda(\mathbb{B}_\varepsilon)|\boldsymbol{z}|^2 \qquad \text{a.e. in } Q,$$

we get

$$0 \geq \frac{(\Lambda(\mathbb{B}_\varepsilon)-\varepsilon)_+}{\Lambda(\mathbb{B}_\varepsilon)(1+|\mathbb{B}_\varepsilon|)}(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_- \geq \frac{(\Lambda(\mathbb{B}_\varepsilon)-\varepsilon)_+}{\Lambda(\mathbb{B}_\varepsilon)(1+|\mathbb{B}_\varepsilon|)}(\Lambda(\mathbb{B}_\varepsilon)-\varepsilon)_-\,|\boldsymbol{z}|^2 = 0.$$

Consequently, using the definition of r_ε in (2.16), the definition (2.19) of $\mathbb{A}$ and the above equality, we have that

$$r_\varepsilon\mathbb{A} = 0 \qquad \text{a.e. in } Q. \tag{2.23}$$

Consequently, with the choice (2.19) of $\mathbb{A}$ in (2.18), we see that the right hand side is identically zero. Therefore, using (2.20), (2.21), (2.22) and (2.23) yields

$$\begin{aligned}&\left\|(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\right\|_2^2(\tau) \\ &\quad\leq \left\|(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\right\|_2^2(\tau) + 2\int_0^\tau \left\|\nabla(\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} - \varepsilon|\boldsymbol{z}|^2)_-\right\|_2^2 = 0,\end{aligned}$$

which implies

$$\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z} \geq \varepsilon|\boldsymbol{z}|^2 \quad \text{a.e. in } Q. \tag{2.24}$$

Since $\boldsymbol{z}\in\mathbb{R}^3$ can be arbitrary, we have the following estimate form the minimal eigenvalue

$$\Lambda(\mathbb{B}_\varepsilon) \geq \inf_{0\neq\boldsymbol{z}\in\mathbb{R}^3} \frac{\mathbb{B}_\varepsilon \boldsymbol{z}\cdot\boldsymbol{z}}{|\boldsymbol{z}|^2} \geq \varepsilon.$$

Consequently, the inverse matrix $\mathbb{B}_\varepsilon^{-1}$ is well defined and satisfies

$$|\mathbb{B}_\varepsilon^{-1}| \leq \frac{C}{\varepsilon}. \tag{2.25}$$

Furthermore, for $\nabla\mathbb{B}_\varepsilon^{-1}$ we can compute

$$\nabla\mathbb{B}_\varepsilon^{-1} = \mathbb{B}_\varepsilon^{-1}\mathbb{B}_\varepsilon\nabla\mathbb{B}_\varepsilon^{-1} = \mathbb{B}_\varepsilon^{-1}\nabla(\mathbb{B}_\varepsilon\mathbb{B}_\varepsilon^{-1}) - \mathbb{B}_\varepsilon^{-1}(\nabla\mathbb{B}_\varepsilon)\mathbb{B}_\varepsilon^{-1} = -\mathbb{B}_\varepsilon^{-1}(\nabla\mathbb{B}_\varepsilon)\mathbb{B}_\varepsilon^{-1}$$

Hence, combining (2.7) and (2.25), we obtain

$$\int_Q |\nabla\mathbb{B}_\varepsilon^{-1}|^2 \leq \int_Q |\mathbb{B}_\varepsilon^{-1}|^4|\nabla\mathbb{B}_\varepsilon|^2 \leq C(\varepsilon) \tag{2.26}$$

and we see that $\mathbb{B}_\varepsilon$ as well as $\mathbb{B}_\varepsilon^{-1}$ can be used as test functions in (2.18).

2.3.2. *Estimates independent of* (ε, k). Next, we focus on the final estimate. Not only it is uniform with respect to ε but it also does not depend on k. To do so, we set

$$\mathbb{A} := \mathbb{J}_\varepsilon = ((1-\gamma)(\mathbb{I}-\mathbb{B}_\varepsilon^{-1}) + \gamma(\mathbb{B}_\varepsilon - \mathbb{I})) \tag{2.27}$$

in (2.18). Thanks to (2.14) and (2.26), we know that $\mathbb{A} \in L^2(0,T;W^{1,2}(\Omega))$ and thus it is an admissible setting. Hence, we obtained

$$\begin{aligned}\langle \partial_t \mathbb{B}_\varepsilon, \mathbb{J}_\varepsilon\rangle &+ ((\boldsymbol{v}_\varepsilon\cdot\nabla)\mathbb{B}_\varepsilon, \mathbb{J}_\varepsilon) + (\nabla\mathbb{B}_\varepsilon, \mathbb{J}_\varepsilon)\\ &= 2(r_\varepsilon\mathbb{B}_\varepsilon(a\mathbb{D}\boldsymbol{v}_\varepsilon - \mathbb{W}\boldsymbol{v}_\varepsilon), \mathbb{J}_\varepsilon) - (r_\varepsilon R(\mathbb{B}_\varepsilon), \mathbb{J}_\varepsilon).\end{aligned}$$

Next, we evaluate all terms. Here, we follow very closely the procedure in Section 1.1, see the derivation of (1.18) and consequent identities. Since

$$\mathbb{J}_\varepsilon = \frac{\partial\psi(\mathbb{B}_\varepsilon)}{\partial\mathbb{B}_\varepsilon},$$

where ψ is defined in (1.15), it is clear that

$$\begin{aligned}\langle \partial_t\mathbb{B}_\varepsilon, \mathbb{J}_\varepsilon\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi(\mathbb{B}_\varepsilon),\\ ((\boldsymbol{v}_\varepsilon\cdot\nabla)\mathbb{B}_\varepsilon, \mathbb{J}_\varepsilon) &= \int_\Omega \boldsymbol{v}_\varepsilon\cdot\nabla\psi(\mathbb{B}_\varepsilon) = 0.\end{aligned}$$

In addition, recalling (1.19), we get

$$\begin{aligned}(r_\varepsilon R(\mathbb{B}_\varepsilon), \mathbb{J}_\varepsilon) = \int_\Omega r_\varepsilon \Big(&\delta_1(1-\gamma)|\mathbb{B}_\varepsilon^{\frac12} - \mathbb{B}_\varepsilon^{-\frac12}|^2 + (\delta_1\gamma + \delta_2((1-\gamma))|\mathbb{B}_\varepsilon - \mathbb{I}|^2\\ &+\delta_2\gamma|\mathbb{B}_\varepsilon^{\frac32} - \mathbb{B}_\varepsilon^{\frac12}|^2\Big),\\ (\nabla\mathbb{B}_\varepsilon, \nabla\mathbb{J}_\varepsilon) = \gamma\|\nabla\mathbb{B}_\varepsilon\|_2^2 &+ (1-\gamma)\|\mathbb{B}_\varepsilon^{-\frac12}\nabla\mathbb{B}_\varepsilon\mathbb{B}_\varepsilon^{-\frac12}\|_2^2\end{aligned}$$

and due to the fact that $\mathbb{B}_\varepsilon\mathbb{J}_\varepsilon = \mathbb{J}_\varepsilon\mathbb{B}_\varepsilon$ we have

$$\begin{aligned}&(r_\varepsilon(\mathbb{W}\boldsymbol{v}_\varepsilon\mathbb{B}_\varepsilon - \mathbb{B}_\varepsilon\mathbb{W}\boldsymbol{v}_\varepsilon), \mathbb{J}_\varepsilon) = 0,\\ &a(r_\varepsilon(\mathbb{D}\boldsymbol{v}_\varepsilon\mathbb{B}_\varepsilon + \mathbb{B}_\varepsilon\mathbb{D}\boldsymbol{v}_\varepsilon), \mathbb{J}_\varepsilon) = 2a(r_\varepsilon\mathbb{D}\boldsymbol{v}_\varepsilon, \mathbb{B}_\varepsilon\mathbb{J}_\varepsilon) = 2a(r_\varepsilon\mathbb{D}\boldsymbol{v}_\varepsilon, (\mathbb{B}_\varepsilon + \gamma(\mathbb{B}_\varepsilon - \mathbb{I})^2)\\ &\qquad\qquad = (r_\varepsilon S(\mathbb{B}_\varepsilon), \mathbb{D}\boldsymbol{v}_\varepsilon),\end{aligned}$$

where we used the fact that the trace of $\mathbb{D}\boldsymbol{v}_\varepsilon$ is identically zero. Hence, using $\mathbb{A}$ defined in (2.27) in (2.18) and taking into account the above identities, we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi(\mathbb{B}_\varepsilon) + (1-\gamma)\Big\|\mathbb{B}_\varepsilon^{-\frac12}\nabla\mathbb{B}_\varepsilon\mathbb{B}_\varepsilon^{-\frac12}\Big\|_2^2 + \gamma\|\nabla\mathbb{B}_\varepsilon\|_2^2\\ &\quad + (\gamma\delta_1 + (1-\gamma)\delta_2)\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon - \mathbb{I})\|_2^2 + (1-\gamma)\delta_1\Big\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon^{\frac12} - \mathbb{B}_\varepsilon^{-\frac12})\Big\|_2^2\\ &\quad + \gamma\delta_2\Big\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon^{\frac32} - \mathbb{B}_\varepsilon^{\frac12})\Big\|_2^2 = 2(r_\varepsilon S(\mathbb{B}_\varepsilon), \mathbb{D}\boldsymbol{v}_\varepsilon).\end{aligned} \tag{2.28}$$

Similarly as in previous section, replacing $\boldsymbol{w}_i$ in (2.17) by $\boldsymbol{v}_\varepsilon$, we get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_\varepsilon\|_2^2 + 2\|\mathbb{D}\boldsymbol{v}_\varepsilon\|_2^2 + \|\mathcal{T}_\varepsilon\boldsymbol{v}_\varepsilon\|_{2,\partial\Omega}^2 = -2a(r_\varepsilon S(\mathbb{B}_\varepsilon), \mathbb{D}\boldsymbol{v}_\varepsilon) + \langle\boldsymbol{f}, \boldsymbol{v}_\varepsilon\rangle, \tag{2.29}$$

Thus summing (2.28) and (2.29) and integrating the result with respect to time $t \in (0,\tau)$, we deduce the final identity

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{v}_\varepsilon(\tau)\|_2^2 + \int_\Omega \psi(\mathbb{B}_\varepsilon(\tau)) \\
&\qquad + \int_0^\tau \Big(2\|\mathbb{D}\boldsymbol{v}_\varepsilon\|_2^2 + \|\mathcal{T}\boldsymbol{v}_\varepsilon\|_{2,\partial\Omega}^2 + (1-\gamma)\Big\|\mathbb{B}_\varepsilon^{-\frac{1}{2}}\nabla\mathbb{B}_\varepsilon\mathbb{B}_\varepsilon^{-\frac{1}{2}}\Big\|_2^2 + \gamma\|\nabla\mathbb{B}_\varepsilon\|_2^2 \\
&\qquad + (\gamma\delta_1 + (1-\gamma)\delta_2)\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon - \mathbb{I})\|_2^2 \\
&\qquad + (1-\gamma)\delta_1\Big\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon^{\frac{1}{2}} - \mathbb{B}_\varepsilon^{-\frac{1}{2}})\Big\|_2^2 + \gamma\delta_2\Big\|\sqrt{r_\varepsilon}(\mathbb{B}_\varepsilon^{\frac{3}{2}} - \mathbb{B}_\varepsilon^{\frac{1}{2}})\Big\|_2^2\Big) \\
&= \frac{1}{2}\|P_k\boldsymbol{v}_0\|_2^2 + \int_\Omega \psi(\mathbb{B}_0^\varepsilon) + \int_0^\tau \langle \boldsymbol{f}, \boldsymbol{v}\rangle \le \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \int_\Omega \psi(\mathbb{B}_0) + \int_0^\tau \langle \boldsymbol{f}, \boldsymbol{v}_\varepsilon\rangle,
\end{aligned}
\tag{2.30}
$$

where, for the last inequality we used the continuity of P_k and the fact that $\psi(\mathbb{I}) = 0$ and the definition of $\mathbb{B}_0^\varepsilon$.

From (2.30), we get, using Korn's and Sobolev's inequalities, that

$$
\|\boldsymbol{v}_\varepsilon\|_{L^\infty L^2} + \|\boldsymbol{v}_\varepsilon\|_{L^2 L^6} + \|\boldsymbol{v}_\varepsilon\|_{L^2 W^{1,2}} + \|\mathbb{B}_\varepsilon\|_{L^2 W^{1,2}} + \|\mathbb{B}_\varepsilon\|_{L^2 L^6} \le C,
\tag{2.31}
$$

where the constant C depends only on Ω, $\boldsymbol{v}_0$ and $\mathbb{B}_0$. Furthermore, the interpolation inequalities yield

$$
\|\boldsymbol{v}_\varepsilon\|_{L^{\frac{10}{3}} L^{\frac{10}{3}}} + \|\boldsymbol{v}_\varepsilon\|_{L^4 L^3} + \|\mathbb{B}_\varepsilon\|_{L^{\frac{10}{3}} L^{\frac{10}{3}}} + \|\mathbb{B}_\varepsilon\|_{L^4 L^3} + \|\mathbb{B}_\varepsilon\|_{L^{\frac{8}{3}} L^4} \le C.
\tag{2.32}
$$

Finally, we focus on the estimate for time derivatives. Let $\boldsymbol{\varphi} \in L^4(0,T; W^{3,2}_{\boldsymbol{n},\mathrm{div}})$ be such that $\|\boldsymbol{\varphi}\|_{L^4 W^{3,2}} \le 1$. Then, since $\boldsymbol{v}_\varepsilon$ is a linear combination of $\{\boldsymbol{w}_i\}_{i=1}^k$, we obtain, using (2.17), Hölder's inequality, (2.30), (2.32) and $W^{3,2}$-continuity of P_k, that

$$
\int_0^T \langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{\varphi}\rangle \le C,
$$

hence

$$
\|\partial_t \boldsymbol{v}_\varepsilon\|_{L^{\frac{4}{3}} W^{-3,2}_{\boldsymbol{n},\mathrm{div}}} \le C.
\tag{2.33}
$$

Similarly, by considering $\mathbb{A} \in L^4(0,T; W^{1,2}(\Omega))$ in (2.18), we get

$$
\|\partial_t \mathbb{B}_\varepsilon\|_{L^{\frac{4}{3}} W^{-1,2}} \le C.
\tag{2.34}
$$

2.3.3. *Limit passage* $\varepsilon \to 0_+$. From (2.30) (where we use Young's and Korn's inequality to estimate $\langle \boldsymbol{f}, \boldsymbol{v}_\varepsilon\rangle$), (2.33), (2.34), Banach-Alaoglu's theorem and the

Aubin-Lions lemma, we obtain that there is a couple $(\boldsymbol{v}_k, \mathbb{B}_k)$ such that the following convergence results[2] hold true

$$
\begin{aligned}
\boldsymbol{v}_\varepsilon &\rightharpoonup \boldsymbol{v}_k && \text{weakly in} \quad L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}),\\
\partial_t\boldsymbol{v}_\varepsilon &\rightharpoonup \partial_t\boldsymbol{v}_k && \text{weakly in} \quad L^{\frac43}(0,T;W^{-3,2}_{\boldsymbol{n},\mathrm{div}}),\\
\mathcal{T}\boldsymbol{v}_\varepsilon &\rightharpoonup \mathcal{T}\boldsymbol{v}_k && \text{weakly in} \quad L^2(0,T;L^2(\partial\Omega)),\\
\mathbb{B}_\varepsilon &\rightharpoonup \mathbb{B}_k && \text{weakly in} \quad L^2(0,T;W^{1,2}(\Omega)),\\
\partial_t\mathbb{B}_\varepsilon &\rightharpoonup \partial_t\mathbb{B}_k && \text{weakly in} \quad L^{\frac43}(0,T;W^{-1,2}(\Omega)),\\
\boldsymbol{v}_\varepsilon &\to \boldsymbol{v}_k && \text{strongly in } L^3(Q) \text{ and a.e. in } Q, \qquad (2.35)\\
\mathbb{B}_\varepsilon &\to \mathbb{B}_k && \text{strongly in } L^3(Q) \text{ and a.e. in } Q. \qquad (2.36)
\end{aligned}
$$

Thus, we can use (2.36) to pass to the limit in (2.24) and obtain

$$\mathbb{B}_k\boldsymbol{z}\cdot\boldsymbol{z} \geq 0 \quad \text{a.e. in } Q \quad \text{for all } \boldsymbol{z}\in\mathbb{R}^3,$$

hence $\Lambda(\mathbb{B}_k) \geq 0$ and $\det\mathbb{B}_k \geq 0$ a.e. in Q. Therefore, using (2.36) and the continuity of ψ, there exists (still possibly infinite) limit

$$\psi(\mathbb{B}_\varepsilon) \to \psi(\mathbb{B}_k) \qquad\qquad \text{a.e. in } Q.$$

However, by Fatou's lemma and $\psi \geq 0$, this limit satisfies

$$\int_\Omega \psi(\mathbb{B}_k)(t) \leq \liminf_{\varepsilon\to0_+}\int_\Omega \psi(\mathbb{B}_\varepsilon)(t) \leq C,$$

for almost every $t\in(0,T)$, hence

$$\|\psi(\mathbb{B}_k)\|_{L^\infty L^1} \leq C. \tag{2.37}$$

Consequently, if there existed a set $E \subset Q$ of a positive measure, where $\Lambda(\mathbb{B}_k) = 0$, then also $-\ln\det\mathbb{B}_k = \infty$ on that set, which contradicts (2.37). Thus, we have

$$\Lambda(\mathbb{B}_k) > 0 \text{ a.e. in } Q. \tag{2.38}$$

Then it directly follows from the continuity of Λ, that $r_\varepsilon \to 1$ a.e. in Q. Then, since $r_\varepsilon \leq 1$, we further get, by Vitali's theorem, that

$$r_\varepsilon \to 1 \qquad\qquad \text{strongly in } L^p \text{ for all } p\in[1,\infty).$$

Using the convergence results above, it is easy to let $\varepsilon \to 0_+$ in (2.17) and (2.18) and obtain that

$$
\begin{aligned}
&\langle\partial_t\boldsymbol{v}_k, \boldsymbol{w}_i\rangle + ((\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k, \boldsymbol{w}_i) + 2(\mathbb{D}\boldsymbol{v}_k, \nabla\boldsymbol{w}_i)\\
&\qquad = -(\mathcal{T}\boldsymbol{v}_k, \mathcal{T}\boldsymbol{w}_i)_{\partial\Omega} - 2a(S(\mathbb{B}_k), \nabla\boldsymbol{w}_i) + \langle\boldsymbol{f}, \boldsymbol{w}_i\rangle
\end{aligned}
$$

for almost all $t\in(0,T)$ and all $i = 1,\ldots,k$, and that

$$
\begin{aligned}
&\langle\partial_t\mathbb{B}_k, \mathbb{A}\rangle + ((\boldsymbol{v}_k\cdot\nabla)\mathbb{B}_k, \mathbb{A}) + (\nabla\mathbb{B}_k, \nabla\mathbb{A})\\
&\qquad = 2(\mathbb{B}_k(a\mathbb{D}\boldsymbol{v}_k - \mathbb{W}\boldsymbol{v}_k), \mathbb{A}) - (R(\mathbb{B}_k), \mathbb{A})
\end{aligned}
$$

for all $\mathbb{A} \in W^{1,2}(\Omega)$ and almost all $t\in(0,T)$. Furthermore, we can pass to the limit in estimates (2.30), (2.32), (2.33) and (2.34). Indeed, in most of the terms we

[2]The convergence results (2.35), (2.36) are true in any space $L^p(Q)$, $1 \leq p < \frac{10}{3}$, as can be seen from (2.32) and Vitali's theorem. The space $L^3(\Omega)$ is chosen for simplicity; in our proof, we need $p > 2$.

use the weak lower semi-continuity of norms and in the terms, which depends only on $\mathbb{B}_\varepsilon$, e.g. $\int_Q r_\varepsilon |\mathbb{B}_\varepsilon^{\frac{3}{2}} - \mathbb{B}_\varepsilon^{\frac{1}{2}}|^2$, we apply (2.38) to obtain the pointwise limit and then use Fatou's lemma. Thus, inequalities (2.30), (2.32), (2.33) and (2.34) continue to hold in the same form, but for $(\boldsymbol{v}_k, \mathbb{B}_k)$ instead of $(\boldsymbol{v}_\varepsilon, \mathbb{B}_\varepsilon)$ and with 1 instead of r_ε. In particular, we have

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{v}_k(\tau)\|_2^2 + \int_\Omega \psi(\mathbb{B}_k(\tau)) \\
&\quad + \int_0^\tau \Big(2\|\mathbb{D}\boldsymbol{v}_k\|_2^2 + \|\mathcal{T}\boldsymbol{v}_k\|_{2,\partial\Omega}^2 + (1-\gamma)\Big\|\mathbb{B}_k^{-\frac{1}{2}} \nabla\mathbb{B}_k \mathbb{B}_k^{-\frac{1}{2}}\Big\|_2^2 + \gamma\|\nabla\mathbb{B}_k\|_2^2 \\
&\qquad + (\gamma\delta_1 + (1-\gamma)\delta_2)\|\mathbb{B}_k - \mathbb{I}\|_2^2 \\
&\qquad + (1-\gamma)\delta_1 \Big\|\mathbb{B}_k^{\frac{1}{2}} - \mathbb{B}_k^{-\frac{1}{2}}\Big\|_2^2 + \gamma\delta_2\Big\|\mathbb{B}_k^{\frac{3}{2}} - \mathbb{B}_k^{\frac{1}{2}}\Big\|_2^2\Big) \\
&\leq \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \int_\Omega \psi(\mathbb{B}_0) + \int_0^\tau \langle \boldsymbol{f}, \boldsymbol{v}_k\rangle
\end{aligned}$$

for almost all $\tau \in (0,T)$. The attainment of initial conditions is in this step standard and we postpone the proof to the last section.

2.4. **Limit** $k \to \infty$**.** Since we have the same a priori estimates as in the previous step, we can proceed with limit completely analogously as with the limit $\varepsilon \to 0_+$. The only difference is that r_ε is not present. Thus, using the density of $\{\boldsymbol{w}_i\}_{i=1}^\infty$ in $W^{3,2}_{\boldsymbol{n},\mathrm{div}}$, we obtain

$$\begin{aligned}
\langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle + ((\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi}) + 2(\mathbb{D}\boldsymbol{v}, \nabla\boldsymbol{\varphi})& \\
= -(\mathcal{T}\boldsymbol{v}, \mathcal{T}\boldsymbol{\varphi})_{\partial\Omega} - 2a(S(\mathbb{B}), \nabla\boldsymbol{\varphi}) + \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle&
\end{aligned} \tag{2.39}$$

for almost all $t \in (0,T)$ and all $\boldsymbol{\varphi} \in W^{3,2}_{\boldsymbol{n},\mathrm{div}}$, and that

$$\langle \partial_t \mathbb{B}, \mathbb{A}\rangle + ((\boldsymbol{v}\cdot\nabla)\mathbb{B}, \mathbb{A}) + (\nabla\mathbb{B}, \nabla\mathbb{A}) = 2(\mathbb{B}(a\mathbb{D}\boldsymbol{v} - \mathbb{W}\boldsymbol{v}), \mathbb{A}) - (R(\mathbb{B}), \mathbb{A})$$

for all $\mathbb{A} \in W^{1,2}(\Omega)$ and almost all $t \in (0,T)$.

Moreover, from the weak lower semi-continuity of norms, we obtain the energy inequality (1.28) for almost all $t \in (0,T)$. Furthermore, using analogous argument as previously, we obtain that $\mathbb{B}$ is positive definite a.e. in Q. Now observe that, by Hölder's inequality and (2.32), all the terms in (2.39) except the first one, are integrable for every $\boldsymbol{\varphi} \in L^4(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}) \hookrightarrow L^4(0,T;L^6(\Omega))$. Indeed, for example for the non-linear terms, we get

$$\int_Q |(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{\varphi}| \leq \|\boldsymbol{v}\|_{L^4L^3}\|\nabla\boldsymbol{v}\|_{L^2L^2}\|\boldsymbol{\varphi}\|_{L^2L^6}$$

and

$$\int_Q |S(\mathbb{B})\cdot\nabla\boldsymbol{\varphi}| \leq C\|\mathbb{B}\|^2_{L^{\frac{8}{3}}L^4}\|\nabla\boldsymbol{\varphi}\|_{L^4L^2}.$$

Hence, the functional $\partial_t \boldsymbol{v}$ can be uniquely extended to $\partial_t\boldsymbol{v} \in L^{\frac{4}{3}}(0,T;W^{-1,2}_{\boldsymbol{n},\mathrm{div}})$ and we can use the density argument to conclude (1.25). Analogously, we obtain (1.26). Hence, it remains to show that (1.28) holds for all $t \in (0,T)$ and that the initial data fulfill (1.27).

2.4.1. *Energy inequality for all $t \in (0,T)$.* First, we notice, that thanks to (2.31), (2.33) and (2.34), we have that[3]

$$\begin{aligned} &\boldsymbol{v} \in \mathcal{C}_{\text{weak}}(0,T;L^2(\Omega)),\\ &\mathbb{B} \in \mathcal{C}_{\text{weak}}(0,T;L^2(\Omega)). \end{aligned} \tag{2.40}$$

Let us begin by noticing that the function ψ is convex when restricted on the convex set $\mathbb{R}^{3\times 3}_{>0}$ (the set of positive definite matrices of the size 3×3). Indeed, evaluating the second Fréchet derivative of ψ, we get for arbitrary $\mathbb{A} \in \mathbb{R}^{3\times 3}_{>0}$

$$\frac{\partial^2 \psi(\mathbb{A})}{\mathbb{A}^2} = (1-\gamma)\mathbb{A}^{-1} \otimes \mathbb{A}^{-1} + \gamma \mathbb{I} \otimes \mathbb{I}, \quad \mathbb{A} \in \mathbb{R}^{3\times 3}_{>0},$$

which is obviously a positive definite operator for any $\gamma \in [0,1]$ and consequently, ψ must be convex on $\mathbb{R}^{3\times 3}_{>0}$.

Next, we integrate (1.28) with respect to $\tau \in (t_1, t_1+\delta)$ and divide the result by δ. After neglecting some parts of the integration in the terms on the left hand side (which preserves the sign), we get

$$\begin{aligned} &\frac{1}{2\delta}\int_{t_1}^{t_1+\delta} \|\boldsymbol{v}(\tau)\|_2^2 + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(\mathbb{B}(\tau))\\ &\quad + \int_0^{t_1} \Big(2\|\mathbb{D}\boldsymbol{v}\|_2^2 + \|\mathcal{T}\boldsymbol{v}\|_{2,\partial\Omega}^2 + (1-\gamma)\Big\|\mathbb{B}^{-\frac12}\nabla\mathbb{B}\mathbb{B}^{-\frac12}\Big\|_2^2 + \gamma\|\nabla\mathbb{B}\|_2^2\\ &\quad + (\gamma\delta_1 + (1-\gamma)\delta_2)\|\mathbb{B}-\mathbb{I}\|_2^2\\ &\quad + (1-\gamma)\delta_1\Big\|\mathbb{B}^{\frac12} - \mathbb{B}^{-\frac12}\Big\|_2^2 + \gamma\delta_2\Big\|\mathbb{B}^{\frac32} - \mathbb{B}^{\frac12}\Big\|_2^2\Big)\\ &\le \frac12\|\boldsymbol{v}_0\|_2^2 + \int_\Omega \psi(\mathbb{B}_0) + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_0^\tau \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \end{aligned}$$

Finally, we let $\delta \to 0_+$. The limit on the right hand side is standard and consequently, if we show that

$$\frac12\|\boldsymbol{v}(t_1)\|_2^2 + \int_\Omega \psi(\mathbb{B}(t_1)) \le \liminf_{\delta\to 0_+} \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\left(\frac{\|\boldsymbol{v}(\tau)\|_2^2}{2} + \int_\Omega \psi(\mathbb{B}(\tau))\right), \tag{2.41}$$

then (1.28) will hold for all $t \in (0,T)$. To show it, we notice that thanks to (2.40)

$$\begin{aligned} &\boldsymbol{v}(t) \rightharpoonup \boldsymbol{v}(t_1) \quad \text{weakly in } L^2(\Omega) \text{ as } t \to t_1,\\ &\mathbb{B}(t) \rightharpoonup \mathbb{B}(t_1) \quad \text{weakly in } L^2(\Omega) \text{ as } t \to t_1, \end{aligned} \tag{2.42}$$

Consequently, due to the weak lower semicontinuity and the convexity of ψ we also have for all $t \in (0,T)$

$$\int_\Omega |\boldsymbol{v}(t)|^2 + \psi(\mathbb{B}(t)) \le C.$$

[3] Here the space $\mathcal{C}_{\text{weak}}(0,T;X) \subset L^\infty(0,T;X)$ denotes a space of weakly continuous function, i.e., for every $f \in \mathcal{C}_{\text{weak}}(0,T;X)$ and every $g \in X^*$ there holds

$$\lim_{t\to t_0} \langle f(t), g\rangle = \langle f(t_0), g\rangle.$$

Hence denoting by $\Omega_M \subset \Omega$ the set where $|\boldsymbol{v}(t_1)| + |\mathbb{B}(t)| + |\mathbb{B}^{-1}(t)| \le M$, it follows from the previous estimate that $|\Omega \setminus \Omega_M| \to 0$ as $M \to \infty$. Hence, since ψ is nonnegative and convex, we have for all $t \in (t_1, t_1 + \delta)$

$$\int_\Omega \frac{|\boldsymbol{v}(t)|^2}{2} + \psi(\mathbb{B}(t)) \ge \int_{\Omega_M} \frac{|\boldsymbol{v}(t)|^2}{2} + \psi(\mathbb{B}(t))$$
$$\ge \int_{\Omega_M} \frac{|\boldsymbol{v}(t_1)|^2}{2} + \psi(\mathbb{B}(t_1)) + \int_{\Omega_M} \boldsymbol{v}(t_1) \cdot (\boldsymbol{v}(t) - \boldsymbol{v}(t_1)) + \frac{\partial \psi(B(t_1))}{\partial \mathbb{B}} \cdot (\mathbb{B}(t) - \mathbb{B}(t_1)).$$

Since, $\boldsymbol{v}(t_1)$ and $\partial_{\mathbb{B}} \psi(\mathbb{B}(t_1))$ are bounded on Ω_M, we can integrate the above estimate over $(t_1, t_1 + \delta)$ and it follows from (2.42) that

$$\liminf_{\delta \to 0_+} \frac{1}{\delta} \int_{t_1}^{t_1+\delta} \int_\Omega \frac{|\boldsymbol{v}(t)|^2}{2} + \psi(\mathbb{B}(t)) \ge \int_{\Omega_M} \frac{|\boldsymbol{v}(t_1)|^2}{2} + \psi(\mathbb{B}(t_1)).$$

Hence, letting $M \to \infty$, we deduce (2.41) and the proof of (1.28) is complete.

2.4.2. *Attainment of initial conditions.* Here, we give only a sketch of the proof. First, it is standard to show from the construction and from the weak continuity (2.42), that for arbitrary $\boldsymbol{\varphi}, \mathbb{A} \in L^2(\Omega)$

$$\begin{aligned} &\lim_{t \to 0_+} (\boldsymbol{v}(t), \boldsymbol{\varphi}) = (\boldsymbol{v}_0, \boldsymbol{\varphi}), \\ &\lim_{t \to 0_+} (\mathbb{B}(t), \mathbb{A}) = (\mathbb{B}_0, \mathbb{A}). \end{aligned} \tag{2.43}$$

Next, using the convexity of ψ and (2.43) (and consequently weak lower semicontinuity) and letting $t \to 0_+$ in (1.28), we deduce

$$\begin{aligned} \|\boldsymbol{v}_0\|_2^2 + 2\int_\Omega \psi(\mathbb{B}_0) &\le \liminf_{t \to 0_+} \left(\|\boldsymbol{v}(t)\|_2^2 + 2\int_\Omega \psi(\mathbb{B}(t)) \right) \\ &\le \limsup_{t \to 0_+} \left(\|\boldsymbol{v}(t)\|_2^2 + 2\int_\Omega \psi(\mathbb{B}(t)) \right) \le \|\boldsymbol{v}_0\|_2^2 + 2\int_\Omega \psi(\mathbb{B}_0) \end{aligned} \tag{2.44}$$

Next, we split the information from (2.44). Assume for a moment that

$$\|\boldsymbol{v}_0\|_2^2 < \liminf_{t \to 0_+} \|\boldsymbol{v}(t)\|_2^2.$$

But then it follows from (2.44) that

$$\int_\Omega \psi(\mathbb{B}_0) > \liminf_{t \to 0_+} \int_\Omega \psi(\mathbb{B}(t)).$$

However, the second inequality contradicts (2.43) and convexity of ψ. Consequently, we obtain

$$\begin{aligned} \|\boldsymbol{v}_0\|_2^2 &= \lim_{t \to 0_+} \|\boldsymbol{v}(t)\|_2^2, \\ \int_\Omega \psi(\mathbb{B}_0) &= \lim_{t \to 0_+} \int_\Omega \psi(\mathbb{B}(t)). \end{aligned} \tag{2.45}$$

Thus, it an easy consequence of $(2.43)_1$ and $(2.45)_1$ that

$$\lim_{t \to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0.$$

To claim the same result also for $\mathbb{B}$, we simply split ψ as follows

$$\psi(\mathbb{A}) = \frac{\gamma}{2}|\mathbb{A} - \mathbb{I}|^2 + \gamma(\operatorname{tr} \mathbb{A} - 3 - \ln \det \mathbb{A}) =: \gamma \psi_1(\mathbb{A}) + (1 - \gamma)\psi_2(\mathbb{A}).$$

Similarly as above, it is easy to observe that ψ_1 as well as ψ_2 are convex on the set of positive definite matrices. Therefore, $(2.45)_2$ and $(2.43)_2$ imply

$$\begin{aligned}\int_\Omega |\mathbb{B}_0 - \mathbb{I}|^2 &= 2\int_\Omega \psi_1(\mathbb{B}_0) = 2\lim_{t\to 0_+}\int_\Omega \psi_1(\mathbb{B}(t)) = \lim_{t\to 0_+}\int_\Omega |\mathbb{B}(t) - \mathbb{I}|^2,\\ \int_\Omega \psi_2(\mathbb{B}_0) &= \lim_{t\to 0_+}\int_\Omega \psi_2(\mathbb{B}(t)).\end{aligned} \tag{2.46}$$

Finally, (2.43) and $(2.46)_1$ lead to

$$\begin{aligned}\lim_{t\to 0_+}\|\mathbb{B}(t) - \mathbb{B}_0\|_2^2 &= \lim_{t\to 0_+}\|(\mathbb{B}(t) - \mathbb{I}) + (\mathbb{I} - \mathbb{B}_0)\|_2^2\\ &= \lim_{t\to 0_+}\left(\|\mathbb{B}(t) - \mathbb{I}\|_2^2 + \|\mathbb{B}_0 - \mathbb{I}\|_2^2 - 2\int_\Omega (\mathbb{B}(t) - \mathbb{I})\cdot(\mathbb{B}_0 - \mathbb{I})\right)\\ &= 0,\end{aligned}$$

which finishes the proof of (1.27) and consequently also the proof of Theorem 2.

References

[1] J.W. Barrett and S. Boyaval, *Existence and approximation of a (regularized) Oldroyd-B model*, Mathematical Models and Methods in Applied Sciences **21** (2011), no. 09, 1783–1837.

[2] J. Blechta, J. Málek, and K.R. Rajagopal, *On the classification of incompressible fluids and a mathematical analysis of the equations that govern their motion*, http://arxiv.org/abs/1902.04853v2 (2019), SIAM J. Mathematical Analysis (to appear).

[3] M. Bulíček, E. Feireisl, and J. Málek, *On a class of compressible viscoelastic rate-type fluids with stress-diffusion*, Nonlinearity **32** (2019), no. 12, 4665–4681.

[4] M. Bulíček, J. Málek, and K.R. Rajagopal, *Mathematical analysis of unsteady flows of fluids with pressure, shear-rate, and temperature dependent material moduli that slip at solid boundaries*, SIAM Journal on Mathematical Analysis **41** (2009), no. 2, 665–707.

[5] M. Bulíček and J. Málek, *Internal flows of incompressible fluids subject to stick-slip boundary conditions*, Vietnam J. Math. **45** (2017), no. 1-2, 207–220. MR 3600423

[6] ______, *Large data analysis for Kolmogorov's two-equation model of turbulence*, Nonlinear Anal. Real World Appl. **50** (2019), 104–143. MR 3948897

[7] M. Bulíček, J. Málek, V. Průša, and E. Süli, *PDE analysis of a class of thermodynamically compatible viscoelastic rate-type fluids with stress-diffusion*, Mathematical analysis in fluid mechanics—selected recent results, Contemp. Math., vol. 710, Amer. Math. Soc., Providence, RI, 2018, pp. 25–51.

[8] M. Bulíček, J. Málek, and J. Žabenský, *On generalized Stokes' and Brinkman's equations with a pressure-and shear-dependent viscosity and drag coefficient*, Nonlinear Anal. Real World Appl. **26** (2015), 109–132. MR 3384328

[9] M. Bulíček and J. Žabenský, *Large data existence theory for unsteady flows of fluids with pressure- and shear-dependent viscosities*, Nonlinear Anal. **127** (2015), 94–127. MR 3392360

[10] P. Constantin and M. Kliegl, *Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress*, Archive for Rational Mechanics and Analysis **206** (2012), no. 3, 725–740.

[11] M. Dostalík, V. Průša, and T. Skřivan, *On diffusive variants of some classical viscoelastic rate-type models*, AIP Conference Proceedings **2107** (2019).

[12] A.W. El-Kareh and G.L. Leal, *Existence of solutions for all deborah numbers for a non-newtonian model modified to include diffusion*, Journal of Non-Newtonian Fluid Mechanics **33**, no. 3, 257–287.

[13] C. Guillopé and J.-C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal **15** (1990), no. 9, 849–869.

[14] J. Hron, V. Miloš, V. Průša, O. Souček, and K. Tůma, *On thermodynamics of incompressible viscoelastic rate type fluids with temperature dependent material coefficients*, International Journal of Non-Linear Mechanics **95** (2017), 193–208.

[15] O. Kreml, Pokorný M., and P. Šalom, *On the global existence for a regularized model of viscoelastic non-Newtonian fluid*, Colloquium Mathematicum **139** (2015), no. 2, 149–163.

[16] P. L. Lions and N. Masmoudi, *Global solutions for some oldroyd models of non-newtonian flows*, Chinese Annals of Mathematics **21** (2000), no. 2, 131–146.
[17] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, *Weak and Measure-valued Solutions to Evolutionary PDEs*, Springer US, 1996.
[18] J. Málek, V. Průša, T. Skřivan, and E. Süli, *Thermodynamics of viscoelastic rate-type fluids with stress diffusion*, Physics of Fluids **30** (2018).
[19] J. Málek and V. Průša, *Derivation of equations for continuum mechanics and thermodynamics of fluids*, Handbook of mathematical analysis in mechanics of viscous fluids, Springer, Cham, 2018, pp. 3–72.
[20] J. Málek, K.R. Rajagopal, and K. Tůma, *On a variant of the Maxwell and Oldroyd-B models within the context of a thermodynamic basis*, International Journal of Non-Linear Mechanics **76** (2015), 42 – 47.
[21] ———, *Derivation of the variants of the Burgers model using a thermodynamic approach and appealing to the concept of evolving natural configurations*, Fluids **3** (2018), no. 4.
[22] E. Maringová and J. Žabenský, *On a Navier-Stokes-Fourier-like system capturing transitions between viscous and inviscid fluid regimes and between no-slip and perfect-slip boundary conditions*, Nonlinear Anal. Real World Appl. **41** (2018), 152–178.
[23] N. Masmoudi, *Global existence of weak solutions to macroscopic models of polymeric flows*, Journal de Mathématiques Pures et Appliquées **96** (2011), no. 5, 502–520.
[24] K.R. Rajagopal and A. R. Srinivasa, *A thermodynamic frame work for rate type fluid models*, J. Non-Newton. Fluid Mech. **88** (2000), no. 3, 207–227.
[25] K.R. Rajagopal and A. R. Srinivasa, *On thermomechanical restrictions of continua*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **460** (2004), no. 2042, 631–651.

Mathematical Institute of Charles University, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
Email address: bathory@karlin.mff.cuni.cz

Mathematical Institute of Charles University, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
Email address: mbul8060@karlin.mff.cuni.cz

Mathematical Institute of Charles University, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
Email address: malek@karlin.mff.cuni.cz