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Preprint no. 2013-05

http://ncmm.karlin.mff.cuni.cz

Two-sided bounds for eigenvalues of differential operators with applications to Friedrichs', Poincaré, trace, and similar constants*

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March 29, 2013

Abstract

We present a general numerical method for computing guaranteed two-sided bounds for principal eigenvalues of symmetric linear elliptic differential operators. The approach is based on the Galerkin method, on the method of a priori-a posteriori inequalities, and on a complementarity technique. The two-sided bounds are formulated in a general Hilbert space setting and as a byproduct we prove an abstract inequality of Friedrichs'–Poincaré type. The abstract results are then applied to Friedrichs', Poincaré, and trace inequalities and fully computable two-sided bounds on the optimal constants in these inequalities are obtained. Accuracy of the method is illustrated on numerical examples.

Keywords: bounds on spectrum, a posteriori error estimate, optimal constant, Friedrichs' inequality, Poincaré inequality, trace inequality, Hilbert space
MSC2010: 35P15, 35J15, 65N25, 65N30

1 Introduction

Eigenvalue problems for differential operators have attracted a lot of attention as they have many applications. These include the dynamic analysis of mechanical systems [3, 15, 25], linear stability of flows in fluid mechanics [24], and electronic band structure calculations [29]. In this paper, we concentrate on guaranteed two-sided bounds of the principal (smallest) eigenvalue of symmetric linear elliptic operators. The standard Galerkin method for solution of eigenproblems is efficient and its convergence and other properties are well analysed [5, 6, 11]. It is also well known for providing upper bounds on eigenvalues. However, in many applications a reliable *lower bound* of the smallest eigenvalue is the key piece of information and, unfortunately, the Galerkin method cannot provide it.

The question of lower bounds on the smallest eigenvalue has already been studied for several decades. For example see [30], where the second order elliptic eigenvalue problems with Dirichlet boundary conditions are considered. Another technique that gives the lower bounds not only for the first eigenvalue is the method of intermediate problems. It is based

*The support of I. Šebestová by the Grant SVV-2013-267316 and by the project MathMAC - University center for mathematical modeling, applied analysis and computational mathematics of the Charles University in Prague and the support of T. Vejchodský by RVO 67985840 are gratefully acknowledged.

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on finding a base problem and subsequently introducing intermediate problems that give lower bounds for eigenvalues of the original problem and at the same time can be resolved explicitly, see for example [8, 9, 39]. A survey of this technique can be found in [16].

Nonconforming methods have been used for computing lower bounds on eigenvalues, see for example [4, 26, 28, 31, 40]. However, these lower bounds are valid asymptotically only and hence these methods *do not guarantee* that the computed approximation is really below the exact value. Recently, sufficient conditions for producing lower bounds for eigenvalues of symmetric elliptic operators by nonconforming methods have been provided in [19]. The described technique is based on satisfying the saturation assumption and on the condition saying that the local approximation property of the underlying finite element space have to be better than its global continuity property. It is proved that the second condition is met by most commonly used nonconforming methods such as the Wilson element, linear nonconforming Crouzeix–Raviart element, the enriched nonconforming rotated Q_1 element, the Morley element, the Adini element, and the Morley–Wang–Xu element. The saturation assumption is proved for the Morley–Wang–Xu element, the enriched Crouzeix–Raviart element, and the Wilson element. Furthermore, new nonconforming methods satisfying these properties are proposed. However, no numerical experiments are presented.

Further, let us point out a recent result [27], where two-sided a priori bounds for the discretization error of eigenvectors are given.

The method of *a priori-a posteriori inequalities* that can be used for computation of lower bounds on eigenvalues was described and published in [22, 23, 34]. However, in these original publications C^2-continuous test and trial functions has been used in order to compute the actual lower bound. These functions are difficult to work with and therefore, we couple the original method of a priori-a posteriori inequalities with the complementarity technique, where a certain flux function has to be reconstructed, see, e.g., [2, 14, 18, 32, 36, 38]. This flux reconstruction can be done in many ways. We choose the most straightforward approach that can be handled by standard Raviart–Thomas finite element method.

Moreover, we generalize the original method of a priori-a posteriori inequalities to the case of a compact operator between a pair of Hilbert spaces. This generalization is especially useful for computing two-sided bounds of the optimal constant in the Friedrichs', Poincaré, trace, and similar inequalities. It is based on the fact that the optimal constant in these inequalities is inversely proportional to the square root of the smallest eigenvalue of the corresponding symmetric linear elliptic partial differential operator.

Further, the generalization we have made enables to set up an abstract framework in the Hilbert space setting. The abstract results can be then easily applied to symmetric linear elliptic partial differential operators and consequently to the optimal constant in the inequalities of Friedrichs'–Poincaré type. Furthermore, as a byproduct of the abstract setting, we obtain a simple proof of the validity of an abstract inequality of Friedrichs'–Poincaré type in the Hilbert space setting. The particular choices of the pair of Hilbert spaces, corresponding scalar products, and the compact operator then naturally yield the validity of the Friedrichs', Poincaré, trace, Korn's and other inequalities.

The main motivation for our interest in two-sided bounds of the constants in the Friedrichs', Poincaré, trace, and similar inequalities stems from the need of these bounds in a posteriori error estimation for numerical solutions of partial differential equations. In particular, the existing guaranteed upper bounds on the energy norm of the error utilize a kind of complementarity technique, see, e.g., [2, 12, 20, 32, 36, 38]. Estimates of this kind contain constants from the Friedrichs', Poincaré, trace, and similar inequalities. Optimal values of these constants are often unknown and therefore suitable approximations have to be used in the error estimates. These approximations have to provide upper bounds on these constants in order to guarantee that the total error estimator is an upper bound on the error. Moreover, they have to be accurate due to the accuracy and efficiency of the error estimates.

The method presented in this paper provides accurate upper bounds on these constants. In addition, this method naturally considers the dependence of the optimal constants on the equation coefficients and on the boundary conditions. This dependence can be strong [37] and its capturing might be crucial for accuracy and robustness of a posteriori error bounds.

The rest of the paper is organized as follows. Section 2 introduces a general variational eigenvalue problem in the Hilbert space setting. It uses the spectral theory of compact operators to prove several properties of this eigenvalue problem including the existence of the principal eigenvalue. In Section 3 we naturally prove the abstract inequality of Friedrichs'–Poincaré type and show the relation between the optimal constant and the principal eigenvalue. Further, we briefly describe the Galerkin method that yields an upper bound on the principal eigenvalue and concentrate on the method of a priori-a posteriori inequalities and on an abstract complementarity result leading to a lower bound on the principal eigenvalue. Sections 4–6 apply the abstract results to the case of Friedrichs', Poincaré, and trace inequality and fully computable two-sided bounds on the optimal constants in these inequalities are obtained. Presented numerical experiments illustrate accuracy of the method and dependence of the optimal constants on a nonhomogeneous diffusion parameter. Finally, Section 7 draws the conclusions.

2 Variational eigenvalue problem in the Hilbert space setting

Let V and H be two real Hilbert spaces with scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$, respectively. The norms induced by these scalar products are denoted by $\|\cdot\|_V$ and $\|\cdot\|_H$. Further, let $\gamma : V \to H$ be a continuous, linear, and *compact* operator. The center of our interest is the following eigenvalue problem. Find $\lambda_i \in \mathbb{R}$, $u_i \in V$, $u_i \neq 0$ such that

$$(u_i, v)_V = \lambda_i (\gamma u_i, \gamma v)_H \quad \forall v \in V. \tag{1}$$

First, let us show that eigenvalues λ_i of (1) are positive.

Lemma 1. *If $u_i \in V$ is an eigenvector corresponding to an eigenvalue λ_i of (1) then $\gamma u_i \neq 0$ and $\lambda_i > 0$.*

Proof. Since $u_i \neq 0$, we have by (1) that $0 \neq \|u_i\|_V^2 = \lambda_i \|\gamma u_i\|_H^2$. Thus, $\gamma u_i \neq 0$ and λ_i has to be positive. $\square$

Below we show that eigenvalues λ_i and eigenvectors u_i of (1) correspond to eigenvalues and eigenvectors of a compact operator, respectively. Consequently, these eigenvalues form a countable sequence that can be ordered as $\lambda_1 \leq \lambda_2 \leq \cdots$. To show this correspondence, we define a *solution operator* $S : H \to V$. If $f \in H$ is arbitrary then the mapping $v \mapsto (f, \gamma v)_H$ is a continuous linear form on V and, hence, the Riesz representation theorem yields existence of a unique element $Sf \in V$ such that

$$(Sf, v)_V = (f, \gamma v)_H \quad \forall v \in V. \tag{2}$$

Consequently, the solution operator S is linear and continuous.

The composition of operators S and γ is a linear, continuous, and compact operator $S\gamma : V \to V$, see [33, Theorem 4.18 (f)]. In addition, this operator is selfadjoint, because definition (2) yields

$$(S\gamma u, v)_V = (\gamma u, \gamma v)_H = (\gamma v, \gamma u)_H = (S\gamma v, u)_V = (u, S\gamma v)_V.$$

Therefore, we can use the Hilbert–Schmidt spectral theorem for $S\gamma$, see [17, Theorem 4, Chapter II, Section 3] and obtain that V can be decomposed into a direct sum of two subspaces

$$V = \mathcal{M} \oplus \ker(S\gamma), \tag{3}$$

where $\ker(S\gamma) = \{v \in V : S\gamma v = 0\}$ is the kernel of $S\gamma$ and $\mathcal{M}$ is generated by all eigenvectors of the operator $S\gamma$ corresponding to nonzero eigenvalues. Let us recall that $u_i \in V$, $u_i \neq 0$ is an eigenvector of $S\gamma$ corresponding to an eigenvalue $\mu_i \in \mathbb{R}$ if

$$S\gamma u_i = \mu_i u_i. \tag{4}$$

Furthermore, the Hilbert–Schmidt spectral theorem implies that the system $u_1, u_2, \dots$ of eigenvectors corresponding to nonzero eigenvalues of (4) is countable and orthogonal in V. A simple consequence of (2) and (4) is the orthogonality of images γu_i, $i = 1, 2, \dots$ in H. In this paper we consider the following normalization of these eigenvectors:

$$(\gamma u_i, \gamma u_j)_H = \delta_{ij} \quad \forall i, j = 1, 2, \dots, \tag{5}$$

where δ_{ij} stands for the Kronecker's delta.

Now, let us observe that eigenproblems (1) and (4) correspond to each other and, therefore, the spectral properties of the compact operator $S\gamma$ translate to the properties of the eigenproblem (1).

Lemma 2. *Considering the above setting, the following statements hold true.*

1. *Number $\lambda_i \in \mathbb{R}$ is an eigenvalue corresponding to the eigenvector $u_i \in V$ of (1) if and only if $\mu_i = 1/\lambda_i$ is a nonzero eigenvalue corresponding to the eigenvector u_i of the operator $S\gamma$, see (4).*
2. *The number of eigenvalues λ_i of (1) such that $\lambda_i \leq M$ is finite for any $M > 0$.*
3. *The value $\lambda_1 = \inf_{u \in V, u \neq 0} \|u\|_V^2 / \|\gamma u\|_H^2$ is the smallest eigenvalue of (1).*

Proof. 1. Definition (2) yields identity $(\gamma u_i, \gamma v)_H = (S\gamma u_i, v)_V$ for all $v \in V$. Hence, the equality (1) can be rewritten as $(u_i, v)_V = \lambda_i (S\gamma u_i, v)_V$, which is equivalent to (4) with $\mu_i = 1/\lambda_i$ provided $\lambda_i \neq 0$ and $\mu_i \neq 0$. Since Lemma 1 guarantees $\lambda_i > 0$ for all $i = 1, 2, \dots,$ the only condition is $\mu_i \neq 0$.

2. If we denote the spectrum of $S\gamma$ by $\sigma(S\gamma)$ then the compactness of $S\gamma$ implies that the set $[\varepsilon, \infty) \cap \sigma(S\gamma)$ is finite for any $\varepsilon > 0$, see [33, Theorem 4.24 (b)]. The statement follows immediately from the fact that $\lambda_i = 1/\mu_i$ for $\mu_i \neq 0$.

3. Since $S\gamma$ is selfadjoint, the Courant–Fischer–Weyl min-max principle implies that

$$\mu_1 = \sup\{(S\gamma v, v)_V : \|v\|_V = 1\} = \sup_{v \in V, v \neq 0} \frac{(S\gamma v, v)_V}{\|v\|_V^2} = \sup_{v \in V, v \neq 0} \frac{\|\gamma v\|_H^2}{\|v\|_V^2}$$

is finite and it is the largest eigenvalue of the operator $S\gamma$. Consequently,

$$\lambda_1 = \mu_1^{-1} = \left(\sup_{v \in V, v \neq 0} \frac{\|\gamma v\|_H^2}{\|v\|_V^2} \right)^{-1} = \inf_{v \in V, v \neq 0} \frac{\|v\|_V^2}{\|\gamma v\|_H^2} \tag{6}$$

is the smallest eigenvalue of problem (1). $\square$

3 Abstract inequality of Friedrichs'–Poincaré type

3.1 The proof of the abstract inequality

Properties of eigenproblem (1) can be utilized in a simple way to derive an abstract inequality of Friedrichs'–Poincaré type. The Hilbert space versions of the particular Friedrichs', Poincaré, trace, Korn's and similar inequalities easily follow from this abstract result. For examples see Sections 4–6.

Theorem 3 (Abstract inequality)**.** *Let $\gamma : V \to H$ be a continuous, linear, and compact operator between Hilbert spaces V and H. Let λ_1 be the smallest eigenvalue of problem (1). Then*

$$\|\gamma v\|_H \leq C_\gamma \|v\|_V \quad \forall v \in V \tag{7}$$

with $C_\gamma = \lambda_1^{-1/2}$. Moreover, this constant is optimal in the sense that it is the smallest possible constant such that (7) holds for all $v \in V$.

Proof. The validity of the abstract inequality follows immediately from (6):

$$\|\gamma v\|_H^2 \leq \lambda_1^{-1} \|v\|_V^2 \quad \forall v \in V.$$

This inequality holds as equality for $v = u_1$ and thus, the constant $C_\gamma = \lambda_1^{-1/2}$ is optimal. $\square$

3.2 Upper bound on the smallest eigenvalue

The upper bound on λ_1 can be computed by the standard Galerkin method, which is both accurate and efficient [5, 6, 11]. The Galerkin method is based on the projection of the eigenproblem (1) into a finite dimensional subspace $V^h \subset V$. We seek eigenvectors $u_i^h \in V^h$, $u_i^h \neq 0$, and eigenvalues λ_i^h such that

$$(u_i^h, v^h)_V = \lambda_i^h (\gamma u_i^h, \gamma v^h)_H \quad \forall v^h \in V^h. \tag{8}$$

Considering a basis φ_j, $j = 1, 2, \ldots, N$ in V^h, we can formulate problem (8) equivalently as a generalized eigenvalue problem

$$A y_i = \lambda_i^h M y_i$$

for matrices A and M with entries

$$A_{jk} = (\varphi_k, \varphi_j)_V \quad \text{and} \quad M_{jk} = (\gamma\varphi_k, \gamma\varphi_j)_H, \quad j, k = 1, 2, \ldots, N.$$

The eigenvectors $y_i \in \mathbb{R}^N$ and $u_i^h \in V^h$ are linked by the relation $u_i^h = \sum_{j=1}^N y_{i,j} \varphi_j$. The generalized matrix eigenvalue problem can be solved by efficient methods of numerical linear algebra [7].

The Galerkin method for eigenvalue problems is very well understood. The convergence and the speed of convergence of this method is established for example in [5, 6, 11]. It is well known [11] that the Galerkin method approximates the exact eigenvalues from above, hence

$$\lambda_i \leq \lambda_i^h, \quad \forall i = 1, 2, \ldots.$$

In particular, the upper bound on the smallest eigenvalue λ_1 and the corresponding lower bound on the optimal constant C_γ read

$$\lambda_1 \leq \lambda_1^h \quad \text{and} \quad (\lambda_1^h)^{-1/2} \leq C_\gamma. \tag{9}$$

3.3 Lower bound on the smallest eigenvalue

In this part we concentrate on a computable lower bound on the smallest eigenvalue λ_1. First, we formulate an auxiliary result. It states that the images γu_i, $i = 1, 2, \dots$ of the orthogonal system of eigenvectors u_i satisfy the Parseval's identity.

Lemma 4. *Let u_i, $i = 1, 2, \dots$, be the above specified orthogonal system of eigenvectors of the operator $S\gamma$ corresponding to nonzero eigenvalues. Let these eigenvectors be normalized as in (5). Finally, let $u_* \in V$ be arbitrary. Then*

$$\|\gamma u_*\|_H^2 = \sum_{i=1}^{\infty} |(\gamma u_*, \gamma u_i)_H|^2.$$

Proof. Due to the decomposition (3), there exist unique components $u_*^{\mathcal{M}} \in \mathcal{M}$ and $u_*^0 \in \ker(S\gamma)$ such that $u_* = u_*^{\mathcal{M}} + u_*^0$. Since $S\gamma u_*^0 = 0$, we have $0 = (S\gamma u_*^0, u_*^0)_V = (\gamma u_*^0, \gamma u_*^0)_H$ by (2) and hence $\gamma u_*^0 = 0$. Consequently, $\gamma u_* = \gamma u_*^{\mathcal{M}}$.

System γu_i, $i = 1, 2, \dots$, forms an orthonormal basis in $\mathcal{M}$. Thus, we can use the standard Parseval's identity in $\mathcal{M}$ [41, Theorem 2, Chapter III, Section 4] to obtain

$$\|\gamma u_*\|_H^2 = \|\gamma u_*^{\mathcal{M}}\|_H^2 = \sum_{i=1}^{\infty} |(\gamma u_*^{\mathcal{M}}, \gamma u_i)_H|^2 = \sum_{i=1}^{\infty} |(\gamma u_*, \gamma u_i)_H|^2.$$

□

The derivation of the lower bound on λ_1 is based on the method of a priori-a posteriori inequalities. This method relies on an abstract theorem proved in [22]. We formulate this theorem in the setting of this paper and for the readers' convenience we present its brief proof. Notice that in contrast to [22], Theorem 5 operates with a pair of Hilbert spaces and with a compact operator between them.

Theorem 5. *Let $\gamma : V \to H$ be a continuous, linear, and compact operator between Hilbert spaces V and H. Let $u_* \in V$ and $\lambda_* \in \mathbb{R}$ be arbitrary. Let us consider $w \in V$ such that*

$$(w, v)_V = (u_*, v)_V - \lambda_* (\gamma u_*, \gamma v)_H \quad \forall v \in V. \tag{10}$$

If $\gamma u_* \neq 0$ then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \le \frac{\|\gamma w\|_H}{\|\gamma u_*\|_H}. \tag{11}$$

Proof. Using Lemma 4 and definitions (1) and (10), we obtain

$$\begin{aligned}
\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \|\gamma u_*\|_H^2 &\le \sum_{i=1}^{\infty} \left| \frac{\lambda_i - \lambda_*}{\lambda_i} (\gamma u_*, \gamma u_i)_H \right|^2 \\
&= \sum_{i=1}^{\infty} \left| \frac{(u_i, u_*)_V}{\lambda_i} - \frac{(u_* - w, u_i)_V}{\lambda_i} \right|^2 \\
&= \sum_{i=1}^{\infty} \left| \frac{(w, u_i)_V}{\lambda_i} \right|^2 = \sum_{i=1}^{\infty} |(\gamma w, \gamma u_i)_H|^2 = \|\gamma w\|_H^2.
\end{aligned}$$

□

In order to obtain a computable lower bound on λ_1, we combine the estimate (11) with a complementarity technique, see Sections 4–6. The bounds derived by the complementarity technique depend on the particular choice of spaces V and H, but they have a common general structure. The following theorem utilizes this general structure and presents an abstract lower bound on λ_1.

Theorem 6 (Abstract complementarity estimate)**.** *Let $u_* \in V$, $\lambda_* \in \mathbb{R}$ be arbitrary and let $w \in V$ satisfy (10). Let λ_1 be the smallest eigenvalue of (1) and let the relatively closest eigenvalue to λ_* be λ_1, i.e. let*

$$\left| \frac{\lambda_1 - \lambda_*}{\lambda_1} \right| \le \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \quad \forall i = 1, 2, \dots. \tag{12}$$

Finally, let us consider $A \ge 0$ and $B \ge 0$ such that

$$\|w\|_V \le A + C_\gamma B, \tag{13}$$

where $B < \lambda_* \|\gamma u_*\|_H$ and C_γ is the optimal constant from (7). Then

$$X_2^2 \le \lambda_1 \quad \text{and} \quad C_\gamma \le 1/X_2, \tag{14}$$

where

$$X_2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4(\lambda_* - \beta)} \right), \quad \alpha = \frac{A}{\|\gamma u_*\|_H}, \quad \text{and} \quad \beta = \frac{B}{\|\gamma u_*\|_H}. \tag{15}$$

Proof. Using the fact that $C_\gamma = \lambda_1^{-1/2}$, the estimate (11), assumptions of the theorem, and inequality (7), we obtain the validity of the following relation

$$\lambda_* C_\gamma^2 - 1 = \frac{\lambda_* - \lambda_1}{\lambda_1} \le \min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \le \frac{\|\gamma w\|_H}{\|\gamma u_*\|_H} \le C_\gamma \frac{\|w\|_V}{\|\gamma u_*\|_H} \le C_\gamma \alpha + C_\gamma^2 \beta.$$

This is equivalent to the quadratic inequality

$$0 \le C_\gamma^2(\beta - \lambda_*) + C_\gamma \alpha + 1.$$

Solving it for C_γ under the assumption $\beta < \lambda_*$, we conclude that this inequality is not satisfied for $C_\gamma > 1/X_2$. Thus, C_γ has to be at most $1/X_2$. □

The particular complementarity estimates have the form (13), where the numbers A and B are obtained by an approximate minimization procedure, see Sections 4–6. The better approximation of the exact minimizer we compute the tighter bound (13) and consequently (14) we obtain. The exact minimizer yields equality in (13) and $B = 0$. Therefore, the assumption $B < \lambda_* \|\gamma u_*\|_H$ can always be satisfied by computing sufficiently accurate minimizer.

The assumption (12) is crucial and it cannot be guaranteed unless lower bounds on λ_1 and λ_2 are known. However, since the Galerkin method is known to converge [5, 6, 11] with a known speed, very accurate approximations of λ_1 and λ_2 can be computed. If these approximations are well separated then they can be used in (12) to verify its validity with a good confidence.

In order to increase this confidence, we propose a test. This test is based on the following observation. Let $\lambda_1^{\mathrm{low}} \le \lambda_1 \le \lambda^* \le \lambda_2^{\mathrm{low}} \le \lambda_2 \le \lambda_2^{\mathrm{up}}$, $D_1 = (\lambda_* - \lambda_1^{\mathrm{low}})/\lambda_1^{\mathrm{low}}$ and $D_2 = (\lambda_2^{\mathrm{low}} - \lambda_*)/\lambda_2^{\mathrm{up}}$. Then inequality $D_1 \le D_2$ implies the assumption (12). Indeed, if all these inequalities are satisfied then, clearly, $|(\lambda_1 - \lambda_*)/\lambda_1| \le D_1 \le D_2 \le |(\lambda_2 - \lambda_*)/\lambda_2| \le |(\lambda_i - \lambda_*)/\lambda_i|$ for all $i = 2, 3, \dots$ and (12) holds true.

Thus, if we knew guaranteed lower bounds λ_1^{low} and λ_2^{low} on the first eigenvalue λ_1 and the second eigenvalue λ_2 of (1), respectively, then we could use the Galerkin method to compute upper bounds λ^* and λ_2^{up}, check if $\lambda^* \leq \lambda_2^{\mathrm{low}}$ and the inequality $D_1 \leq D_2$ would then guarantee the validity of (12). However, the guaranteed lower bounds λ_1^{low} and λ_2^{low} are not available.

Therefore, in practice we propose to set $\lambda_1^{\mathrm{low}} = X_2^2$, see (14), $\lambda_2^{\mathrm{low}} = (\lambda_* + \lambda_2^{\mathrm{up}})/2$, and compute D_1 and D_2 with these values. This yields the following diagnostics indicating the validity of assumption (12). If $D_2 \leq 0$ or $D_1 > D_2$ then it is highly probable that some of the assumptions is not satisfied and the results should not be trusted. On the other hand if $D_2 > 0$ and D_1 is several times smaller than D_2 then we can have a good confidence about the validity of (12). This diagnostics performs very well in all numerical experiments presented below. At the early stages of the computations the approximations are not very precise and the diagnostics showed that the results are untrustworthy. However, at the final stages of the computations the value D_1 was at least ten times smaller than D_2 providing good confidence about the validity of (12).

4 Application to the Friedrichs' inequality

In this section we apply the above general theory to the case of Friedrichs' inequality. We will consider the variant of Friedrichs' inequality that is suitable for general symmetric second-order linear elliptic differential operators. First, we introduce differential operators of this general type in Subsection 4.1. In Subsection 4.2 we derive two-sided bounds on the optimal constant in the Friedrichs' inequality that are based on the general theory and on the complementarity technique. In Subsection 4.3 certain computational issues are discussed and finally, in Subsection 4.4 we present numerical results.

4.1 A general symmetric second-order linear elliptic operator

Let us consider a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. Let $\partial\Omega$ consist of two relatively open parts Γ_{D} and Γ_{N} such that $\partial\Omega = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$ and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. Note that we admit the case where either Γ_{D} or Γ_{N} is empty. Further, let us consider a matrix function $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}$, coefficients $c \in L^\infty(\Omega)$, and $\alpha \in L^\infty(\Gamma_{\mathrm{N}})$. Matrix $\mathcal{A}$ is assumed to be symmetric and uniformly positive definite, i.e. there exists a constant $C > 0$ such that

$$\boldsymbol{\xi}^T \mathcal{A}(x)\boldsymbol{\xi} \geq C|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d \text{ and for a.e. } x \in \Omega,$$

where $|\cdot|$ stands for the Euclidean norm. Coefficients c and α are considered to be nonnegative.

Further, we consider a subspace

$$H^1_{\Gamma_{\mathrm{D}}}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ in the sense of traces}\}$$

of the Sobolev space $H^1(\Omega)$ of $L^2(\Omega)$ functions with square integrable distributional derivatives. In what follows, we use the notation $(\cdot,\cdot)$ and $(\cdot,\cdot)_{L^2(\Gamma_{\mathrm{N}})}$ for the $L^2(\Omega)$ and $L^2(\Gamma_{\mathrm{N}})$ scalar products, respectively. Using this notation, we define a bilinear form

$$a(u,v) = (\mathcal{A}\nabla u, \nabla v) + (cu, v) + (\alpha u, v)_{L^2(\Gamma_{\mathrm{N}})}. \tag{16}$$

This bilinear form is a scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ under the conditions presented in the following lemma. Its proof follows for instance from [21, Theorem 5.11.2].

Lemma 7. *The bilinear form defined in (16) is a scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ provided that at least one of the following conditions is satisfied:*

(a) $\operatorname{meas}_{d-1}\Gamma_{\mathrm{D}} > 0$,

(b) *there exists a nonempty ball* $B \subset \Omega$ *such that* $c > 0$ *on* B,

(c) *there exists a subset* $\Gamma_\alpha \subset \Gamma_{\mathrm{N}}$ *such that* $\operatorname{meas}_{d-1}\Gamma_\alpha > 0$ *and* $\alpha > 0$ *on* Γ_α.

The equivalence of the norm induced by $a(\cdot,\cdot)$ and the standard H^1 norm yields completeness and therefore the space $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ equipped with the scalar product $a(\cdot,\cdot)$ is a Hilbert space. This enables to use the theory from Section 2. We set

$$V = H^1_{\Gamma_{\mathrm{D}}}(\Omega), \quad (u,v)_V = a(u,v), \quad H = L^2(\Omega), \quad (u,v)_H = (u,v), \tag{17}$$

and we define $\gamma : H^1_{\Gamma_{\mathrm{D}}}(\Omega) \to L^2(\Omega)$ as the identity, which is compact due to the Rellich theorem [1, Theorem 6.3]. With this setting and with the notation $\|\cdot\|_a$ for the norm induced by $a(\cdot,\cdot)$, we obtain the validity of the Friedrichs' inequality.

Theorem 8. *Let the bilinear form $a(\cdot,\cdot)$ given by (16) form a scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Then there exists a constant $C_{\mathrm{F}} > 0$ such that*

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}}\|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{18}$$

Moreover, the optimal value of this constant is $C_{\mathrm{F}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue of the following problem: find $u_i \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_i \ne 0$, and $\lambda_i \in \mathbb{R}$ such that

$$a(u_i, v) = \lambda_i(u_i, v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{19}$$

Proof. Lemma 7 guarantees that $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ is a Hilbert space with scalar product $a(\cdot,\cdot)$ given by (16). The statement then follows immediately from Theorem 3. ☐

Let us note that the most common version of Friedrichs' inequality

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}}\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

follows from (18) with $\Gamma_{\mathrm{D}} = \partial\Omega$, $\Gamma_{\mathrm{N}} = \emptyset$, $\mathcal{A}$ being identity matrix, $c = 0$, and $\alpha = 0$. As usual, we denote by $H^1_0(\Omega)$ the space $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with $\Gamma_{\mathrm{D}} = \partial\Omega$.

4.2 Two-sided bounds on Friedrichs' constant

A lower bound on the Friedrichs' constant C_{F} can be efficiently computed by the Galerkin method. We use the setting (17) and proceed as it is described in Section 3.2. The upper bound on C_{F} is obtained by the complementarity technique presented in the following theorem.

Let $\mathbf{H}(\operatorname{div},\Omega)$ stands for the space of d-dimensional vector fields with square integrable divergence and let $\boldsymbol{n}$ be the unit outward-facing normal vector to the boundary $\partial\Omega$. Further, let $\|\boldsymbol{q}\|^2_{\mathcal{A}} = (\mathcal{A}\boldsymbol{q},\boldsymbol{q})$ be a norm in $[L^2(\Omega)]^d$ induced by the matrix $\mathcal{A}$.

Theorem 9. *Let $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_* \in V$ and $\lambda_* \in \mathbb{R}$. Let the bilinear form $a(\cdot,\cdot)$ given by (16) form a scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Let $w \in V$ satisfy*

$$a(w, v) = a(u_*, v) - \lambda_*(u_*, v) \quad \forall v \in V, \tag{20}$$

where recall that $(\cdot,\cdot)$ stands for the $L^2(\Omega)$ scalar product. Then

$$\|w\|_a \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}} + C_{\mathrm{F}}\|\lambda_* u_* - c u_* + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)} \quad \forall \boldsymbol{q} \in W, \tag{21}$$

where $W = \{\boldsymbol{q} \in \mathbf{H}(\operatorname{div},\Omega) : \boldsymbol{q}\cdot\boldsymbol{n} = -\alpha u_*$ on $\Gamma_{\mathrm{N}}\}$.

Proof. Let us fix any $\boldsymbol{q} \in W$, test (20) by $v = w$ and use the divergence theorem to express

$$\|w\|_a^2 = (\mathcal{A}\nabla u_*, \nabla w) + (cu_*, w) + (\alpha u_*, w)_{L^2(\Gamma_\mathrm{N})} - \lambda_*(u_*, w) - (\boldsymbol{q}, \nabla w) - (\operatorname{div} \boldsymbol{q}, w) \\ + (\boldsymbol{q}\cdot\boldsymbol{n}, w)_{L^2(\Gamma_\mathrm{N})} = \big(\mathcal{A}(\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}), \nabla w\big) - (\lambda_* u_* - cu_* + \operatorname{div}\boldsymbol{q}, w).$$

The Cauchy–Schwarz inequality and Friedrichs' inequality (18) yield

$$\|w\|_a^2 \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_\mathcal{A}\|\nabla w\|_\mathcal{A} + C_\mathrm{F}\|\lambda_* u_* - cu_* + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}\|w\|_a.$$

Inequality $\|\nabla w\|_\mathcal{A} \le \|w\|_a$ finishes the proof. □

The estimate (21) is of type (13) with

$$A = \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_\mathcal{A}, \quad B = \|\lambda_* u_* - cu_* + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}, \tag{22}$$

and $C_\gamma = C_\mathrm{F}$. The numbers A and B can be readily computed as soon as suitable approximations λ_*, u_*, and a suitable vector field $\boldsymbol{q} \in W$ are in hand. Estimate (14) then gives a guaranteed upper bound on the Friedrichs' constant C_F.

The crucial part is the computation of suitable approximations λ_* and u_* of the smallest eigenvalue λ_1 of problem (19) and its corresponding eigenvector u_1 such that the inequality (12) is satisfied. Equally crucial is a suitable choice of the vector field $\boldsymbol{q} \in W$ in such a way that estimate (21) provides a tight upper bound on $\|w\|_a$. A possible approach and practical details about these issues are provided in the next section.

4.3 Flux reconstruction

In order to compute two-sided bounds on Friedrichs' constant, we proceed as follows. First, we use the Galerkin method, see Section 3.2, to compute an approximation λ_1^h and u_1^h of the eigenpair λ_1 and u_1 of (19). We set $\lambda_* = \lambda_1^h$, $u_* = u_1^h$, compute a suitable vector field $\boldsymbol{q} \in W$, and evaluate the numbers A and B by (22). Estimate (14) then provides the upper bound on the Friedrichs' constant C_F and the Galerkin approximation λ_1^h yields the lower bound (9) on C_F.

The key point is the computation of the suitable vector field $\boldsymbol{q}$. For simplicity, we choose a straightforward approach of approximate minimization of the upper bound (21) with respect to a suitable subset of W. First, we exploit the affine structure of W. Let us choose an arbitrary but fixed $\overline{\boldsymbol{q}} \in W$. The practical construction of this $\overline{\boldsymbol{q}}$ is a geometrical issue depending on Ω and coefficient α. It suffices to construct a vector field $\overline{\boldsymbol{q}}_1 \in \mathbf{H}(\operatorname{div}, \Omega)$ such that $\overline{\boldsymbol{q}}_1 \cdot \boldsymbol{n} = 1$ on Γ_N and a function $\overline{\alpha} \in H^1(\Omega)$ such that $\overline{\alpha} = \alpha$ on Γ_N. Let us note that usually Raviart–Thomas–Nédélec space is considered for construction of vector fields in $\mathbf{H}(\operatorname{div}, \Omega)$. Then we can simply set $\overline{\boldsymbol{q}} = -\overline{\alpha} u_* \overline{\boldsymbol{q}}_1$. This $\overline{\boldsymbol{q}}$ obviously satisfies the boundary condition $\overline{\boldsymbol{q}} \cdot \boldsymbol{n} = -\alpha u_*$ on Γ_N and if $\overline{\boldsymbol{q}}_1$ and $\overline{\alpha}$ are sufficiently smooth then $\overline{\boldsymbol{q}} \in \mathbf{H}(\operatorname{div}, \Omega)$.

In any case, having a $\overline{\boldsymbol{q}} \in W$, we can express the affine space W as $W = \overline{\boldsymbol{q}} + W_0$, where

$$W_0 = \{\boldsymbol{q} \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{q}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_\mathrm{N}\}$$

is already a linear space. The idea is to minimize the upper bound (21) over the set $\overline{\boldsymbol{q}} + W_0^h$, where $W_0^h \subset W_0$ is a finite dimensional subspace. However, the right-hand side of (21) is a nonlinear functional in $\boldsymbol{q}$ and, thus, in order to simplify the computation, we use the idea from [32] and approximate it by a quadratic functional. We rewrite inequality (21) using the notation (22) and apply an elementary estimate:

$$\|w\|_a^2 \le (A + C_\mathrm{F}B)^2 \le (1 + \varrho^{-1})A^2 + (1+\varrho)C_\mathrm{F}^2 B^2 \quad \forall \varrho > 0.$$

The right-hand side of this inequality is already a quadratic functional for a fixed ϱ, but the exact value of C_{F} is unknown in general. However, it is sufficient to find an approximate minimizer only. Therefore we approximate C_{F} by the available value $(\lambda_1^h)^{-1/2}$. This leads us to the minimization of

$$(1+\varrho^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}}^2 + (1+\varrho)(\lambda_1^h)^{-1}\|\lambda_1^h u_1^h - c u_1^h + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}^2 \tag{23}$$

over the affine set $\overline{\boldsymbol{q}}+W_0^h$ with a fixed value of $\varrho > 0$. This minimization problem is equivalent to seeking $\boldsymbol{q}_0^h \in W_0^h$ satisfying

$$\mathcal{B}(\boldsymbol{q}_0^h, \boldsymbol{w}_0^h) = \mathcal{F}(\boldsymbol{w}_0^h) - \mathcal{B}(\overline{\boldsymbol{q}}, \boldsymbol{w}_0^h) \quad \forall \boldsymbol{w}_0^h \in W_0^h, \tag{24}$$

where

$$\mathcal{B}(\boldsymbol{q},\boldsymbol{w}) = (\operatorname{div}\boldsymbol{q}, \operatorname{div}\boldsymbol{w}) + \frac{\lambda_1^h}{\varrho}(\mathcal{A}^{-1}\boldsymbol{q},\boldsymbol{w}),$$
$$\mathcal{F}(\boldsymbol{w}) = \frac{\lambda_1^h}{\varrho}(\nabla u_1^h, \boldsymbol{w}) - (\lambda_1^h u_1^h - c u_1^h, \operatorname{div}\boldsymbol{w}).$$

The computed vector field $\boldsymbol{q}^h = \overline{\boldsymbol{q}} + \boldsymbol{q}_0^h \in W$ is then used in (22) to evaluate A and B and consequently the two-sided bounds $C_{\mathrm{F}}^{\mathrm{low}} \le C_{\mathrm{F}} \le C_{\mathrm{F}}^{\mathrm{up}}$, where

$$C_{\mathrm{F}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2} \quad \text{and} \quad C_{\mathrm{F}}^{\mathrm{up}} = 1/X_2, \tag{25}$$

X_2 is given by (15) with $\gamma u_* = u_1^h$, see (9) and (14).

Note that problem (24) can be naturally approached by standard Raviart–Thomas finite elements [13]. Further we note that a particular value of the constant ϱ can influence the accuracy of the final bound. However, this influence was minor in all cases we numerically tested and the natural value $\varrho = 1$ yielded accurate results. If necessary a simultaneous minimization of (23) with respect to both $\varrho > 0$ and $\boldsymbol{q} \in \overline{\boldsymbol{q}} + W_0^h$ can be performed.

4.4 Numerical experiment

In order to illustrate the capabilities of the above described approach for computation of two-sided bounds on Friedrichs' constant, we present numerical results showing the dependence of Friedrichs' constant on piecewise constant values of $\mathcal{A}$. We consider the general setting from Section 4.1 and in particular we set $\Omega = (-1,1)^2$, $\Gamma_{\mathrm{N}} = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 1 \text{ and } -1 < x_2 < 1\}$, $\Gamma_{\mathrm{D}} = \partial\Omega \setminus \overline{\Gamma}_{\mathrm{N}}$, $c = 0$, and $\alpha = 0$. The matrix $\mathcal{A}$ is piecewise constant, defined as $\mathcal{A}(x_1,x_2) = I$ for $x_1 x_2 \le 0$ and $\mathcal{A}(x_1,x_2) = \tilde{a} I$ for $x_1 x_2 > 0$, where $I \in \mathbb{R}^{d\times d}$ stands for the identity matrix and the value of the constant $\tilde{a}$ is specified below.

We employ the standard lowest-order (piecewise linear) triangular finite elements to discretize eigenvalue problem (19). Thus, we consider a triangular mesh $\mathcal{T}_h$ and seek Galerkin solution of problem (19) in space

$$V^h = \{v^h \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : v^h|_K \in P^1(K),\ \forall K \in \mathcal{T}_h\}, \tag{26}$$

where $P^1(K)$ denotes the space of affine functions on triangle K, see Section 3.2. We point out that the discontinuity of $\mathcal{A}(x_1,x_2)$ causes a strong singularity of the eigenvectors at the origin. Therefore, we employ a standard adaptive algorithm, see Algorithm 1, and construct adaptively refined meshes in order to approximate the singularity well. We use the localized version of (23) to define the error indicators

$$\eta_K^2 = (1+\varrho^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A},K}^2 + (1+\varrho)(\lambda_1^h)^{-1}\|\lambda_1^h u_1^h - c u_1^h + \operatorname{div}\boldsymbol{q}\|_{L^2(K)}^2 \tag{27}$$

Adaptive algorithm

Step 1 Construct an initial mesh $\mathcal{T}_h$.

Step 2 Use the space (26) and compute Galerkin approximations $\lambda_1^h \in \mathbb{R}$ and $u_1^h \in V^h$ of the smallest eigenvalue and the corresponding eigenvector of problem (19), see (8).

Step 3 Use the space of second-order Raviart–Thomas [13] finite elements $W_0^h = \{\boldsymbol{w}_h \in W_0 : \boldsymbol{w}_h \in [P^2(K)]^2,\ \forall K \in \mathcal{T}_h\}$ and solve (24).

Step 4 Evaluate two-sided bounds (25). Set $C_{\mathrm{F}}^{\mathrm{avg}} = (C_{\mathrm{F}}^{\mathrm{up}} + C_{\mathrm{F}}^{\mathrm{low}})/2$, $E_{\mathrm{REL}} = (C_{\mathrm{F}}^{\mathrm{up}} - C_{\mathrm{F}}^{\mathrm{low}})/C_{\mathrm{F}}^{\mathrm{avg}}$, and stop the algorithm as soon as $E_{\mathrm{REL}} \leq E_{\mathrm{TOL}}$.

Step 5 Compute error indicators (27) for all $K \in \mathcal{T}_h$ and sort them in descending order: $\eta_{K_1} \geq \eta_{K_2} \geq \cdots \geq \eta_{K_{N_{\mathrm{el}}}}$, where N_{el} is the number of elements in $\mathcal{T}_h$.

Step 6 (Bulk criterion.) Find the smallest n such that $\theta^2 \sum_{i=1}^{N_{\mathrm{el}}} \eta_{K_i}^2 \leq \sum_{i=1}^{n} \eta_{K_i}^2$, where $\theta \in (0,1)$ is a parameter.

Step 7 Construct a new mesh $\mathcal{T}_h$ by refining elements $K_1, K_2, \ldots, K_n$.

Step 8 Go to Step 2.

Algorithm 1: Mesh adaptation algorithm for two-sided bounds on Friedrichs' constant.

$\tilde{a}$	$C_{\mathrm{F}}^{\mathrm{low}}$	$C_{\mathrm{F}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	9.0086	9.0939	0.0094	4 832
0.01	2.8697	2.8971	0.0095	5 003
0.1	1.0035	1.0124	0.0088	7 866
1	0.5693	0.5743	0.0086	4 802
10	0.3173	0.3201	0.0088	7 866
100	0.2870	0.2897	0.0095	5 003
1000	0.2849	0.2876	0.0094	4 832

Table 1: Friedrichs' constant. The lower bound $C_{\mathrm{F}}^{\mathrm{low}}$, upper bound $C_{\mathrm{F}}^{\mathrm{up}}$, relative error E_{REL}, and the number of degrees of freedom $N_{\mathrm{DOF}} = \dim V^h$ for particular values of $\tilde{a}$.

for all $K \in \mathcal{T}_h$, where $\|\boldsymbol{w}\|_{\mathcal{A},K}^2 = (\mathcal{A}\boldsymbol{w}, \boldsymbol{w})_{L^2(K)}$.

The unknown value of Friedrichs' constant lies between bounds $C_{\mathrm{F}}^{\mathrm{low}}$ and $C_{\mathrm{F}}^{\mathrm{up}}$ computed in Step 4 of Algorithm 1. These bounds are utilized for the relative error stopping criterion. When the algorithm stops, the relative error is guaranteed to be at most the given tolerance E_{TOL}.

In this particular numerical experiment we have chosen the initial mesh with eight triangles as shown in Figure 1. The marking parameter in Step 6 and the tolerance for the relative error in Step 4 were chosen as $\theta = 0.75$ and $E_{\mathrm{TOL}} = 0.01$. Further, we naturally set $\varrho = 1$.

The results for a series of values of $\tilde{a}$ are summarized in Table 1. For each particular value of $\tilde{a}$, we run the adaptive algorithm until the relative error drops below $E_{\mathrm{TOL}} = 0.01$. This error level was reached in all cases using several thousands of degrees of freedom. Notice the considerable dependence of the optimal value of Friedrichs' constant on $\tilde{a}$. The values for $\tilde{a} = 0.001$ and $\tilde{a} = 1000$ differ more than thirty times. Further notice that the exact value of Friedrichs' constant for $\tilde{a} = 1$ is $C_{\mathrm{F}} = 4/(\pi\sqrt{5}) \approx 0.5694$.

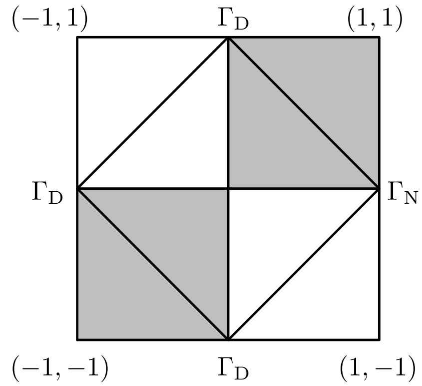

Figure 1: The initial mesh for the adaptive algorithm. The Dirichlet and Neumann parts of the boundary are indicated as well as the piecewise constant matrix $\mathcal{A}$: $\mathcal{A} = I$ in white elements and $\mathcal{A} = \tilde{a}I$ in gray elements.

5 Application to the Poincaré inequality

5.1 Poincaré inequality and the optimal constant

In this section we consider the case when none of conditions (a)–(c) of Lemma 7 is satisfied. Therefore, we assume the general symmetric second-order elliptic operator as described in Section 4.1 with $c = 0$, $\alpha = 0$, $\Gamma_{\mathrm{D}} = \emptyset$, and $\Gamma_{\mathrm{N}} = \partial\Omega$. We apply the general theory of Section 3 with $V = \widetilde{H}^1(\Omega) = \{v \in H^1(\Omega) : (v, 1) = 0\}$, $(u, v)_V = a(u, v) = (\mathcal{A}\nabla u, \nabla v)$, $H = L^2(\Omega)$, and $(u, v)_H = (u, v)$. It is an easy exercise to verify that $a(\cdot, \cdot)$ forms a scalar product on $\widetilde{H}^1(\Omega)$ and that it induces a norm $\|\cdot\|_a$ equivalent to the standard H^1-seminorm and H^1-norm. The operator $\gamma : V \to H$ is simply the identity mapping $I : \widetilde{H}^1(\Omega) \to L^2(\Omega)$. This mapping is clearly compact, because the identity mapping from $\widetilde{H}^1(\Omega)$ to $H^1(\Omega)$ is linear and continuous and the identity mapping from $H^1(\Omega)$ to $L^2(\Omega)$ is compact due to the Rellich theorem [1, Theorem 6.3]. This setting enables to use the general result of Theorem 3 and obtain the following result.

Theorem 10. *There exists a constant $C_{\mathrm{P}} > 0$ such that*

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{P}} \|v\|_a \quad \forall v \in \widetilde{H}^1(\Omega). \tag{28}$$

Moreover, the optimal value of this constant is $C_{\mathrm{P}} = \lambda_1^{-1/2}$, where λ_1 is the smallest positive *eigenvalue of the following problem: find $u_i \in H^1(\Omega)$, $u_i \neq 0$, and $\lambda_i \in \mathbb{R}$ such that*

$$a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in H^1(\Omega). \tag{29}$$

Proof. The inequality (28) follows immediately from Theorem 3. This theorem also implies that the optimal constant is $C_{\mathrm{P}} = \tilde{\lambda}_1^{-1/2}$, where $\tilde{\lambda}_1$ is the smallest eigenvalue of the following problem: find $\tilde{u}_i \in \widetilde{H}^1(\Omega)$, $\tilde{u}_i \neq 0$, and $\tilde{\lambda}_i \in \mathbb{R}$ such that

$$a(\tilde{u}_i, v) = \tilde{\lambda}_i (\tilde{u}_i, v) \quad \forall v \in \widetilde{H}^1(\Omega). \tag{30}$$

Notice that $0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots$. Similarly, the eigenvalues of (29) satisfy $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$ and the zero eigenvalue corresponds to a constant eigenvector $u_0 = 1$. It can be easily shown that $\tilde{\lambda}_i = \lambda_i$ and $\tilde{u}_i = u_i$ for all $i = 1, 2, \dots$. Thus, the smallest eigenvalue $\tilde{\lambda}_1$ of (30) is equal to the smallest *positive* eigenvalue λ_1 of (29) and the proof is finished. $\square$

Let us note that the Poincaré inequality (28) can be equivalently formulated as

$$\|v - \tilde{v}\|_{L^2(\Omega)} \le C_{\mathrm{P}} \|v\|_a \quad \forall v \in H^1(\Omega), \quad \tilde{v} = (v,1)/|\Omega|. \tag{31}$$

Further, it is clear that the common version of the Poincaré inequality

$$\|v - \tilde{v}\|_{L^2(\Omega)} \le C_{\mathrm{P}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega)$$

follows from (31) and consequently from (28) with $\mathcal{A}$ being the identity matrix.

5.2 Two-sided bounds on the Poincaré constant

The lower bound on the Poincaré constant C_{P} can be computed in the standard way by the Galerkin method, see Section 3.2. The only difference is that here we compute the approximation λ_1^h of the second smallest (the smallest positive) eigenvalue of (29), because the smallest eigenvalue is $\lambda_0 = 0$. In order to compute the upper bound, we employ the complementarity technique as follows.

Theorem 11. *Let $V = \widetilde{H}^1(\Omega)$, $u_* \in V$, and $\lambda_* \in \mathbb{R}$. Let $w \in V$ satisfy*

$$a(w,v) = a(u_*,v) - \lambda_*(u_*,v) \quad \forall v \in V. \tag{32}$$

Then

$$\|w\|_a \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}} + C_{\mathrm{P}} \|\lambda_* u_* + \operatorname{div} \boldsymbol{q}\|_{L^2(\Omega)} \quad \forall \boldsymbol{q} \in W_0, \tag{33}$$

where $W_0 = \{\boldsymbol{q} \in \mathbf{H}(\operatorname{div},\Omega) : \boldsymbol{q}\cdot\boldsymbol{n} = 0$ on $\partial\Omega\}$.

Proof. Fixing arbitrary $\boldsymbol{q} \in W_0$, testing (32) by $v = w$ and using the divergence theorem, we obtain

$$\|w\|_a^2 = (\mathcal{A}\nabla u_*, \nabla w) - \lambda_*(u_*,w) - (\boldsymbol{q}, \nabla w) - (\operatorname{div}\boldsymbol{q}, w)$$
$$= \big(\mathcal{A}(\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}), \nabla w\big) - (\lambda_* u_* + \operatorname{div}\boldsymbol{q}, w).$$

The Cauchy–Schwarz inequality and Poincaré inequality (28) yield

$$\|w\|_a^2 \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}} \|\nabla w\|_{\mathcal{A}} + C_{\mathrm{P}} \|\lambda_* u_* + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)} \|w\|_a.$$

Since $\|\nabla w\|_{\mathcal{A}} = \|w\|_a$, the proof is finished. ☐

We observe that complementarity estimate (33) is of type (13) with

$$A = \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}}, \quad B = \|\lambda_* u_* + \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}, \tag{34}$$

and $C_\gamma = C_{\mathrm{P}}$. As soon as a suitable vector field $\boldsymbol{q} \in W_0$ is available, the numbers A and B can be computed and used in (14)–(15) to obtain a guaranteed upper bound on the Poincaré constant C_{P}.

A straightforward approach for computing a suitable vector field $\boldsymbol{q}^h \in W_0$ was described in Section 4.3 for the case of Friedrichs' constant. It can be directly used also for the Poincaré constant. It is even simpler, because $c = 0$, $\alpha = 0$, and $\Gamma_{\mathrm{N}} = \partial\Omega$. The only difference is that the approximation λ_1^h of the second smallest (the first positive) eigenvalue of (29) has to be used. In particular, the approximation $\boldsymbol{q}^h \in W_0$ is computed using (24). This vector field is then used in (34) to evaluate A and B and consequently the two-sided bounds $C_{\mathrm{P}}^{\mathrm{low}} \le C_{\mathrm{P}} \le C_{\mathrm{P}}^{\mathrm{up}}$, where

$$C_{\mathrm{P}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2} \quad \text{and} \quad C_{\mathrm{P}}^{\mathrm{up}} = 1/X_2, \tag{35}$$

X_2 is given by (15) with $\gamma u_* = u_1^h$ being the approximate eigenvector of (29) corresponding to λ_1^h, see also (9) and (14).

$\tilde{a}$	$C_{\mathrm{P}}^{\mathrm{low}}$	$C_{\mathrm{P}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	14.2390	14.3690	0.0091	3 400
0.01	4.5199	4.5623	0.0093	3 510
0.1	1.4849	1.4989	0.0094	4 382
1	0.6365	0.6424	0.0092	3 009
10	0.4696	0.4740	0.0094	4 382
100	0.4520	0.4562	0.0093	3 510
1000	0.4503	0.4544	0.0091	3 400

Table 2: Poincaré constant. The lower bound $C_{\mathrm{P}}^{\mathrm{low}}$, upper bound $C_{\mathrm{P}}^{\mathrm{up}}$, relative error E_{REL}, and the number of degrees of freedom $N_{\mathrm{DOF}} = \dim V^h$ for particular values of $\tilde{a}$.

5.3 Numerical experiment

We consider the same setting as in Section 4.4. The only difference is that in the case of Poincaré constant we assume $\Gamma_{\mathrm{N}} = \partial\Omega$ and $\Gamma_{\mathrm{D}} = \emptyset$. We use the adaptive algorithm as before (Algorithm 1) with clear modifications. We use error indicators (27), where λ_1^h and u_1^h are the Galerkin approximations of the second smallest (the smallest positive) eigenvalue λ_1 of (29) and its corresponding eigenvector u_1. The relative error in Step 4 of Algorithm 1 is computed using two-sided bounds (35).

The obtained two-sided bounds for a series of values of $\tilde{a}$ are presented in Table 2. The guaranteed 0.01 relative error tolerance was reached in all cases using several thousands degrees of freedom. As in the case of Friedrichs' constant, we observe considerable dependence of the Poincaré constant C_{P} on $\tilde{a}$. Finally, we point out that the exact value of the Poincaré constant for $\tilde{a} = 1$ is $C_{\mathrm{P}} = 2/\pi \approx 0.6366$.

6 Application to the trace inequality

6.1 Trace inequality and the optimal constant

In order to apply the general theory from Sections 2–3 to the case of the trace inequality, we consider the same general symmetric second-order elliptic operator as in Section 4.1. In addition we assume $\operatorname{meas}_{d-1} \Gamma_{\mathrm{N}} > 0$. The general theory is applied with $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $(u,v)_V = a(u,v)$, $H = L^2(\Gamma_{\mathrm{N}})$, and $(u,v)_H = (u,v)_{L^2(\Gamma_{\mathrm{N}})}$. The operator $\gamma : V \to H$ is the standard trace operator. Its compactness and other properties are provided in [21, Theorem 6.10.5], see also [10]. The general result from Theorem 3 then translates as follows.

Theorem 12. *Let the bilinear form $a(\cdot,\cdot)$ given by (16) form a scalar product in $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Then there exists a constant $C_{\mathrm{T}} > 0$ such that*

$$\|v\|_{L^2(\Gamma_{\mathrm{N}})} \le C_{\mathrm{T}} \|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{36}$$

Moreover, the optimal value of this constant is $C_{\mathrm{T}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue of problem: find $u_i \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_i \neq 0$, and $\lambda_i \in \mathbb{R}$ such that

$$a(u_i, v) = \lambda_i (u_i, v)_{L^2(\Gamma_{\mathrm{N}})} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{37}$$

Proof. The statement follows immediately from Lemma 7 and Theorem 3. □

Clearly, the common version of the trace inequality

$$\|v\|_{L^2(\partial\Omega)} \le C_{\mathrm{T}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega)$$

follows from (36) with $\Gamma_{\mathrm{D}} = \emptyset$, $\Gamma_{\mathrm{N}} = \partial\Omega$, $\mathcal{A}$ being the identity matrix, $c = 0$, and $\alpha = 0$.

6.2 Two-sided bounds on the trace constant

As in the case of Friedrichs' inequality, we compute the lower bound of the optimal value for the constant C_{T} by the Galerkin method, see Section 3.2. In order to compute an upper bound on C_{T} we employ the complementarity technique as follows.

Theorem 13. *Let $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega)$, $u_* \in V$ and $\lambda_* \in \mathbb{R}$. Let the bilinear form $a(\cdot,\cdot)$ given by (16) form a scalar product in V. Let $w \in V$ satisfy*

$$a(w, v) = a(u_*, v) - \lambda_*(u_*, v)_{L^2(\Gamma_{\mathrm{N}})} \quad \forall v \in V. \tag{38}$$

Then

$$\|w\|_a \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}} + C_{\mathrm{F}}\|cu_* - \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)} + C_{\mathrm{T}}\|\alpha u_* - \lambda_* u_* + \boldsymbol{q}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\mathrm{N}})} \tag{39}$$

for all $\boldsymbol{q} \in \mathbf{H}(\operatorname{div},\Omega)$.

Proof. Let us fix any $\boldsymbol{q} \in \mathbf{H}(\operatorname{div},\Omega)$, test (38) by $v = w$ and use the divergence theorem to express

$$\begin{aligned}\|w\|_a^2 &= (\mathcal{A}\nabla u_*, \nabla w) + (cu_*, w) + (\alpha u_*, w)_{L^2(\Gamma_{\mathrm{N}})} - \lambda_*(u_*, w)_{L^2(\Gamma_{\mathrm{N}})}\\ &\qquad - (\boldsymbol{q}, \nabla w) - (\operatorname{div}\boldsymbol{q}, w) + (\boldsymbol{q}\cdot\boldsymbol{n}, w)_{L^2(\Gamma_{\mathrm{N}})}\\ &= \big(\mathcal{A}(\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}), \nabla w\big) + (cu_* - \operatorname{div}\boldsymbol{q}, w) + (\alpha u_* - \lambda_* u_* + \boldsymbol{q}\cdot\boldsymbol{n}, w)_{L^2(\Gamma_{\mathrm{N}})}.\end{aligned}$$

The Cauchy–Schwarz inequality, Friedrichs' inequality (18), and trace inequality (36) yield

$$\begin{aligned}\|w\|_a^2 \le \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}}\|\nabla w\|_{\mathcal{A}} + C_{\mathrm{F}}\|cu_* - \operatorname{div}\boldsymbol{q}\|_0\|w\|_a\\ + C_{\mathrm{T}}\|\alpha u_* - \lambda_* u_* + \boldsymbol{q}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\mathrm{N}})}\|w\|_a.\end{aligned}$$

Inequality $\|\nabla w\|_{\mathcal{A}} \le \|w\|_a$ finishes the proof. □

As in the case of Friedrichs' inequality, the bound (39) is of the type (13) with

$$A = \|\nabla u_* - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}} + C_{\mathrm{F}}\|cu_* - \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}, \quad B = \|\alpha u_* - \lambda_* u_* + \boldsymbol{q}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\mathrm{N}})} \tag{40}$$

and $C_\gamma = C_{\mathrm{T}}$. Let us note that the complementarity estimate (39) is just one out of several possibilities. This bound comes from [32] and contains the Friedrichs' constant C_{F}. Instead of its exact value an upper bound as computed in Section 4 can be used here. However, there exist other variants of the complementarity technique that can be used to obtain a bound on $\|w\|_a$ without the Friedrichs' constant C_{F}. See for example [2, 12, 35, 38].

A suitable vector field $\boldsymbol{q} \in \mathbf{H}(\operatorname{div},\Omega)$ is computed by approximate minimization of the right-hand side of (39). In a similar way as we obtained the functional (23), we obtain the quadratic functional

$$\begin{aligned}(1+\varrho^{-1})(1+\sigma^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A}}^2 + (1+\varrho)(1+\sigma^{-1})(C_{\mathrm{F}}^{\mathrm{up}})^2\|cu_1^h - \operatorname{div}\boldsymbol{q}\|_{L^2(\Omega)}^2\\ + (1+\sigma)(\lambda_1^h)^{-1}\|\alpha u_1^h - \lambda_1^h u_1^h + \boldsymbol{q}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\mathrm{N}})}^2 \quad \forall\varrho > 0,\ \sigma > 0,\end{aligned} \tag{41}$$

where $C_{\mathrm{F}}^{\mathrm{up}}$ is an upper bound of C_{F} computed as described in Sections 4.2–4.3, λ_1^h is the approximation of the smallest eigenvalue of (37) obtained by the Galerkin method and u_1^h is the corresponding approximate eigenvector. We look for the minimum of (41) over a finite dimensional subspace W^h of $\mathbf{H}(\operatorname{div},\Omega)$. This minimization problem is equivalent to seeking $\boldsymbol{q}^h \in W^h$ such that

$$\mathcal{B}(\boldsymbol{q}^h, \boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in W^h,$$

$\tilde{a}$	$C_{\mathrm{T}}^{\mathrm{low}}$	$C_{\mathrm{T}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	17.8110	17.9760	0.0092	5 523
0.01	5.6490	5.7047	0.0098	5 418
0.1	1.8433	1.8593	0.0086	7 775
1	0.7963	0.8033	0.0088	5 499
10	0.5829	0.5880	0.0086	7 775
100	0.5649	0.5705	0.0098	5 421
1000	0.5632	0.5685	0.0092	5 523

Table 3: Trace constant. The lower bound $C_{\mathrm{T}}^{\mathrm{low}}$, upper bound $C_{\mathrm{T}}^{\mathrm{up}}$, relative error E_{REL}, and the number of degrees of freedom $N_{\mathrm{DOF}} = \dim V^h$ for particular values of $\tilde{a}$.

where

$$\mathcal{B}(\boldsymbol{q},\boldsymbol{w}) = \frac{1+\varrho}{\sigma}\lambda_1^h (C_{\mathrm{F}}^{\mathrm{up}})^2(\operatorname{div}\boldsymbol{q},\operatorname{div}\boldsymbol{w}) + \frac{1+\varrho}{\varrho\sigma}\lambda_1^h(\mathcal{A}^{-1}\boldsymbol{q},\boldsymbol{w}) + (\boldsymbol{q}\cdot\boldsymbol{n},\boldsymbol{w}\cdot\boldsymbol{n})_{L^2(\Gamma_{\mathrm{N}})},$$
$$\mathcal{F}(\boldsymbol{w}) = \frac{1+\varrho}{\varrho\sigma}\lambda_1^h(\nabla u_1^h,\boldsymbol{w}) + \frac{1+\varrho}{\sigma}\lambda_1^h(C_{\mathrm{F}}^{\mathrm{up}})^2(cu_1^h,\operatorname{div}\boldsymbol{w}) + (\lambda_1^h u_1^h - \alpha u_1^h,\boldsymbol{w}\cdot\boldsymbol{n})_{L^2(\Gamma_{\mathrm{N}})}.$$

Practically, the classical Raviart–Thomas finite element method [13] can be used to solve this problem. Note that the natural values for ϱ and σ are $\varrho = 1$ and $\sigma = 2$, because then $(1+\varrho^{-1})(1+\sigma^{-1}) = (1+\varrho)(1+\sigma^{-1}) = (1+\sigma) = 3$, see (41). Similarly to the case of Friedrichs' constant, these natural values often yield accurate results. If not, a simultaneous minimization of (41) with respect to $\varrho > 0$, $\sigma > 0$, and $\boldsymbol{q} \in W^h$ can be performed.

The computed vector field $\boldsymbol{q}^h \in W^h$ is then substituted to (40) to get A and B and consequently the two-sided bounds $C_{\mathrm{T}}^{\mathrm{low}} \le C_{\mathrm{T}} \le C_{\mathrm{T}}^{\mathrm{up}}$, where

$$C_{\mathrm{T}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2} \quad \text{and} \quad C_{\mathrm{T}}^{\mathrm{up}} = 1/X_2, \tag{42}$$

X_2 is given by (15) with $\gamma u_* = u_1^h$, see (9) and (14).

6.3 Numerical experiment

To illustrate the numerical performance of the above described method, we consider the same example as in Section 4.4. We proceed in the same way as in Section 4.4 with clear modifications in order to compute two-sided bounds (42) on the trace constant. As before, the adaptive algorithm is steered by error indicators that are defined by a localized version of (41):

$$\eta_K^2 = (1+\varrho^{-1})(1+\sigma^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\boldsymbol{q}\|_{\mathcal{A},K}^2 + (1+\varrho)(1+\sigma^{-1})(C_{\mathrm{F}}^{\mathrm{up}})^2\|cu_1^h - \operatorname{div}\boldsymbol{q}\|_{L^2(K)}^2$$
$$+ (1+\sigma)(\lambda_1^h)^{-1}\|\alpha u_1^h - \lambda_1^h u_1^h + \boldsymbol{q}\cdot\boldsymbol{n}\|_{L^2(\partial K\cap\Gamma_{\mathrm{N}})}^2 \quad \forall K \in \mathcal{T}_h.$$

The parameters ϱ and σ are naturally chosen as $\varrho = 1$ and $\sigma = 2$. The values of the upper bound $C_{\mathrm{F}}^{\mathrm{up}}$ are taken from Table 1.

The obtained results are presented in Table 3. The method succeeded in obtaining guaranteed two-sided bounds on the trace constant with a relative error at most 0.01 in all cases using several thousands of degrees of freedom. The particular value of the trace constant C_{T} depends considerably on $\tilde{a}$. The values for $\tilde{a} = 0.001$ and $\tilde{a} = 1000$ differ more than thirty times. Finally, notice that the exact value of the trace constant for $\tilde{a} = 1$ is $C_{\mathrm{T}} = (2/(\pi\coth\pi))^{1/2} \approx 0.7964$.

7 Conclusions

We present a method for computing guaranteed lower and upper bounds of principal eigenvalues of elliptic operators and consequently for computing guaranteed two-sided bounds of the optimal constants in Friedrichs', Poincaré, trace, and similar inequalities. The bounds are guaranteed provided there are no round-off errors and all integrals are evaluated exactly. Further, the bounds are guaranteed only if the domain Ω is represented exactly by used finite elements. Furthermore, the upper bounds on eigenvalues computed by the Galerkin method are guaranteed only if the corresponding matrix-eigenvalue problems are solved exactly. On the other hand, the lower bounds on eigenvalues obtained by the complementarity technique are guaranteed even if the matrix-eigenvalue problems and linear algebraic systems are solved approximately only. In any case, the crucial assumption for having guaranteed lower bounds on eigenvalues is (12).

These two-sided bounds can be of interest if the corresponding eigenvalue problem cannot be solved analytically and if analytical estimates are not available or they are too inaccurate. In particular, this is the case of complicated geometry of the domain Ω, mixed boundary conditions, presence of non-constant and/or anisotropic diffusion coefficient $\mathcal{A}$, presence of reaction coefficient c, and presence of coefficient α.

The method is quite general and it can be used for a wide variety of problems. The general Hilbert space setting presented in Section 2 enables a variety of applications including linear elasticity. We believe that this approach can be further generalized. Nonlinear eigenvalue problems and nonsymmetric operators can be of particular interest and generalizations in these directions can be subject for further research.

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