On planar flows of viscoelastic fluids of Burgers type*

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Abstract

Viscoelastic rate-type fluid models involving the stress and its observer-invariant time derivatives of higher order are used to describe the behaviour of materials with complex microstructure, for example geomaterials like asphalt, biomaterials such as vitreous in the eye, synthetic rubbers such as SBR (styrene butadiene rubber). A standard model that belongs to the category of viscoelastic rate-type fluid models of the second order is the model due to Burgers, which can be viewed as a mixture of two Oldroyd–B models of the first order. This viewpoint allows one to develop the whole hierarchy of generalized models of a Burgers type. We study one such generalization. Carrying on the study by Masmoudi [1], where he made a sketch of the proof of weak sequential stability of (hypothetical) weak solutions to the so called Giesekus model, we prove long time and large data existence of weak solutions to a Burgers-type model that can be written as a mixture of two Giesekus models in two spatial dimensions.

1. Introduction

Viscoelastic rate-type fluid models involving the stress and its observer-invariant time derivatives of higher order are used to describe the behaviour of materials with complex microstructure. This is due to the fact that higher order viscoelastic rate-type fluid models are capable of capturing several different relaxation mechanisms (as well as other non-Newtonian phenomena). Geomaterials such as asphalt, biomaterials such as vitreous in the eye, synthetic rubbers such as SBR (styrene butadiene rubber), can serve as examples, see Monismith, Secor [2], Narayan et al. [3], Málek, Rajagopal, Tůma [4], Sharif-Kashani et al. [5], Řehoř et al. [6] for experimental data and for corroboration this data using higher order viscoelastic rate type fluid models. A standard model belonging to the category of viscoelastic rate type fluids of the second order is the model due to Burgers. Burgers [7] developed a one dimensional model; its d-dimensional variant, $d \geq 2$, can be written (see e.g. [8]) as the following system of equations satisfied in

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$Q_T := (0,T)\times\Omega$, where $T>0$ is a fixed number and $\Omega\subset\mathbb{R}^d$ is a domain:

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0, && (1.1)\\ \rho\left(\partial_t\boldsymbol{v}+\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\right) &= \operatorname{div}\mathbb{T}+\rho\boldsymbol{f}, && (1.2)\\ \mathbb{T} &= -p\mathbb{I}+2\nu\mathbb{D}+\mathbb{S}, && (1.3)\\ \overset{\triangledown\triangledown}{\mathbb{S}}+\alpha_1\overset{\triangledown}{\mathbb{S}}+\alpha_0\mathbb{S} &= \beta_1\overset{\triangledown}{\mathbb{D}}+\beta_0\mathbb{D}. && (1.4) \end{aligned}$$

Here, ∂_t denotes the partial time derivative, $\nabla=(\partial_{x_1},...,\partial_{x_d})$ denotes the gradient with respect to the space variables, the operator div denotes the divergence with respect to the space variables, i.e.

$$\operatorname{div}\boldsymbol{u}=\sum_{j=1}^{d}\partial_{x_j}u_j$$

for any vector function $\boldsymbol{u}=(u_1,...,u_d)$. Next, $\boldsymbol{v}$ is the velocity, $\mathbb{D}:=\frac{1}{2}\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\right)$ is the symmetric part of the velocity gradient, $\mathbb{T}$ is the Cauchy stress tensor, $\mathbb{I}$ is the identity tensor and p (often called the pressure) is a scalar quantity associated with the fact that the fluid is incompressible, i.e. with the constraint (1.1). The given vector function $\boldsymbol{f}$ represents the external forces acting on the body, the parameter $\rho>0$ stands for the density, 2ν, α_0, α_1, β_0, β_1 are positive material coefficients. Finally, for any tensor $\mathbb{A}$, the nonlinear differential operator $\overset{\triangledown}{\mathbb{A}}$ stands for

$$\overset{\triangledown}{\mathbb{A}}:=\partial_t\mathbb{A}+\sum_{j=1}^{d}v_j\partial_{x_j}\mathbb{A}-\nabla\boldsymbol{v}\mathbb{A}-\mathbb{A}(\nabla\boldsymbol{v})^T,$$

where $(\nabla\boldsymbol{v})^T$ denotes the transpose of $\nabla\boldsymbol{v}$, and $\overset{\triangledown\triangledown}{\mathbb{A}}:=\overset{\triangledown}{\overset{\triangledown}{\mathbb{A}}}$. This article concerns a robust PDE analysis of equations describing the mechanical behaviour of viscoelastic rate-type fluid models of a Burgers type (of models that come from simmilar thermodynamical principles as (1.1)–(1.4)). By a robust PDE analysis we mean the development of mathematical results for any regular data (domain, time interval, boundary and initial data, external forces, material coefficients). Obviously, the system (1.1) – (1.4) is much more complicated than the incompressible Navier-Stokes equations. Due to the structure of the nonlinear equation (1.4), for example it does not contain any diffusion term (second order spatial differential operator), it is nontrivial to achieve apriori estimates for the unknowns $\boldsymbol{v}$ and $\mathbb{S}$ controlled by data of the problem. Also, due to the presence of the second order time derivative of $\mathbb{S}$, one should assign not only initial data for $\mathbb{S}$, but also for $\partial_t\mathbb{S}$, which again seems uneasy task (from the point of view of physical interpretation, see [9]). There are additional more general questions (such as how to appropriately extend this model to compressible setting or how to appropriately include thermal effects) that call for a detailed understanding of physical underpinings of the governing equations. Before going towards this direction, it is worth mentioning that – for detailed computations see Málek, Rajagopal and Tůma [4] – the setting (1.1)–(1.4) follows from the following setting satisfied in $Q_T = (0,T)\times\Omega$:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0, && (1.5)\\
\rho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\right) - \operatorname{div}\mathbb{T} - \rho\boldsymbol{f} &= \mathbf{0}, \quad \rho > 0, && (1.6)\\
\overset{\triangledown}{\mathbb{B}}_i + \frac{1}{\tau_i}(\mathbb{B}_i - \mathbb{I}) &= \mathbb{O}, \quad \tau_i > 0, \quad i = 1,2, && (1.7)\\
-p\mathbb{I} + 2\nu\mathbb{D} + \sum_{i=1}^{2} G_i(\mathbb{B}_i - \mathbb{I}) &= \mathbb{T}, \quad 2\nu, G_1, G_2 > 0 && (1.8)
\end{aligned}
$$

provided that we set

$$\mathbb{S} := \sum_{i=1}^{2} G_i(\mathbb{B}_i - \mathbb{I})$$

and

$$\alpha_1 := \frac{\tau_1+\tau_2}{\tau_1\tau_2}, \quad \alpha_0 := \frac{1}{\tau_1\tau_2}, \quad \beta_1 := \frac{2}{G_1+G_2}, \quad \beta_0 := 2\left(\frac{G_1}{\tau_1} + \frac{G_2}{\tau_2}\right).$$

The setting (1.5) – (1.8) describes a mixture of two Oldroyd-B models of the first order, see [10].

Carrying on the thermodynamical approach developed by Rajagopal and Srinivasa [11] (see also Rajagopal and Srinivasa [12] for a general description of their approach) and strengthened, more recently, by Málek, Rajagopal and Tůma [13] and Málek and Průša [14], Málek, Rajagopal and Tůma [8], using also the ideas from Karra and Rajagopal [15], developed an hierarchy of Burgers-type models that stems from four main concepts:

1. There is an underlying natural configuration that evolves together with the current configuration and that splits the total deformation in a multiplicative way into the part that is elastic (reversible) and the part that takes into account all irreversible changes.
2. There are more than one natural configurations associated with the current configuration and these natural configurations coexist in the sense of the theory of interacting continua, see Truesdell [16], Samohýl [17] or Rajagopal and Tao [18].
3. The basic governing equations stem from the balance equations (for mass, momenta, energy) and from the formulation of the second law of thermodynamics at a continuum level. The resulting system is not closed and, besides the state variables such as the density, the velocity, the temperature, contains other quantities, such as the Cauchy stress, entropy and entropy fluxes. In order to close the system, the equations relating these quantities to the state variables and their derivatives must be formulated. These additional equations are called constitutive equations.
4. The constitutive equations can be fully specified provided that we know the constitutive relations for two scalars: the Helmholtz free energy (or other thermodynamical potential such as the Gibbs potential, internal energy or enthalpy, or the entropy itself) and the rate of entropy production. These two scalar constitutive relations, characterizing how the fluid (material) stores the energy and how the energy is dissipated, suffice to determine the constitutive equations for the Cauchy stress etc.

Using the above mentioned concepts and ideas, Málek, Rajagopal and Tůma [8] developed the hierarchy of viscoelastic rate-type fluid models capturing two different relaxation mechanisms, satisfying the following system of equations in Q_T:

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, && (1.9)\\ \rho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\right) - \operatorname{div}\mathbb{T} - \rho \boldsymbol{f} &= \mathbf{0}, \quad \rho > 0, && (1.10)\\ \overset{\triangledown}{\mathbb{B}}_{\kappa_i} + \frac{1}{\tau_i}(\mathbb{B}_{\kappa_i}^{2-\lambda_i} - \mathbb{B}_{\kappa_i}^{1-\lambda_i}) &= \mathbb{O}, \quad \tau_i = \frac{G_i}{\nu_i} > 0,\ \lambda_i \in \mathbb{R},\ i = 1,2, && (1.11)\\ -p\mathbb{I} + 2\nu\mathbb{D} + \sum_{i=1}^{2} G_i(\mathbb{B}_{\kappa_i} - \mathbb{I}) &= \mathbb{T}, \quad 2\nu, G_1, G_2 > 0 && (1.12) \end{aligned}$$

supposed that $\mathbb{B}_{\kappa_i}$ can be written as

$$\mathbb{B}_{\kappa_i} = \mathbb{F}_{\kappa_i}\mathbb{F}_{\kappa_i}^T, \quad \mathbb{F}_{\kappa_i} \in \mathbb{R}^{d\times d},\ \det \mathbb{F}_{\kappa_i} > 0 \text{ in } Q_T,\ i = 1,2. \tag{1.13}$$

Let us note that $\mathbb{F}_{\kappa_i}$ represent the deformation tensors between the natural configurations κ_i and the current configuration κ_t, the deformation between κ_i and κ_t takes into account the instantaneous elastic response of the i-th component of the body upon the unloading. Notice that (1.9)–(1.12) coincides with (1.5)–(1.8) if we set $\lambda_1 = \lambda_2 = 1$.

To achieve (1.9)–(1.13), Málek, Rajagopal and Tůma [8] started with the following assumption: the Helmholtz free energy is considered to be of the form

$$\Psi(\rho, \mathbb{B}_{\kappa_1}, \mathbb{B}_{\kappa_2}) = \tilde{\Psi}(\rho) + \sum_{i=1}^{2} \frac{G_i}{2\rho}(\operatorname{tr}\mathbb{B}_{\kappa_i} - d - \ln\det\mathbb{B}_{\kappa_i}) \tag{1.14}$$

and the rate of the entropy production ζ takes the form

$$\zeta(|\mathbb{D}|, \mathbb{F}_{\kappa_1}, \mathbb{F}_{\kappa_2}, \mathbb{B}_{\kappa_1}, \mathbb{B}_{\kappa_2}) = \frac{1}{\theta}\left(2\nu|\mathbb{D}|^2 + \sum_{i=1}^{2} 2\nu_i \left|\mathbb{D}_{\kappa_i}(\mathbb{F}_{\kappa_i}^T\mathbb{F}_{\kappa_i})^{\frac{\lambda_i}{2}}\right|^2\right), \tag{1.15}$$

where

$$\mathbb{D}_{\kappa_i} := -\frac{1}{2}\mathbb{F}_{\kappa_i}^{-1}\overset{\triangledown}{\mathbb{B}}_{\kappa_i}\mathbb{F}_{\kappa_i}^{-T}$$

and $\theta > 0$ is the temperature, in this article assumed to be constant. We find advantageous that the approach based on the constitutive equations (1.14) and (1.15) provides the a-priori estimates. This is due to the fact that in the considered setting the reduced thermodynamical identity holds, it has the form

$$\mathbb{T} : \mathbb{D} - \rho(\partial_t \Psi + \nabla\Psi \cdot \boldsymbol{v}) = \theta\zeta. \tag{1.16}$$

The identity (1.16) can be derived (see e.g. [8]) from the balance of internal energy e

$$\rho\left(\partial_t e + \nabla e \cdot \boldsymbol{v}\right) = \mathbb{T} : \mathbb{D}, \tag{1.17}$$

from the balance of entropy η

$$\rho\left(\partial_t \eta + \nabla\eta \cdot \boldsymbol{v}\right) = \zeta \tag{1.18}$$

and from the definition of the Helmholtz free energy Ψ

$$\Psi := e - \theta\eta. \tag{1.19}$$

Since we consider θ to be constant, subtracting (1.18) multiplied by θ from (1.17) yields

$$\rho\partial_t(e-\theta\eta)+\rho\nabla(e-\theta\eta)\cdot\boldsymbol{v}=\mathbb{T}:\mathbb{D}-\theta\zeta. \tag{1.20}$$

Putting (1.19) into (1.20), we arrive at (1.16). Now, since the presence of the body force $\boldsymbol{f}$ does not involve the idea of achieving the a-priori estimates, let us suppose for simplicity that $\boldsymbol{f}\equiv\boldsymbol{0}$ in Q_T. Multiplying the equation (1.10) scalarly by $\boldsymbol{v}$, integrating it over Ω, using the integration by parts, the constraint (1.9), the boundary condition (1.24) and the symmetry of $\mathbb{T}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho|\boldsymbol{v}|^2+\int_\Omega\mathbb{T}:\mathbb{D}=0. \tag{1.21}$$

Integrating (1.16) over Ω, using the integration by parts, (1.9) and (1.24), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho\Psi+\int_\Omega(-\mathbb{T}:\mathbb{D}+\theta\zeta)=0. \tag{1.22}$$

Summing (1.21) together with (1.22), where Ψ is expressed by (1.14) and ζ is expressed by (1.15) and integrating the result over $(0,t)$ leads to

$$\begin{aligned}\frac{1}{2}\int_\Omega\rho|\boldsymbol{v}(t)|^2&+\sum_{i=1}^{2}\int_\Omega\rho\frac{G_i}{2}(\mathrm{tr}\mathbb{B}_{\kappa_i}(t)-d-\ln\det\mathbb{B}_{\kappa_i}(t))\\&+\int_0^t\int_\Omega\left(2\nu|\mathbb{D}|^2+\sum_{i=1}^{2}2\nu_i|\mathbb{D}_{\kappa_i}(\mathbb{F}_{\kappa_i}^T\mathbb{F}_{\kappa_i})^{\frac{\lambda_i}{2}}|^2\right)\\=\frac{1}{2}\int_\Omega\rho|\boldsymbol{v}(0)|^2&+\sum_{i=1}^{2}\int_\Omega\rho\frac{G_i}{2}(\mathrm{tr}\mathbb{B}_{\kappa_i}(0)-d-\ln\det\mathbb{B}_{\kappa_i}(0)).\end{aligned} \tag{1.23}$$

The general aim for PDE analysis is to establish for a given number $T>0$ and domain $\Omega\subset\mathbb{R}^d$ long time and large data existence of weak solutions to the unsteady internal flows governed by the equations (1.9)–(1.13) in $Q_T:=(0,T)\times\Omega$ completed with the boundary condition

$$\boldsymbol{v}=\boldsymbol{0}\text{ on }\Sigma_T:=(0,T)\times\partial\Omega \tag{1.24}$$

and with the initial conditions

$$\boldsymbol{v}(0,\cdot)=\boldsymbol{v}_0,\quad\mathbb{B}_{\kappa_i}(0,\cdot)=\mathbb{B}_{i_0}\quad\text{in }\Omega,\ i=1,2, \tag{1.25}$$

where $\boldsymbol{v}_0$ and $\mathbb{B}_{i_0}$ are given functions satisfying suitable compatibility assumptions. The reason for the choice of weak solution as a suitable concept of solution is twofold: First, it is the concept that might be well defined for $\boldsymbol{v}$ and $\mathbb{B}_{\kappa_i}$, $i=1,2$, fulfilling (1.23) (or (1.23), where instead of equality the inequality "$\le$" holds true), and second, several numerical methods are based on this concept of solution.

Let us note that even in the case when $\mathbb{B}_{\kappa_2}\equiv\mathbb{O}$, there are only few studies regarding the long-time and large-data existence theory. Lions and Masmoudi [19] analyzed the system (1.9)–(1.12) with $\lambda_1=1$, $\mathbb{B}_{\kappa_2}\equiv\mathbb{O}$, but instead of $\overset{\triangledown}{\mathbb{B}}_{\kappa_1}$ they considered the term ($\mathbb{B}_\kappa:=\mathbb{B}_{\kappa_1}$)

$$\partial_t\mathbb{B}_\kappa+\sum_{j=1}^{d}\boldsymbol{v}_j\partial_{x_j}\mathbb{B}_\kappa-\mathbb{W}\mathbb{B}_\kappa-\mathbb{B}_\kappa\mathbb{W}^T,$$

where $\mathbb{W} := \frac{1}{2}(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^T)$. This type of observer-invariant time derivative simplifies the analysis, but it does not come out naturally from the thermodynamical approach described above. Later on, Masmoudi [1], carrying on some ideas developed in Hu and Lelièvre [20] that are close to the thermodynamical set-up described above, presented the theorem regarding the long time and large data existence of weak solutions to the system (1.9)–(1.12) with $\mathbb{B}_{\kappa_2} \equiv \mathbb{O}$ and $\lambda := \lambda_1 = 0$, $\mathbb{B}_\kappa := \mathbb{B}_{\kappa_1} = \mathbb{F}_\kappa \mathbb{F}_\kappa^T$. This leads to the model due to Giesekus [21]. Masmoudi reduced his proof to a sketch of the proof of the weak sequential stability of hypothetical weak solutions in function spaces coming from apriori estimates. Despite bringing original ideas, Masmoudi did not give the right mathematical sense to most of the statements, which is due to the presence of highly nonlinear terms a nontrivial task and requires additional work. Masmoudi also did not introduce suitable approximations to the considered problem and consequently did not show their existence and convergence to the solution of problem in interest. He also provided a proof of the property $\det \mathbb{F}_\kappa > 0$ (the requirement (1.13)), but it contains mistakes at some crucial points. We are not aware of any other results for viscoelastic rate-type fluid models fulfilling even the equations (1.9)–(1.12) with $\mathbb{B}_{\kappa_2} \equiv \mathbb{O}$. In particular, the case of the Oldroyd-B model ($\lambda_1 = 1$) is open.

Our goal is to develop a robust mathematical theory for (1.9)–(1.13) for large class of λ_1, λ_2.

However, the only apriori estimate (1.23) may not suffice to obtain even physically acceptable regularity properties of hypothetical weak solutions to (1.9)–(1.12), at least the integrability of the solutions over time and space, the integrability of their time derivatives over time, nor to obtain the weak sequential stability of these (hypothetical) weak solutions. Let us show that the system (1.9)–(1.12) (even with $\mathbb{B}_{\kappa_2} \equiv \mathbb{O}$, $\mathbb{B}_\kappa := \mathbb{B}_{\kappa_1}$) directly provides the apriori estimates even of $\int_\Omega \mathrm{tr}\mathbb{B}_\kappa(t)$ for all $t \in (0,T)$ by the initial data only if $\lambda := \lambda_1 \le 1$ and the apriori estimate of $\int_\Omega |\partial_t(\mathrm{tr}\mathbb{B}_\kappa)|$ by the initial data only if $\lambda \le 0$. For simplicity let us set all material constants to be equal to one and $\boldsymbol{f} \equiv \mathbf{0}$.

Let $\lambda \le 1$. Summing (1.10) multiplied scalarly by $2\boldsymbol{v}$ with (1.11) multiplied scalarly by $\mathbb{I}$, integrating over $(0,t) \times \Omega$, using the integration by parts, the constraint (1.9), the definition (1.12), the boundary condition (1.24), the symmetry of $\mathbb{D}$ and of $\mathbb{B}_\kappa = \mathbb{F}_\kappa \mathbb{F}_\kappa^T$, we get

$$\int_\Omega \big(|\boldsymbol{v}(t)|^2 + \mathrm{tr}\mathbb{B}_\kappa(t)\big) + \int_0^t \int_\Omega \big(2|\mathbb{D}|^2 + \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda}\big) = \int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{1-\lambda} + \int_\Omega \big(|\boldsymbol{v}(0)|^2 + \mathrm{tr}\mathbb{B}_\kappa(0)\big)\,. \tag{1.26}$$

The matrix $\mathbb{B}_\kappa = \mathbb{F}_\kappa \mathbb{F}_\kappa^T$ is symmetric and positive semidefinite, hence it is a diagonalizable matrix and the corresponding diagonal matrix $\mathbb{J}_\kappa$ has nonnegative diagonal terms. If $\lambda < 1$, then $\frac{2-\lambda}{1-\lambda} > 1$. The Hölder inequality with the exponents $p := \frac{2-\lambda}{1-\lambda}$ and $\frac{p}{p-1}$, combined with the Young inequality of the form ($K \in (0,\infty)$ depends on p)

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}}) \le K(a+b)^{\frac{1}{p}} \quad \forall a.b \ge 0$$

then implies

$$\int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{1-\lambda} = \int_0^t \int_\Omega \mathrm{tr}\mathbb{J}_\kappa^{1-\lambda} \le C \left(\int_0^t \int_\Omega \mathrm{tr}\mathbb{J}_\kappa^{2-\lambda} \right)^{\frac{1-\lambda}{2-\lambda}} = C \left(\int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda} \right)^{\frac{1-\lambda}{2-\lambda}}, \tag{1.27}$$

where

$$C = C(T, \Omega, \lambda) \in (0, \infty).$$

If $\lambda = 1$, (1.27) holds trivially. If $\int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda} > 1$, we divide (1.26) by $\left(\int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda}\right)^{\frac{1-\lambda}{2-\lambda}}$, where in the case $\lambda \le 1$ it holds $\frac{1-\lambda}{2-\lambda} \in [0, 1)$, use the nonnegativity of all terms on both handsides of (1.26) (which follows from the positive semidefinitness of $\mathbb{B}_\kappa$), the estimate (1.27) and Korn's inequality to conclude

$$\int_\Omega \left(|\boldsymbol{v}(t)|^2 + \mathrm{tr}\mathbb{B}_\kappa(t)\right) + \int_0^t \int_\Omega \left(|\nabla \boldsymbol{v}|^2 + \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda}\right) \le \tilde{C}(T, \Omega, \lambda, \boldsymbol{v}(0), \mathrm{tr}\mathbb{B}_\kappa(0)). \tag{1.28}$$

If $\int_0^t \int_\Omega \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda} \in [0, 1]$, we conclude the estimate (1.28) from (1.26) and (1.27) directly. Hence whenever $\lambda \le 1$, the estimate (1.28) holds true. On the other hand, the inequality $\frac{1-\lambda}{2-\lambda} \in [0, 1)$, which was crucial for deriving (1.28), would not be satisfied if we considered $\lambda > 1$.

Let now $\lambda \le 0$. If $\lambda < 0$, then from Hölder's inequality with the exponents $p := \frac{2-\lambda}{2}$ and $\frac{p}{p-1}$ (let us note that $p > 1$), from the inequality ($K \in (0, \infty)$ depends on p)

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}}) \le K(a + b)^{\frac{1}{p}} \quad \forall a, b \ge 0$$

and from the symmetry of $\mathbb{B}_\kappa$ it follows ($\hat{C} \in (0, \infty)$ depends on T, Ω, λ)

$$\int_{Q_T} |\mathbb{B}_\kappa|^2 = \int_{Q_T} \mathrm{tr}\mathbb{B}_\kappa^2 = \int_{Q_T} \mathrm{tr}\mathbb{J}_\kappa^2 \le \hat{C} \left(\int_{Q_T} \mathrm{tr}\mathbb{J}_\kappa^{2-\lambda}\right)^{\frac{2}{2-\lambda}} = \hat{C} \left(\int_{Q_T} \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda}\right)^{\frac{2}{2-\lambda}}. \tag{1.29}$$

If $\lambda = 0$, then (1.29) follows from the symmetry of $\mathbb{B}_\kappa$ directly. Integrating (1.11) multiplied scalarly by $\mathbb{I}$ over Q_T, using the integration by parts, the relations (1.9), (1.24), (1.27), (1.28), (1.29) and the Hölder inequality, we get ($C \in (0, \infty)$ coincides with C in (1.27))

$$\int_{Q_T} |\partial_t(\mathrm{tr}\mathbb{B}_\kappa)| \le \left(\int_{Q_T} |\nabla \boldsymbol{v}|^2\right)^{\frac{1}{2}} \left(\int_{Q_T} |\mathbb{B}_\kappa|^2\right)^{\frac{1}{2}} + (1+C) \int_{Q_T} \mathrm{tr}\mathbb{B}_\kappa^{2-\lambda} \le \overline{C}(T, \Omega, \lambda, \boldsymbol{v}(0), \mathrm{tr}\mathbb{B}(0)).$$

Let us note that the inequality $\frac{2-\lambda}{2} \ge 1$, which was crucial for deriving the last apriori estimate, would not be satisfied if we considered $\lambda > 0$.

In this article, as the starting point, we show the long time and large data existence of weak solutions to the Burgers-type model (1.9)–(1.13) with $\lambda_i = 0$, $i = 1, 2$, in two spatial dimensions, carrying on some ideas developed by Masmoudi [1]. Moreover, we show that the solutions are strongly continuous with respect to time with values in (multidimensional) Lebesgue spaces.

The structure of the paper is the following. In Section 2 we fix notations and formulate the main result. In Section 3 we introduce some general mathematical tools used in the existence proof. In Sections 4–7 we treat the system considering $\mathbb{B}_{\kappa_2} \equiv \mathbb{O}$. In Section 8 we conclude the existence result without the restriction $\mathbb{B}_{\kappa_2} \equiv \mathbb{O}$.

2. Notation and Formulation of the problem

In order to define weak solutions to the considered problem and formulate the main result we need to fix notations. The operator " $\cdot$ " denotes the scalar product of two vectors, the operator ":" denotes the scalar product of two tensors. The operator " $\otimes$ " denotes the tensor product of two vectors. For a matrix $\mathbb{A} = \{A_{ij}\}_{i,j=1}^d$ and a vector $\boldsymbol{b} = (b_1, ..., b_d)$ we define the third order tensor $\mathbb{A} \otimes \boldsymbol{b} = \{(\mathbb{A} \otimes \boldsymbol{b})_{ijk}\}_{i,j,k=1}^d$ as

$$(\mathbb{A} \otimes \boldsymbol{b})_{ijk} := A_{ij} b_k.$$

The Euclidean norm of a vector, the Frobenius norm of a tensor, or the Lebesgue measure of the given measurable subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, is denoted as $|\cdot|$. Next we define for a matrix function $\mathbb{A} = (A_{ij})_{i,j=1}^d$ and a vector function $\boldsymbol{b} = (b_1, ..., b_d)$ the operator Div acting on the third order tensor $\mathbb{A} \otimes \boldsymbol{b}$ as

$$\operatorname{Div}(\mathbb{A} \otimes \boldsymbol{b}) := \sum_{j=1}^{d} \partial_{x_j}(b_j \mathbb{A}).$$

Let $\Omega \subset \mathbb{R}^d$ be a domain of class $C^{0,1}$, let $\partial\Omega$ be its boundary. Let $T > 0$ be a fixed number, according to the Introduction let us denote $Q_T := (0,T) \times \Omega$, $\Sigma_T := (0,T) \times \partial\Omega$. For $q \in [1, \infty]$ the symbol $\|\cdot\|_q$ stands for the norm in the usual Lebesgue space $L^q(\Omega)$ (or in its multidimensional variant $(L^q(\Omega))^d$, $(L^q(\Omega))^{d\times d}$, etc.), while the symbol $\|\cdot\|_{1,q}$ stands for the norm in the usual Sobolev space $W^{1,q}(\Omega)$ (or in its multidimensional variant $(W^{1,q}(\Omega))^d$, $(W^{1,q}(\Omega))^{d\times d}$, etc.). The symbol $\mathcal{M}(\overline{Q_T})$ stands for the space of the Radon measures defined on the closure of Q_T. If X is a Banach space, then X^* denotes its dual space. The dualities between Banach spaces and their duals are denoted as $\langle\cdot,\cdot\rangle$. Being X a Banach space, $L^q(0,T;X)$ for $q \in [1,\infty]$ is the relevant Bochner space, $C([0,T];X)$ is the space of functions continuous in $[0,T]$ with values in X, $C_{weak}([0,T];X)$ is the space of functions weakly continuous in $[0,T]$ with values in X. For an open set $O \subset \mathbb{R}^d$, $C_c^\infty(O)$ is the space of smooth functions compactly supported in O, $L^q_{loc}(O)$ is the space of functions, whose q-power is locally integrable over O. For any $q \in [1, \infty)$ we introduce the function spaces

$$\begin{aligned}
W_0^{1,q}(\Omega) &:= \{u \in W^{1,q}(\Omega); u = 0 \text{ on } \partial\Omega\},\\
W^{1,q}_{\mathbf{0},\operatorname{div}} &:= \{\boldsymbol{u} \in (W^{1,q}(\Omega))^d; \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega; \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\},\\
L^q_{\boldsymbol{n},\operatorname{div}} &:= \overline{\{\boldsymbol{u} \in (C_c^\infty(\Omega))^d; \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}}^{\|\cdot\|_q},
\end{aligned}$$

$$\|u\|_{W_0^{1,q}(\Omega)} := \|\nabla u\|_q, \quad \|\boldsymbol{u}\|_{W^{1,2}_{\mathbf{0},\operatorname{div}}} := \|\nabla \boldsymbol{u}\|_q, \quad \|\boldsymbol{u}\|_{L^q_{\boldsymbol{n},\operatorname{div}}} := \|\boldsymbol{u}\|_q.$$

If it does not cause any misunderstanding, we write the integrals over time and space without the symbols dt, d$\boldsymbol{x}$, for example, if $g = g(t, \boldsymbol{x})$ is a given function defined in Q_T, we write $\int_{Q_T} g$ instead of $\int_{Q_T} g \; \mathrm{d}t\mathrm{d}\boldsymbol{x}$. We denote the positive constants of uniform bounds, whose exact values are not essential for our aims, as K, C, $\tilde{C}$, $\hat{C}$, $\overline{C}$, C^*, their values can change troughout the text.

2.1. ***Formulation of the main result***

Starting from here, we write $\mathbb{F}_i$ instead of $\mathbb{F}_{\kappa_i}$, $i = 1, 2$. Let us recall that $\mathbb{D}$ denotes the symmetric part of $\nabla \boldsymbol{v}$, i.e. $\mathbb{D} := \frac{1}{2}\left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right)$.

Definition 2.1 (Generalized Burgers model)**.** In the rest of this paper by the Generalized Burgers model we understand the following system of equations satisfied in Q_T with unknown quantities $\boldsymbol{v}$, p, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{B}_1$, $\mathbb{B}_2$:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{2.1}$$

$$\rho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) + \nabla p - 2\nu \operatorname{div} \mathbb{D} - \sum_{i=1}^{2} G_i \operatorname{div} \mathbb{B}_i - \rho \boldsymbol{f} = \boldsymbol{0}, \qquad \rho, 2\nu, G_1, G_2 > 0, \tag{2.2}$$

$$\partial_t \mathbb{B}_i + \operatorname{Div}\, (\mathbb{B}_i \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{B}_i - \mathbb{B}_i (\nabla \boldsymbol{v})^T + \frac{1}{\tau_i}\left(\mathbb{B}_i^2 - \mathbb{B}_i\right) = \mathbb{O}, \qquad \tau_i > 0, i = 1, 2, \tag{2.3}$$

$$\mathbb{B}_i = \mathbb{F}_i \mathbb{F}_i^T, \quad i = 1, 2, \tag{2.4}$$

where

$$\det \mathbb{F}_i > 0 \quad \text{if } \mathbb{B}_i \not\equiv \mathbb{O}, \quad i = 1, 2. \tag{2.5}$$

The system is completed with the boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma_T \tag{2.6}$$

and with the initial conditions

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0, \quad \mathbb{F}_i(0, \cdot) = \mathbb{F}_{i_0}, \quad \mathbb{B}_i(0, \cdot) = \mathbb{B}_{i_0} := \mathbb{F}_{i_0} \mathbb{F}_{i_0}^T \quad \text{in } \Omega, \ i = 1, 2. \tag{2.7}$$

Before introducing the weak formulation of the system (2.1) – (2.7) let us set for simplicity the positive constants ρ, 2ν, τ_1, τ_2 to be equal to one and the external forces $\boldsymbol{f}$ to be identically equal to zero. As one may check, if we took ρ, 2ν, τ_1, $\tau_2 > 0$ and $\boldsymbol{f} \in L^2(0, T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)$ arbitrary, the proof of the existence of weak solutions to the system (2.1)–(2.7) would be made essentially in the same way as the proof that we present, there would only be more technicalities, distracting, in our opinion, the reader from the main ideas of the proof. We do not set G_1, G_2 to be equal to one since we study the system with two different relaxation mechanisms with different weights.

Definition 2.2 (Generalized Burgers – weak formulation)**.** Let us assume $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_{i_0} \in (L^2(\Omega))^{d \times d}$, $\mathbb{B}_{i_0} := \mathbb{F}_{i_0} \mathbb{F}_{i_0}^T$, $\det \mathbb{F}_{i_0} > 0$ a.e. in Ω, $\ln \det \mathbb{F}_{i_0} \in L^1(\Omega)$ and $G_i > 0$, $i = 1, 2$. By a weak solution to the Generalized Burgers problem we call a quintuple $[\boldsymbol{v}, \mathbb{F}_1, \mathbb{F}_2, \mathbb{B}_1, \mathbb{B}_2]$ fulfilling for $i = 1, 2$

$$\begin{aligned}
\boldsymbol{v} &\in C\left([0, T]; L^2_{\boldsymbol{n},\mathrm{div}}\right) \cap L^2\left(0, T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}\right), \\
\partial_t \boldsymbol{v} &\in L^2\left(0, T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), \\
\mathbb{F}_i &\in C\left([0, T]; (L^2(\Omega))^{d \times d}\right) \cap (L^4(Q_T))^{d \times d}, \\
\partial_t \mathbb{F}_i &\in L^{\frac{4}{3}}\left(0, T; ((W^{1,2}(\Omega))^{d \times d})^*\right), \\
\mathbb{B}_i &\in C\left([0, T]; (L^1(\Omega))^{d \times d}\right) \cap (L^2(Q_T))^{d \times d}, \\
\partial_t \mathbb{B}_i &\in L^1\left(0, T; ((W^{1,4}(\Omega))^{d \times d})^*\right)
\end{aligned}$$

and satisfying for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,4}(\Omega))^{d\times d}$ and a.a. $t \in (0,T)$

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i \mathbb{B}_i : \nabla\boldsymbol{w} = 0, \quad (2.8)$$

$$\langle \partial_t \mathbb{B}_i, \mathbb{A}\rangle - \int_\Omega (\mathbb{B}_i \otimes \boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega (\nabla\boldsymbol{v}\ \mathbb{B}_i) : \mathbb{A} - \int_\Omega (\mathbb{B}_i(\nabla\boldsymbol{v})^T) : \mathbb{A} + \int_\Omega (\mathbb{B}_i^2 - \mathbb{B}_i) : \mathbb{A} = 0, \quad (2.9)$$

where

$$\mathbb{B}_i = \mathbb{F}_i\mathbb{F}_i^T, \quad \det\mathbb{F}_i > 0 \text{ a.e. in } Q_T \text{ if } \mathbb{B}_i \not\equiv \mathbb{O}, \quad (2.10)$$

with the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_{i_0}$, $\mathbb{B}_{i_0}$ fulfilled in the sense

$$\lim_{t\to 0+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = 0, \quad (2.11)$$

$$\lim_{t\to 0+} \|\mathbb{F}_i(t) - \mathbb{F}_{i_0}\|_2 = 0, \quad (2.12)$$

$$\lim_{t\to 0+} \|\mathbb{B}_i(t) - \mathbb{B}_{i_0}\|_1 = 0. \quad (2.13)$$

The aim of this paper is to prove the following theorem.

Theorem 2.3. *Let $d = 2$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_{i_0} \in (L^2(\Omega))^{2\times 2}$, $\mathbb{B}_{i_0} := \mathbb{F}_{i_0}\mathbb{F}_{i_0}^T$, $\det \mathbb{F}_{i_0} > 0$ a.e. in Ω, $\ln\det \mathbb{F}_{i_0} \in L^1(\Omega)$ and $G_i > 0$, $i = 1, 2$. Then there exists a weak solution to the Generalized Burgers problem in the sense of Definition 2.2.*

The proof of Theorem 2.3 is split into Sections 4–8. In the following Section 3 we introduce mathematical tools, which will be employed in the own proof of the theorem. In Sections 4–7 we make the proof of Theorem 2.3 restricting ourselves to $G_1 = 1$, $\mathbb{B}_2 \equiv \mathbb{O}$. In the last Section 8 we conclude the result for G_1, $G_2 > 0$ arbitrary, without the restriction $\mathbb{B}_2 \equiv \mathbb{O}$.

3. Mathematical tools

In this section we present two lemmata useful for the proof of Theorem 2.3. The first lemma is the Friedrichs lemma on commutators, see e.g. [22]. The second lemma concerns the monotonicity of one special matrix function.

Lemma 3.1 (Friedrichs lemma on commutators, [22])**.** *Let $O \subset \mathbb{R}^d$ be a domain, $d \in \mathbb{N}$, $p, q, r \in \mathbb{R}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. Let $f \in L^p(O)$, $\boldsymbol{g} \in (W^{1,q}(O))^d$. For any $\boldsymbol{x} \in \mathbb{R}^d$ and $h \in L^1_{loc}(\mathbb{R}^d)$ let us denote*

$$h_\delta(\boldsymbol{x}) := \int_{\mathbb{R}^d} \omega_\delta(\boldsymbol{x} - \boldsymbol{y}) h(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y},$$

where ω_δ is the standard mollifying kernel. Then

$$\| \operatorname{div}(f_\delta \boldsymbol{g}) - \operatorname{div}(f\boldsymbol{g})_\delta \|_{L^r_{loc}(O)} \leq C \|f\|_{L^p(O)} \|\boldsymbol{g}\|_{(W^{1,q}(O))^d}.$$

Moreover, if $r < \infty$, then

$$\operatorname{div}(f_\delta \boldsymbol{g}) - \operatorname{div}\,(f\boldsymbol{g})_\delta \to 0 \quad \text{strongly in } L^r_{loc}(O).$$

Lemma 3.2 (Monotonicity)**.** *Let $d \in \mathbb{N}$. The function $S : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ given by*

$$S(\mathbb{X}) := \mathbb{X}\mathbb{X}^T\mathbb{X}$$

is monotone, i.e.

$$(S(\mathbb{X}) - S(\mathbb{Y})) : (\mathbb{X} - \mathbb{Y}) \geq 0 \quad \forall \mathbb{X}, \mathbb{Y} \in \mathbb{R}^{d\times d}. \tag{3.1}$$

Proof. In the whole proof the symbol δ_{ij}, where $i, j \in \{1, ..., d\}$, stands for the Kronecker symbol, i.e.

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

The i, j component of any matrix $\mathbb{X} \in \mathbb{R}^{d\times d}$ is denoted either as X_{ij}, either as $(\mathbb{X})_{ij}$. For brevity in the computations the Einstein summation convention is used, i.e. all sum indices are omitted.

For all matrices $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{d\times d}$ it holds

$$S(\mathbb{X}) - S(\mathbb{Y}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left(S\left(\mathbb{Y} + s(\mathbb{X} - \mathbb{Y})\right)\right) \,\mathrm{d}s \tag{3.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(S\left(\mathbb{Y} + s(\mathbb{X} - \mathbb{Y})\right)\right) = \frac{\partial S\left(\mathbb{K}(s)\right)}{\partial \left(\mathbb{K}(s)\right)_{ab}} L_{ab}, \tag{3.3}$$

where $\mathbb{K}(s) := \mathbb{Y} + s(\mathbb{X} - \mathbb{Y})$, $\mathbb{L} := \mathbb{X} - \mathbb{Y}$. Collecting (3.2) and (3.3), one concludes

$$(S(\mathbb{X}) - S(\mathbb{Y})) : (\mathbb{X} - \mathbb{Y}) = \int_0^1 \frac{\partial \left(S\left(\mathbb{K}(s)\right)\right)_{ij}}{\partial \left(\mathbb{K}(s)\right)_{ab}} L_{ab} L_{ij} \,\mathrm{d}s,$$

thus in order to prove (3.1) it suffices to show

$$\frac{\partial \left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} L_{ab} L_{ij} \geq 0 \quad \forall \mathbb{K}, \mathbb{L} \in \mathbb{R}^{d\times d}. \tag{3.4}$$

We write

$$\begin{aligned}
\frac{\partial \left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} &= \frac{\partial}{\partial K_{ab}} (K_{im} K_{km} K_{kj}) \\
&= \delta_{ia}\delta_{mb} K_{km} K_{kj} + \delta_{ak}\delta_{mb} K_{im} K_{kj} + \delta_{ak}\delta_{bj} K_{im} K_{km} \\
&= \delta_{ia} K_{kb} K_{kj} + \delta_{ak} K_{ib} K_{kj} + \delta_{bj} K_{im} K_{am},
\end{aligned}$$

and finally

$$\begin{aligned}
\frac{\partial \left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} L_{ab} L_{ij} &= \left(\delta_{ia} K_{kb} K_{kj} + \delta_{ak} K_{ib} K_{kj} + \delta_{bj} K_{im} K_{am}\right) L_{ab} L_{ij} \\
&= K_{kb} K_{kj} L_{ib} L_{ij} + K_{ib} K_{kj} L_{kb} L_{ij} + K_{im} K_{am} L_{aj} L_{ij} \\
&= (\mathbb{K}\mathbb{L}^T) : (\mathbb{K}\mathbb{L}^T) + (\mathbb{K}\mathbb{L}^T) : (\mathbb{K}^T\mathbb{L}) + (\mathbb{K}^T\mathbb{L}) : (\mathbb{K}^T\mathbb{L}) \\
&\geq \frac{1}{2}\left(|\mathbb{K}\mathbb{L}^T|^2 + |\mathbb{K}^T\mathbb{L}|^2\right) \geq 0,
\end{aligned}$$

where the first inequality follows from Young's inequality. The lemma is proved. □

4. System with evolutionary equation for the tensor $\mathbb{F}$

As introduced above, first we prove Theorem 2.3 with restrictions $G_1 = 1$, $\mathbb{B}_2 \equiv \mathbb{O}$, we denote $\mathbb{B} := \mathbb{B}_1$. Carrying on the ideas developed by Masmoudi [1], we start with the setting containing the evolutionary equation for the tensor $\mathbb{F} := \mathbb{F}_1$ instead of the evolutionary equation for $\mathbb{B} = \mathbb{F}\mathbb{F}^T$. More specifically, we start with the following setting supposed to be satisfied in Q_T:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.1}$$

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \operatorname{div} \mathbb{D} - \operatorname{div} \mathbb{F}\mathbb{F}^T = \mathbf{0}, \tag{4.2}$$

$$\partial_t \mathbb{F} + \operatorname{Div}(\mathbb{F} \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{F} + \frac{1}{2}(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) = \mathbb{O}, \tag{4.3}$$

$$\det \mathbb{F} > 0 \tag{4.4}$$

completed with the boundary condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T \tag{4.5}$$

and the initial conditions

$$\begin{aligned} \boldsymbol{v}(0,\cdot) &= \boldsymbol{v}_0 \quad \text{in } \Omega, &(4.6)\\ \mathbb{F}(0,\cdot) &= \mathbb{F}_0 \quad \text{in } \Omega. &(4.7)\end{aligned}$$

Formally, multiplying (4.3) by $\mathbb{F}^T$ from right, multiplying the transpose of (4.3) by $\mathbb{F}$ from left and summing, we obtain the equation

$$\partial_t(\mathbb{F}\mathbb{F}^T) + \operatorname{Div}\big((\mathbb{F}\mathbb{F}^T)\otimes\boldsymbol{v}\big) - \nabla\boldsymbol{v}(\mathbb{F}\mathbb{F}^T) - (\mathbb{F}\mathbb{F}^T)(\nabla\boldsymbol{v})^T + (\mathbb{F}\mathbb{F}^T)^2 - \mathbb{F}\mathbb{F}^T = \mathbb{O}. \tag{4.8}$$

Setting $\mathbb{B} := \mathbb{F}\mathbb{F}^T$, $G_1 = 1$, $\mathbb{B}_2 \equiv \mathbb{O}$, the equation (4.8) is equivalent to (2.3) and hence the system (4.1), (4.2), (4.4)–(4.8) is equivalent to the system (2.1)–(2.7) with $G_1 = 1$, $\mathbb{B} := \mathbb{B}_1$, $\mathbb{B}_2 \equiv \mathbb{O}$.

We find two advantages of this approach. First, as one may expect, the tensor $\mathbb{F}$ has better regularity properties than $\mathbb{B} = \mathbb{F}\mathbb{F}^T$. Formally, multiplying (4.2) scalarly by $\boldsymbol{v}$, multiplying (4.3) scalarly by $\mathbb{F}$, integrating over Q_T, summing, using (4.1), (4.5) and standard analytical tools (for deducing the details see the rigorous computations in next two sections), we get for all $t \in (0,T)$ the apriori estimate

$$\|\boldsymbol{v}(t)\|_2^2 + \|\mathbb{F}(t)\|_2^2 + \int_0^t \big(\|\nabla\boldsymbol{v}\|_2^2 + \|\mathbb{F}\|_4^4\big) \le C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2).$$

Better regularity properties of (hypothetical) weak solutions to the system extend the set of admissible test functions in the corresponding equations, which increases the chance to obtain, for example, weak sequential stability of these solutions, or to make a short proof of the property $\det\mathbb{F} > 0$ (the condition (2.5)). Second, after a rigorous proceeding from (4.3) to (4.8) we immediately obtain $\mathbb{B}$ of the form $\mathbb{B} = \mathbb{F}\mathbb{F}^T$ (the condition (2.4)), satisfying (2.3).

Definition 4.1 (System with equation for $\mathbb{F}$ - weak formulation)**.** Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_0 \in (L^2(\Omega))^{2\times2}$, $\det\mathbb{F}_0 > 0$ a.e. in Ω and $\ln\det\mathbb{F}_0 \in L^1(\Omega)$. By a weak solution to the system (4.1)–(4.7) we understand a couple $[\boldsymbol{v},\mathbb{F}]$ fulfilling

$$\begin{aligned} \boldsymbol{v} &\in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}}),\\ \partial_t\boldsymbol{v} &\in L^2(0,T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*),\\ \mathbb{F} &\in C([0,T]; (L^2(\Omega))^{2\times2}) \cap (L^4(Q_T))^{2\times2},\\ \partial_t\mathbb{F} &\in L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times2})^*\right),\\ \det\mathbb{F} &> 0 \quad \text{a.e. in } Q_T \end{aligned}$$

and satisfying for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times2}$ and a.a. $t \in (0,T)$

$$\langle\partial_t\boldsymbol{v},\boldsymbol{w}\rangle - \int_\Omega(\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \int_\Omega(\mathbb{F}\mathbb{F}^T) : \nabla\boldsymbol{w} \;=\; 0, \tag{4.9}$$

$$\langle\partial_t\mathbb{F},\mathbb{A}\rangle - \int_\Omega(\mathbb{F}\otimes\boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega((\nabla\boldsymbol{v})\mathbb{F}) : \mathbb{A} + \frac{1}{2}\int_\Omega(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \mathbb{A} \;=\; 0 \tag{4.10}$$

with the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_0$ fulfilled in the sense

$$\lim_{t\to 0+} \|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2 = 0, \quad (4.11)$$
$$\lim_{t\to 0+} \|\mathbb{F}(t)-\mathbb{F}_0\|_2 = 0. \quad (4.12)$$

Theorem 4.2. *Let* $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_0 \in (L^2(\Omega))^{2\times 2}$, $\det \mathbb{F}_0 > 0$ *a.e. in* Ω, $\ln\det\mathbb{F}_0 \in L^1(\Omega)$. *Then there exists a weak solution to the system* (4.1)–(4.7) *in the sense of Definition 4.1.*

The following two sections are devoted to the proof of Theorem 4.2.

5. Approximations

We start with the system approximating (4.1)–(4.7), where on the right handside of the equation (4.3) the term representing small stress diffusion is added. The system, where all the equations are supposed to be satisfied in Q_T, reads as follows:

$$\operatorname{div}\boldsymbol{v} = 0, \quad (5.1)$$
$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) + \nabla p - \operatorname{div}\mathbb{D} - \operatorname{div}\mathbb{F}\mathbb{F}^T = \mathbf{0}, \quad (5.2)$$
$$\partial_t \mathbb{F} + \operatorname{Div}(\mathbb{F}\otimes\boldsymbol{v}) - (\nabla\boldsymbol{v})\mathbb{F} + \frac{1}{2}\left(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}\right) = \varepsilon\Delta\mathbb{F}. \quad (5.3)$$

The system is completed with the boundary condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T \quad (5.4)$$

and with the initial conditions

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 \quad \text{in } \Omega, \quad (5.5)$$
$$\mathbb{F}(0,\cdot) = \mathbb{F}_0 \quad \text{in } \Omega. \quad (5.6)$$

Let us note that the functions $\boldsymbol{v}_0$, $\mathbb{F}_0$ introduced in (5.5) and (5.6) coincide with the functions $\boldsymbol{v}_0$, $\mathbb{F}_0$ introduced in Sections 2 and 4.

The reason for our choice of approximations is twofold. First, as we will show, the presence of the term $\varepsilon\Delta\mathbb{F}$ provides the uniform estimate

$$\varepsilon\|\nabla\mathbb{F}_n\|^2_{2,Q_T} \le C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2), \quad (5.7)$$

where $\{\mathbb{F}_n\}_{n\in\mathbb{N}}$ is a sequence of Galerkin's approximations to $\mathbb{F}$ (their existence is proved in the following subsection). The estimate (5.7) (together with the uniform bounds of $\mathbb{F}_n$ and $\partial_t\mathbb{F}_n$ in appropriate norms proved bellow and the Aubin-Lions compactness lemma) leads to the compactness of $\{\mathbb{F}_n\}_{n\in\mathbb{N}}$ in $(L^2(Q_T))^{2\times 2}$, which makes the proof of the existence of weak solutions to (5.1)–(5.6) relatively simple (the system (5.1)–(5.3) is close to an advection-diffusion equation, see e.g. [23]). Second, considering the sequence

$\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_\varepsilon\}$ of weak solutions to (5.1)–(5.6) and taking the limit $\varepsilon \to 0+$, the stress diffusion terms, in a weak formulation written for a.a. $t \in (0,T)$ as $\varepsilon \int_\Omega \nabla\mathbb{F}_\varepsilon : \nabla\mathbb{A}$, where $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ is arbitrary, converge to zero due to the uniform estimate (5.7) and the Hölder inequality ($\sqrt{\varepsilon}\nabla\mathbb{F}_\varepsilon$ is uniformly bounded in $(L^2(Q_T))^{2\times2\times2}$ and $\sqrt{\varepsilon}\nabla\mathbb{A}$ converges to zero strongly in $(L^2(Q_T))^{2\times2\times2}$). Hence we can deduce that the (hypothetical) weak limits of the sequences $\{\boldsymbol{v}_\varepsilon\}, \{\mathbb{F}_\varepsilon\}$ are weak solutions to the system (4.1)–(4.7) if the sequence $\{\mathbb{F}_\varepsilon\}$ is compact in $(L^2(Q_T))^{2\times 2}$. The proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ is the most complicated part of the proof of Theorem 4.2. However, with the introduced approximations, it is not much more complicated than the proof of weak sequential stability of (hypothetical) weak solutions to (4.1)–(4.7). The only difference is that without the presence of the supplementary stress diffusion term $\varepsilon\Delta\mathbb{F}_\varepsilon$ (it is present in (5.3), but not in (4.3)) the relation (6.62) would hold true with equality. However, the achieved inequality does not complicate further computations.

Proposition 5.1. *Let $\varepsilon > 0$, $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_0 \in (L^2(\Omega))^{2\times 2}$. Then there exists a weak solution to the system* (5.1)–(5.6)*, i.e. there exists a couple $[\boldsymbol{v}, \mathbb{F}]$ fulfilling*

$$\begin{aligned}
\boldsymbol{v} &\in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}),\\
\partial_t \boldsymbol{v} &\in L^2(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*),\\
\mathbb{F} &\in C_{weak}([0,T]; (L^2(\Omega))^{2\times 2}) \cap L^2(0,T; (W^{1,2}(\Omega))^{2\times 2}),\\
\partial_t \mathbb{F} &\in L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right)
\end{aligned}$$

and satisfying for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t \in (0,T)$

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \int_\Omega (\mathbb{F}\mathbb{F}^T) : \nabla\boldsymbol{w} = 0, \quad (5.8)$$

$$\langle \partial_t \mathbb{F}, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}\otimes\boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega ((\nabla\boldsymbol{v})\mathbb{F}) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \mathbb{A} + \varepsilon\int_\Omega \nabla\mathbb{F} : \nabla\mathbb{A} = 0 \quad (5.9)$$

with the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_0$ fulfilled in the sense

$$\lim_{t\to 0+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = 0, \quad (5.10)$$

$$\lim_{t\to 0+} \|\mathbb{F}(t) - \mathbb{F}_0\|_2 = 0. \quad (5.11)$$

We split the proof of Proposition 5.1 into five subsections.

5.1. ***Galerkin's approximations***

Let $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ be a basis of $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ composed of eigenfunctions of the Stokes operator subject to the boundary condition $\boldsymbol{w} = \boldsymbol{0}$ on $\partial\Omega$, orthogonal in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, orthonormal in $L^2_{\boldsymbol{n},\mathrm{div}}$. Let $\{\mathbb{A}_j\}_{j\in\mathbb{N}}$ be a basis of $(W^{1,2}(\Omega))^{2\times 2}$ composed of eigenfunctions of the Laplace operator subject to the boundary condition $\nabla\mathbb{A}\cdot\boldsymbol{n} := \{\nabla A_{kl}\cdot\boldsymbol{n}\}_{k,l=1}^2 = \mathbb{O}$ on $\partial\Omega$, orthogonal in $(W^{1,2}(\Omega))^{2\times 2}$, orthonormal in $(L^2(\Omega))^{2\times 2}$. Let us denote $W_n :=$ span$\{\boldsymbol{w}_1, ..., \boldsymbol{w}_n\}$, $X_n :=$ span$\{\mathbb{A}_1, ..., \mathbb{A}_n\}$. Let us denote the orthogonal projection from $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ to W_n as P_n and the orthogonal projection from $(W^{1,2}(\Omega))^{2\times 2}$ to X_n as Q_n.

The projection P_n is continuous in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, the projection Q_n is continuous in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$. From the Carathéodory theory for ordinary differential equations it follows that there exist time dependent coefficients $\alpha_1^n(t), ..., \alpha_n^n(t)$, $\beta_1^n(t), ..., \beta_n^n(t)$ (but we will write only $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_n$) such that

$$\boldsymbol{v}_n = \sum_{j=1}^{n} \alpha_j \boldsymbol{w}_j \quad \text{and} \quad \mathbb{F}_n = \sum_{j=1}^{n} \beta_j \mathbb{A}_j \tag{5.12}$$

fulfill for all $j \in \{1, ..., n\}$, for all $t \in (0, \tilde{t})$, where $\tilde{t}$ is certain positive number, the following system of equations (we denote $\mathbb{D}_n := \frac{1}{2}\left(\nabla \boldsymbol{v}_n + (\nabla \boldsymbol{v}_n)^T\right)$):

$$\partial_t \left(\int_\Omega \boldsymbol{v}_n \cdot \boldsymbol{w}_j \right) - \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla \boldsymbol{w}_j + \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{w}_j + \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{w}_j = 0, \tag{5.13}$$

$$\begin{aligned} \partial_t \left(\int_\Omega \mathbb{F}_n : \mathbb{A}_j \right) - \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla \mathbb{A}_j - \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{A}_j + \frac{1}{2} \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n) : \mathbb{A}_j \\ - \frac{1}{2} \int_\Omega \mathbb{F}_n : \mathbb{A}_j + \varepsilon \int_\Omega \nabla \mathbb{F}_n : \nabla \mathbb{A}_j = 0. \end{aligned} \tag{5.14}$$

The functions $\boldsymbol{v}_n$ are absolutely continuous in $[0, \tilde{t})$ with values in W_n, the functions $\mathbb{F}_n$ are absolutely continuous in $[0, \tilde{t})$ with values in X_n, they satisfy the initial conditions

$$\boldsymbol{v}_n(0, \cdot) = P_n(\boldsymbol{v}_0) \quad \text{in } \Omega, \tag{5.15}$$

$$\mathbb{F}_n(0, \cdot) = Q_n(\mathbb{F}_0) \quad \text{in } \Omega. \tag{5.16}$$

The fact that $\tilde{t} = T$ is an easy consequence of the uniform estimates that follow.

5.2. ***Uniform estimates***

Multiplying (5.13) by α_j, (5.14) by β_j and taking the sum over $j = 1, ..., n$, we obtain (use also the symmetry of $\mathbb{F}_n \mathbb{F}_n^T$)

$$\frac{\partial_t \|\boldsymbol{v}_n\|_2^2}{2} - \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla \boldsymbol{v}_n + \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{v}_n + \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{v}_n = 0,$$

$$\frac{\partial_t \|\mathbb{F}_n\|_2^2}{2} - \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla \mathbb{F}_n - \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n + \frac{\|\mathbb{F}_n \mathbb{F}_n^T\|_2^2 - \|\mathbb{F}_n\|_2^2}{2} + \varepsilon \|\nabla \mathbb{F}_n\|_2^2 = 0.$$

Integrating both equations over $(0, t)$, where $t \in (0, T)$ is arbitrary, and employing the integration by parts and the properties $\operatorname{div} \boldsymbol{v}_n = 0$ in Q_T, $\boldsymbol{v}_n = \boldsymbol{0}$ on Σ_T yields

$$\begin{aligned} \frac{\|\boldsymbol{v}_n(t)\|_2^2}{2} + \int_0^t \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{v}_n + \int_0^t \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{v}_n &= \frac{\|\boldsymbol{v}_n(0)\|_2^2}{2}, \\ \frac{\|\mathbb{F}_n(t)\|_2^2}{2} - \int_0^t \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n + \int_0^t \frac{\|\mathbb{F}_n \mathbb{F}_n^T\|_2^2 - \|\mathbb{F}_n\|_2^2}{2} + \varepsilon \int_0^t \|\nabla \mathbb{F}_n\|_2^2 &= \frac{\|\mathbb{F}_n(0)\|_2^2}{2}. \end{aligned}$$

By the symmetry of $\mathbb{D}_n$ it holds $\mathbb{D}_n : \nabla \boldsymbol{v}_n = |\mathbb{D}_n|^2$ and by the symmetry of $\mathbb{F}_n\mathbb{F}_n^T$ it holds $(\mathbb{F}_n\mathbb{F}_n^T) : \nabla \boldsymbol{v}_n = (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n$, thus by summing the last two equations (both multiplied by 2), we get for all $t \in (0,T)$

$$\begin{aligned} \|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2 + &\int_0^t \left(2\|\mathbb{D}_n\|_2^2 + \|\mathbb{F}_n\mathbb{F}_n^T\|_2^2 + 2\varepsilon\|\nabla \mathbb{F}_n\|_2^2\right) \\ &\leq \|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2 + \int_0^t \|\mathbb{F}_n\|_2^2 \\ &\leq \|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2 + \int_0^t \left(\|\boldsymbol{v}_n\|_2^2 + \|\mathbb{F}_n\|_2^2\right). \end{aligned} \tag{5.17}$$

Since $\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2$ is estimated by the right handside of (5.17), the Gronwall lemma applied on (5.17) (the functions $\|\boldsymbol{v}_n(\cdot)\|_2$ and $\|\mathbb{F}_n(\cdot)\|_2$ are continuous in $[0,T)$) together with the conditions (5.15), (5.16) and the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ and of Q_n in $(L^2(\Omega))^{2\times 2}$ implies

$$\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2 \leq e^t \left(\|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2\right) \leq e^t \left(\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}_0\|_2^2\right). \tag{5.18}$$

Let us note that the inequality (5.18) will be useful in the proof of attainment of the initial conditions (5.5) and (5.6). The inequality (5.17) together with (5.18) yields for all $t \in (0,T)$

$$\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2 + \int_0^t \left(\|\mathbb{D}_n\|_2^2 + \|\mathbb{F}_n\mathbb{F}_n^T\|_2^2 + \varepsilon\|\nabla \mathbb{F}_n\|_2^2\right) \leq C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{5.19}$$

The matrix $\mathbb{F}_n\mathbb{F}_n^T$ acting in (5.19) is symmetric and positive semidefinite, hence it is a diagonalizable matrix and the corresponding diagonal matrix $\mathbb{J}_n$ has nonnegative diagonal terms. Thus the Young inequality gives in Q_T

$$|\mathbb{F}_n|^4 = \left(\mathrm{tr}(\mathbb{F}_n\mathbb{F}_n^T)\right)^2 = (\mathrm{tr}\ \mathbb{J}_n)^2 \leq 2\ \mathrm{tr}(\mathbb{J}_n^2) = 2\ \mathrm{tr}((\mathbb{F}_n\mathbb{F}_n^T)^2) = 2\ |\mathbb{F}_n\mathbb{F}_n^T|^2, \tag{5.20}$$

hence for all $t \in (0,T)$ it holds

$$\|\mathbb{F}_n\|_4^4 \leq 2\|\mathbb{F}_n\mathbb{F}_n^T\|_2^2. \tag{5.21}$$

Taking supremum over $t \in (0,T)$ at each term of (5.19) and using Korn's inequality and (5.21) leads to

$$\begin{aligned} \sup_{t\in(0,T)} \|\boldsymbol{v}_n(t)\|_2^2 + \sup_{t\in(0,T)} \|\mathbb{F}_n(t)\|_2^2 + \|\nabla \boldsymbol{v}_n\|_{2,Q_T}^2 + \|\mathbb{F}_n\|_{4,Q_T}^4 \\ + \varepsilon\|\nabla \mathbb{F}_n\|_{2,Q_T}^2 \leq \tilde{C}(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \end{aligned} \tag{5.22}$$

It remains to estimate the time derivatives of $\boldsymbol{v}_n$ and $\mathbb{F}_n$. Obviously, we can replace in (5.13) the base functions $\boldsymbol{w}_j$ by any function belonging to W_n and in (5.14) the base functions $\mathbb{A}_j$ by any function belonging to X_n. Let $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, by (5.13) it holds for all $t \in (0,T)$

$$\int_\Omega (\partial_t \boldsymbol{v}_n \cdot P_n(\boldsymbol{w})) = \int_\Omega \left((\boldsymbol{v}_n \otimes \boldsymbol{v}_n) - \mathbb{D}_n - \mathbb{F}_n\mathbb{F}_n^T\right) : \nabla P_n(\boldsymbol{w}). \tag{5.23}$$

Thanks to the orthogonality and the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, employing the Cauchy-Schwartz and the Hölder inequality, we derive from (5.23) for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ and for all $t \in (0,T)$

$$\begin{aligned} |\langle \partial_t \boldsymbol{v}_n, \boldsymbol{w}\rangle| &= \left| \int_\Omega (\partial_t \boldsymbol{v}_n \cdot \boldsymbol{w}) \right| = \left| \int_\Omega (\partial_t \boldsymbol{v}_n \cdot P_n(\boldsymbol{w})) \right| \\ &\le \int_\Omega \left| (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) - \mathbb{D}_n - \mathbb{F}_n \mathbb{F}_n^T \right| |\nabla P_n(\boldsymbol{w})| \\ &\le \left(\|\boldsymbol{v}_n\|_4^2 + \|\nabla \boldsymbol{v}_n\|_2 + \|\mathbb{F}_n\|_4^2 \right) \|\nabla \boldsymbol{w}\|_2. \end{aligned} \tag{5.24}$$

By the Ladyzenskaya inequality and (5.22) it holds

$$\|\boldsymbol{v}_n\|^4_{4,Q_T} = \int_0^T \|\boldsymbol{v}_n\|_4^4 \le \int_0^T \|\boldsymbol{v}_n\|_2^2 \|\nabla \boldsymbol{v}_n\|_2^2 \le \hat{C}\left(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2\right). \tag{5.25}$$

Integrating the second power of (5.24) over $(0,T)$, using (5.22), (5.25) and the Minkowski inequality, we can write

$$\|\partial_t \boldsymbol{v}_n\|^2_{L^2\left(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right)} \le \int_0^T \left(\|\boldsymbol{v}_n\|_4^4 + \|\nabla \boldsymbol{v}_n\|_2^2 + \|\mathbb{F}_n\|_4^4 \right) \le \overline{C}\left(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2\right). \tag{5.26}$$

Analogously we estimate $\|\partial_t \mathbb{F}_n\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)}$. Let $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$, by (5.14) it holds for all $t \in (0,T)$

$$\begin{aligned} \int_\Omega (\partial_t \mathbb{F}_n : Q_n(\mathbb{A})) = \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla Q_n(\mathbb{A}) + \int_\Omega \left(\nabla \boldsymbol{v}_n \mathbb{F}_n - \mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n + \mathbb{F}_n \right) : Q_n(\mathbb{A}) \\ - \varepsilon \int_\Omega \nabla \mathbb{F}_n : \nabla Q_n(\mathbb{A}). \end{aligned} \tag{5.27}$$

Employing the orthogonality and the continuity of Q_n in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$, the Cauchy-Schwartz and the Hölder inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ (it holds $\|a\|_4 \le \hat{C}\|a\|_{1,2}$ for every $a \in W^{1,2}(\Omega)$, where $\hat{C} = \hat{C}(\Omega)$), we obtain from (5.27) for all $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and $t \in (0,T)$

$$\begin{aligned} |\langle \partial_t \mathbb{F}_n, \mathbb{A} \rangle| &= \left| \int_\Omega (\partial_t \mathbb{F}_n : \mathbb{A}) \right| = \left| \int_\Omega (\partial_t \mathbb{F}_n : Q_n(\mathbb{A})) \right| \\ &\le \left(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2 \right) \|\nabla \mathbb{A}\|_2 + \|\mathbb{F}_n\|_2 \|\mathbb{A}\|_2 \\ &\quad + \left(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3 \right) \|Q_n(\mathbb{A})\|_4 \\ &\le \left(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2 + \|\mathbb{F}_n\|_2 \right) \|\mathbb{A}\|_{1,2} \\ &\quad + \hat{C}\left(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3 \right) \|Q_n(\mathbb{A})\|_{1,2} \\ &\le \Big(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2 + \|\mathbb{F}_n\|_2 + \hat{C}(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3) \Big) \|\mathbb{A}\|_{1,2}. \end{aligned}$$

Integrating the $\frac{4}{3}$-power of the last chain over $(0,T)$, using (5.22), (5.25) and Hölder's and Minkowski's inequalities, we conclude

$$\|\partial_t \mathbb{F}_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \le \overline{C}\left(\varepsilon, T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2\right). \tag{5.28}$$

Moreover, since due to (5.22) we have for $\varepsilon \le 1$ (here $C = C(\Omega)$)

$$\int_0^T (\varepsilon \|\nabla \mathbb{F}_n\|_2)^{\frac{4}{3}} \le \varepsilon^{\frac{1}{3}} C \int_0^T \varepsilon \|\nabla \mathbb{F}_n\|_2^2 \le \tilde{C}(T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2),$$

we can omit from (5.28) the dependence on ε and write

$$\|\partial_t \mathbb{F}_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \le \overline{C}\,(T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2)\,. \tag{5.29}$$

5.3. *Limit* $n \to \infty$

The uniform estimates (5.22), (5.26) and (5.28) imply the existence of $\boldsymbol{v}$, $\mathbb{F}$ satisfying the following convergence relations (the relations hold true for suitable subsequences of $\{\boldsymbol{v}_n\}$, $\{\mathbb{F}_n\}$, which we do not relable):

$$\begin{aligned}
\boldsymbol{v}_n &\rightharpoonup^* \boldsymbol{v} \quad \text{weakly-* in } L^\infty(0,T; L^2_{\boldsymbol{n},\mathrm{div}}), && (5.30)\\
\boldsymbol{v}_n &\rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap (L^4(Q_T))^2, && (5.31)\\
\partial_t \boldsymbol{v}_n &\rightharpoonup \partial_t \boldsymbol{v} \text{ weakly in } L^2\left(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), && (5.32)\\
\mathbb{F}_n &\rightharpoonup^* \mathbb{F} \quad \text{weakly-* in } L^\infty\left(0,T; (L^2(\Omega))^{2\times 2}\right), && (5.33)\\
\mathbb{F}_n &\rightharpoonup \mathbb{F} \quad \text{weakly in } L^2\left(0,T; (W^{1,2}(\Omega))^{2\times 2}\right) \cap (L^4(Q_T))^{2\times 2}, && (5.34)\\
\partial_t \mathbb{F}_n &\rightharpoonup \partial_t \mathbb{F} \text{ weakly in } L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right). && (5.35)
\end{aligned}$$

Let us note that thanks to the properties $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^2\left(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right)$, $\mathbb{F} \in L^\infty\left(0,T; (L^2(\Omega))^{2\times 2}\right)$ and $\partial_t \mathbb{F} \in L^{\frac{4}{3}}(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*)$ together with the density of $(W^{1,2}(\Omega))^{2\times 2}$ in $(L^2(\Omega))^{2\times 2}$, the functions $\boldsymbol{v}$, $\mathbb{F}$ after a possible change in a zero-measure subset of $(0,T)$ enjoy

$$\begin{aligned}
\boldsymbol{v} &\in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}), && (5.36)\\
\mathbb{F} &\in C_{weak}\left([0,T]; (L^2(\Omega))^{2\times 2}\right), && (5.37)
\end{aligned}$$

and thus (use also the weak lower semicontinuity of $L^2(\Omega)$ norm)

$$\begin{aligned}
\sup_{t\in(t_0,t_1)} \|\boldsymbol{v}(t)\|_2^2 &= \mathrm{esssup}_{t\in(t_0,t_1)} \|\boldsymbol{v}(t)\|_2^2, \quad 0 \le t_0 < t_1 \le T, && (5.38)\\
\sup_{t\in(t_0,t_1)} \|\mathbb{F}(t)\|_2^2 &\le \mathrm{esssup}_{t\in(t_0,t_1)} \|\mathbb{F}(t)\|_2^2, \quad 0 \le t_0 < t_1 \le T. && (5.39)
\end{aligned}$$

Employing (5.31), (5.32), (5.34), (5.35) and the Aubin-Lions compactness lemma, we get

$$\begin{aligned}
\boldsymbol{v}_n &\to \boldsymbol{v} \text{ strongly in } (L^q(Q_T))^2 \text{ for all } q \in [1,4), && (5.40)\\
\mathbb{F}_n &\to \mathbb{F} \text{ strongly in } (L^q(Q_T))^{2\times 2} \text{ for all } q \in [1,4). && (5.41)
\end{aligned}$$

From (5.31), (5.34), (5.40) and (5.41) we obtain also the following relations:

$$\begin{aligned}
\boldsymbol{v}_n \otimes \boldsymbol{v}_n &\rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} \text{ weakly in } (L^2(Q_T))^{2\times 2}, && (5.42)\\
\mathbb{F}_n \otimes \boldsymbol{v}_n &\rightharpoonup \mathbb{F} \otimes \boldsymbol{v} \text{ weakly in } (L^2(Q_T))^{2\times 2\times 2}, && (5.43)\\
\nabla \boldsymbol{v}_n \mathbb{F}_n &\rightharpoonup \nabla \boldsymbol{v} \mathbb{F} \text{ weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, && (5.44)\\
\mathbb{F}_n \mathbb{F}_n^T &\rightharpoonup \mathbb{F}\mathbb{F}^T \text{ weakly in } (L^2(Q_T))^{2\times 2}, && (5.45)\\
\mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n &\rightharpoonup \mathbb{F}\mathbb{F}^T\mathbb{F} \text{ weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}. && (5.46)
\end{aligned}$$

The convergence results above suffice to conclude from (5.13) and (5.14) for all $n \in \mathbb{N}$, $\boldsymbol{w} \in W_n$, $\mathbb{A} \in X_n$ and $\phi \in C_c^\infty(0,T)$

$$\int_0^T \langle \partial_t \boldsymbol{v}, \phi \boldsymbol{w} \rangle - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : \phi \nabla \boldsymbol{w} + \int_{Q_T} \mathbb{D} : \phi \nabla \boldsymbol{w} + \int_{Q_T} (\mathbb{F}\mathbb{F}^T) : \phi \nabla \boldsymbol{w} = 0,$$

$$\int_0^T \langle \partial_t \mathbb{F}, \phi \mathbb{A} \rangle - \int_{Q_T} (\mathbb{F} \otimes \boldsymbol{v}) : (\phi \nabla \mathbb{A}) - \int_{Q_T} (\nabla \boldsymbol{v} \mathbb{F}) : \phi \mathbb{A} + \frac{1}{2} \int_{Q_T} (\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \phi \mathbb{A} + \varepsilon \int_{Q_T} \nabla \mathbb{F} : (\phi \nabla \mathbb{A}) = 0.$$

Since $\bigcup_{n\in\mathbb{N}} W_n$ is dense in $W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\bigcup_{n\in\mathbb{N}} X_n$ is dense in $(W^{1,2}(\Omega))^{2\times 2}$, by using the Du Bois–Reymond lemma we obtain for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t \in (0,T)$ the equations (5.8) and (5.9), i.e.

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \int_\Omega (\mathbb{F}\mathbb{F}^T) : \nabla \boldsymbol{w} = 0, \quad (5.47)$$

$$\langle \partial_t \mathbb{F}, \mathbb{A} \rangle - \int_\Omega (\mathbb{F} \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v})\mathbb{F}) : \mathbb{A} + \frac{1}{2} \int_\Omega (\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \mathbb{A} + \varepsilon \int_\Omega \nabla \mathbb{F} : \nabla \mathbb{A} = 0. \quad (5.48)$$

5.4. *Attainment of the initial data*

Multiplying (5.13) by any $\phi \in C_c^\infty(-\infty,T)$, $\phi(0) \neq 0$, integrating over $(0,T)$ and employing the orthogonality of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ (together with the condition (5.15)) yields for every $j \le n$, $\boldsymbol{w}_j \in W_j$

$$-\int_\Omega \boldsymbol{v}_0 \cdot \phi(0) \boldsymbol{w}_j - \int_{Q_T} \boldsymbol{v}_n \cdot (\partial_t \phi) \boldsymbol{w}_j + \int_{Q_T} \big(-(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) + \mathbb{D}_n + \mathbb{F}_n \mathbb{F}_n^T \big) : (\phi \nabla \boldsymbol{w}_j) = 0. \quad (5.49)$$

Multiplying (5.47) by $\phi \in C_c^\infty(-\infty,T)$, $\phi(0) \neq 0$, and integrating over $(0,T)$ yields for every $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$

$$-\int_\Omega \boldsymbol{v}(0) \cdot \phi(0) \boldsymbol{w} - \int_{Q_T} \boldsymbol{v} \cdot (\partial_t \phi) \boldsymbol{w} + \int_{Q_T} \big(-(\boldsymbol{v} \otimes \boldsymbol{v}) + \mathbb{D} + \mathbb{F}\mathbb{F}^T \big) : (\phi \nabla \boldsymbol{w}) = 0. \quad (5.50)$$

Subtracting (5.49) from (5.50), applying (5.31), (5.42), (5.45), the density of $\bigcup_{n\in\mathbb{N}} W_n$ in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\mathbf{0},\mathrm{div}}$, passing $n \to \infty$, $j \to \infty$ and dividing the result by $\phi(0)$ leads to

$$\int_\Omega \boldsymbol{v}(0) \cdot \boldsymbol{w} = \int_\Omega \boldsymbol{v}_0 \cdot \boldsymbol{w} \qquad \forall \boldsymbol{w} \in L^2_{\boldsymbol{n},\mathrm{div}}. \quad (5.51)$$

Multiplying (5.14) by $\phi \in C_c^\infty(-\infty,T)$, $\phi(0) \neq 0$, integrating over $(0,T)$ and employing the orthogonality of Q_n in $(L^2(\Omega))^{2\times 2}$ (together with the condition (5.16)) yields for every $j \le n$, $\mathbb{A}_j \in X_j$

$$-\int_\Omega \mathbb{F}_0 : \phi(0) \mathbb{A}_j - \int_{Q_T} \mathbb{F}_n : (\partial_t \phi) \mathbb{A}_j - \int_{Q_T} (\mathbb{F}_n \otimes \boldsymbol{v}_n) : (\phi \nabla \mathbb{A}_j) + \int_{Q_T} \left(-\nabla \boldsymbol{v}_n \mathbb{F}_n + \frac{\mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n - \mathbb{F}_n}{2} \right) : (\phi \mathbb{A}_j) + \varepsilon \int_{Q_T} \nabla \mathbb{F}_n : (\phi \nabla \mathbb{A}_j) = 0. \quad (5.52)$$

Multiplying (5.48) by $\phi \in C_c^\infty(-\infty,T)$, $\phi(0)\neq 0$, and integrating the result over $(0,T)$ yields for every $\mathbb{A}\in (W^{1,2}(\Omega))^{2\times 2}$

$$-\int_\Omega \mathbb{F}(0):\phi(0)\mathbb{A} - \int_{Q_T}\mathbb{F}:(\partial_t\phi)\mathbb{A} - \int_{Q_T}(\mathbb{F}\otimes \boldsymbol{v}):(\phi\nabla\mathbb{A}) + \int_{Q_T}\left(-\nabla \boldsymbol{v}\mathbb{F} + \frac{\mathbb{F}\mathbb{F}^T\mathbb{F}-\mathbb{F}}{2}\right):(\phi\mathbb{A}) \\ + \varepsilon\int_{Q_T}\nabla\mathbb{F}:(\phi\nabla\mathbb{A}) = 0.$$

Subtracting (5.52) from the last equation, applying (5.34), (5.43), (5.44) and (5.46) and the density of $\bigcup_{n\in\mathbb{N}} X_n$ in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$, passing $n\to\infty$, $j\to\infty$ and dividing the result by $\phi(0)$ leads to

$$\int_\Omega \mathbb{F}(0):\mathbb{A} = \int_\Omega \mathbb{F}_0:\mathbb{A} \qquad \forall \mathbb{A}\in (L^2(\Omega))^{2\times 2}. \tag{5.53}$$

In order to prove the attainment of the initial conditions in the sense of (5.10) and (5.11) we take the limit $n\to\infty$ in (5.18). From (5.30) and (5.33) we deduce that $\boldsymbol{v}_n(t)\rightharpoonup \boldsymbol{v}(t)$ weakly in $L^2_{\boldsymbol{n},\mathrm{div}}$ and $\mathbb{F}_n(t)\rightharpoonup\mathbb{F}(t)$ weakly in $(L^2(\Omega))^{2\times 2}$ for a.a. $t\in(0,T)$, hence by the weak lower semicontinuity of $L^2(\Omega)$ norm we have

$$\|\boldsymbol{v}(t)\|_2^2 + \|\mathbb{F}(t)\|_2^2 \le e^t\left(\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}_0\|_2^2\right) \quad \text{for a.a. } t\in(0,T). \tag{5.54}$$

Let $\delta\in(0,T)$ be arbitrary. From (5.38), (5.39) and (5.54) it follows

$$\sup_{t\in(0,\delta)}\left(\|\boldsymbol{v}(t)\|_2^2+\|\mathbb{F}(t)\|_2^2\right) \le \operatorname{esssup}_{t\in(0,\delta)}\left(\|\boldsymbol{v}(t)\|_2^2+\|\mathbb{F}(t)\|_2^2\right) \le e^\delta\left(\|\boldsymbol{v}_0\|_2^2+\|\mathbb{F}_0\|_2^2\right), \tag{5.55}$$

which yields

$$\limsup_{t\to 0+}\left(\|\boldsymbol{v}(t)\|_2^2+\|\mathbb{F}(t)\|_2^2\right) \le \|\boldsymbol{v}_0\|_2^2+\|\mathbb{F}_0\|_2^2. \tag{5.56}$$

Collecting (5.51) with $\boldsymbol{w}:=\boldsymbol{v}_0$, (5.53) with $\mathbb{A}:=\mathbb{F}_0$ and (5.56), we conclude

$$\limsup_{t\to 0+}\left(\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}(t)-\mathbb{F}_0\|_2^2\right)\le 0, \tag{5.57}$$

which immediately implies fulfilling of (5.10) and (5.11).

6. Proof of Theorem 4.2

From Section 5, Proposition 5.1, we have for each $\varepsilon>0$ a couple $[\boldsymbol{v}_\varepsilon,\mathbb{F}_\varepsilon]$ fulfilling

$$\boldsymbol{v}_\varepsilon \in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}})\cap L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}}), \tag{6.1}$$

$$\partial_t\boldsymbol{v}_\varepsilon \in L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*), \tag{6.2}$$

$$\mathbb{F}_\varepsilon \in C_{weak}([0,T];(L^2(\Omega))^{2\times 2})\cap L^2(0,T;(W^{1,2}(\Omega))^{2\times 2}), \tag{6.3}$$

$$\partial_t\mathbb{F}_\varepsilon \in L^{\frac{4}{3}}\left(0,T;((W^{1,2}(\Omega))^{2\times 2})^*\right) \tag{6.4}$$

and satisfying for all $\boldsymbol{w}\in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\mathbb{A}\in(W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t\in(0,T)$ (we denote $\mathbb{D}_\varepsilon := \frac{1}{2}\left(\nabla\boldsymbol{v}_\varepsilon + (\nabla\boldsymbol{v}_\varepsilon)^T\right)$)

$$\langle\partial_t\boldsymbol{v}_\varepsilon,\boldsymbol{w}\rangle - \int_\Omega(\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon):\nabla\boldsymbol{w} + \int_\Omega\mathbb{D}_\varepsilon:\nabla\boldsymbol{w} + \int_\Omega(\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T):\nabla\boldsymbol{w} = 0, \tag{6.5}$$

$$\langle \partial_t \mathbb{F}_\varepsilon, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v}_\varepsilon)\mathbb{F}_\varepsilon) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon) : \mathbb{A} + \varepsilon \int_\Omega \nabla \mathbb{F}_\varepsilon : \nabla \mathbb{A} = 0 \tag{6.6}$$

with the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_0$ fulfilled in the sense

$$\lim_{t\to 0+} \|\boldsymbol{v}_\varepsilon(t) - \boldsymbol{v}_0\|_2 = 0, \tag{6.7}$$

$$\lim_{t\to 0+} \|\mathbb{F}_\varepsilon(t) - \mathbb{F}_0\|_2 = 0. \tag{6.8}$$

6.1. *Limit in approximations*

First, let us mention that whenever we use the results from Section 5, the functions $\boldsymbol{v}$, $\mathbb{F}$ established in Section 5 correspond for a fixed $\varepsilon > 0$ to the functions $\boldsymbol{v}_\varepsilon$, $\mathbb{F}_\varepsilon$ established in (6.1)–(6.8).

Employing the convergences (5.30), (5.33), we get (for suitable subsequences of $\{\boldsymbol{v}_n\}$, $\{\mathbb{F}_n\}$, which we do not relabel)

$$\boldsymbol{v}_n(t) \rightharpoonup \boldsymbol{v}_\varepsilon(t) \quad \text{weakly in } L^2_{\boldsymbol{n},\mathrm{div}} \quad \text{for a.a. } t \in (0,T), \tag{6.9}$$

$$\mathbb{F}_n(t) \rightharpoonup \mathbb{F}_\varepsilon(t) \quad \text{weakly in } (L^2(\Omega))^{2\times 2} \quad \text{for a.a. } t \in (0,T). \tag{6.10}$$

The convergences (5.31), (5.34), (6.9), (6.10) and the estimate (5.22) together with (5.38), (5.39) and weak lower semicontinuity of all norms acting in (5.22) lead to the following uniform estimate for $\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_\varepsilon\}$:

$$\sup_{t\in(0,T)} \|\boldsymbol{v}_\varepsilon\|_2^2 + \sup_{t\in(0,T)} \|\mathbb{F}_\varepsilon\|_2^2 + \|\nabla \boldsymbol{v}_\varepsilon\|_{2,Q_T}^2 + \|\mathbb{F}_\varepsilon\|_{4,Q_T}^4 + \varepsilon \|\nabla \mathbb{F}_\varepsilon\|_{2,Q_T}^2 \le \tilde{C}(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{6.11}$$

In order to derive the uniform estimates for $\partial_t \boldsymbol{v}_\varepsilon$ in $L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)$ and for $\partial_t \mathbb{F}_\varepsilon$ in $L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)$ from (6.5), (6.6) and (6.11), we proceed simmilarly as in Section 5, where we derived from (5.13), (5.14) and (5.22) the estimates for $\partial_t \boldsymbol{v}_n$ and $\partial_t \mathbb{F}_n$ in the same norms, see the passage (5.23)–(5.29) (here it is more simple since $\boldsymbol{w}$ acting in (6.5) belongs to $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, not only to W_n, and $\mathbb{A}$ acting in (6.6) belongs to $(W^{1,2}(\Omega))^{2\times 2}$, not only to X_n). The estimates read as

$$\|\partial_t \boldsymbol{v}_\varepsilon\|_{L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)} + \|\partial_t \mathbb{F}_\varepsilon\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \le \overline{C}(T,\Omega,\|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{6.12}$$

The uniform estimates (6.11) and (6.12) imply the existence of $\boldsymbol{v}$, $\mathbb{F}$ fulfilling the following convergence relations (for suitable subsequences of $\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_\varepsilon\}$, which we do not relable):

$$\boldsymbol{v}_\varepsilon \rightharpoonup^* \boldsymbol{v} \quad \text{weakly-* in } L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}), \tag{6.13}$$

$$\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \tag{6.14}$$

$$\partial_t \boldsymbol{v}_\varepsilon \rightharpoonup \partial_t \boldsymbol{v} \text{ weakly in } L^2\left(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), \tag{6.15}$$

$$\begin{aligned}
\mathbb{F}_\varepsilon &\rightharpoonup^* \mathbb{F} \quad \text{weakly-* in } L^\infty\left(0,T;(L^2(\Omega))^{2\times 2}\right), && (6.16)\\
\mathbb{F}_\varepsilon &\rightharpoonup \mathbb{F} \quad \text{weakly in } (L^4(Q_T))^{2\times 2}, && (6.17)\\
\varepsilon\nabla\mathbb{F}_\varepsilon &\to \mathbb{O} \quad \text{strongly in } (L^2(Q_T))^{2\times 2\times 2}, && (6.18)\\
\partial_t\mathbb{F}_\varepsilon &\rightharpoonup \partial_t\mathbb{F} \text{ weakly in } L^{\frac{4}{3}}\left(0,T;((W^{1,2}(\Omega))^{2\times 2})^*\right). && (6.19)
\end{aligned}$$

Let us note that thanks to the properties $\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^2\left(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*\right)$, $\mathbb{F} \in L^\infty\left(0,T;(L^2(\Omega))^{2\times 2}\right)$ and $\partial_t\mathbb{F} \in L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)$ together with the density of $(W^{1,2}(\Omega))^{2\times 2}$ in $(L^2(\Omega))^{2\times 2}$, the functions $\boldsymbol{v}$, $\mathbb{F}$ after a possible change on a zero-measure subset of $(0,T)$ enjoy

$$\begin{aligned}
\boldsymbol{v} &\in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}}), && (6.20)\\
\mathbb{F} &\in C_{weak}\left([0,T];(L^2(\Omega))^{2\times 2}\right), && (6.21)
\end{aligned}$$

and thus (use also the weak lower semicontinuity of $L^2(\Omega)$ norm)

$$\begin{aligned}
\sup_{t\in(t_0,t_1)} \|\boldsymbol{v}(t)\|_2^2 &= \operatorname{esssup}_{t\in(t_0,t_1)}\|\boldsymbol{v}(t)\|_2^2, \quad 0\le t_0<t_1\le T, && (6.22)\\
\sup_{t\in(t_0,t_1)} \|\mathbb{F}(t)\|_2^2 &\le \operatorname{esssup}_{t\in(t_0,t_1)}\|\mathbb{F}(t)\|_2^2, \quad 0\le t_0<t_1\le T. && (6.23)
\end{aligned}$$

Employing (6.14), (6.15) and the Aubin-Lions compactness lemma, we get

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \text{ strongly in } (L^q(Q_T))^2 \text{ for all } q\in[1,4). \qquad (6.24)$$

The weak convergences (6.14), (6.17) together with the strong convergence (6.24) yield

$$\begin{aligned}
\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon &\rightharpoonup \boldsymbol{v}\otimes\boldsymbol{v} \text{ weakly in } (L^2(Q_T))^{2\times 2}, && (6.25)\\
\mathbb{F}_\varepsilon\otimes\boldsymbol{v}_\varepsilon &\rightharpoonup \mathbb{F}\otimes\boldsymbol{v} \text{ weakly in } (L^2(Q_T))^{2\times 2\times 2}. && (6.26)
\end{aligned}$$

Next, for a weakly or weakly-* convergent subsequence of $\{a_\varepsilon\}$ let us denote the corresponding limit by $\overline{a}$. It holds

$$\begin{aligned}
\nabla\boldsymbol{v}_\varepsilon\mathbb{F}_\varepsilon &\rightharpoonup \overline{(\nabla\boldsymbol{v})\mathbb{F}} \text{ weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, && (6.27)\\
\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T &\rightharpoonup \overline{\mathbb{F}\mathbb{F}^T} \text{ weakly in } (L^2(Q_T))^{2\times 2}, && (6.28)\\
|\mathbb{F}_\varepsilon|^2 &\rightharpoonup \overline{|\mathbb{F}|^2} \text{ weakly in } L^2(Q_T), && (6.29)\\
\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T\mathbb{F}_\varepsilon &\rightharpoonup \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} \text{ weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, && (6.30)\\
|\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T|^2 &\rightharpoonup^* \overline{|\mathbb{F}\mathbb{F}^T|^2} \text{ weakly-* in } \mathcal{M}(\overline{Q_T}), && (6.31)\\
\nabla\boldsymbol{v}_\varepsilon\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T &\rightharpoonup^* \overline{\nabla\boldsymbol{v}\mathbb{F}\mathbb{F}^T} \text{ weakly-* in } (\mathcal{M}(\overline{Q_T}))^{2\times 2}, && (6.32)\\
|\mathbb{D}_\varepsilon|^2 &\rightharpoonup^* \overline{|\mathbb{D}|^2} \text{ weakly-* in } \mathcal{M}(\overline{Q_T}), && (6.33)
\end{aligned}$$

as the sequences in (6.27) – (6.30) are uniformly bounded in the corresponding spaces, the sequences $|\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T|^2$, $|\mathbb{D}_\varepsilon|^2$ are uniformly bounded in $L^1(Q_T)$ and the sequence $\nabla\boldsymbol{v}_\varepsilon\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T$ is uniformly bounded in $(L^1(Q_T))^{2\times 2}$ (it follows from the estimate (6.11) and Hölder's inequality).

The convergence results above applied on (6.5) and (6.6) suffice to conclude for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t \in (0,T)$

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \int_\Omega \overline{\mathbb{F}\mathbb{F}^T} : \nabla \boldsymbol{w} = 0, \tag{6.34}$$

$$\langle \partial_t \mathbb{F}, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}\otimes\boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega \overline{(\nabla\boldsymbol{v})\mathbb{F}} : \mathbb{A} + \frac{1}{2}\int_\Omega (\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} - \mathbb{F}) : \mathbb{A} = 0. \tag{6.35}$$

In order to prove the attainment of the initial conditions (4.11), (4.12), we follow step by step the proof of the corresponding conditions for $\boldsymbol{v}_\varepsilon$ and $\mathbb{F}_\varepsilon$ (the conditions (5.10), (5.11) with $\boldsymbol{v}_\varepsilon$ in the role of $\boldsymbol{v}$, $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$) presented in Subsection 5.4. Let us briefly mention what is different. In the equality corresponding to (5.49) here we consider $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, not only in W_n, in the equality corresponding to (5.52) we consider $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$, not only in X_n. In order to obtain the equality

$$\lim_{t\to 0+} \int_\Omega \boldsymbol{v}(t)\cdot\boldsymbol{w} = \int_\Omega \boldsymbol{v}_0\cdot\boldsymbol{w} \quad \forall \boldsymbol{w} \in L^2_{\boldsymbol{n},\mathrm{div}}, \tag{6.36}$$

instead of the density of $\bigcup_{n\in\mathbb{N}} W_n$ in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\mathbf{0},\mathrm{div}}$, here we use only the density of $W^{1,2}_{\mathbf{0},\mathrm{div}}$ in $L^2_{\boldsymbol{n},\mathrm{div}}$, in order to obtain the equality

$$\lim_{t\to 0+} \int_\Omega \mathbb{F}(t) : \mathbb{A} = \int_\Omega \mathbb{F}_0 : \mathbb{A} \quad \forall \mathbb{A} \in (L^2(\Omega))^{2\times 2}, \tag{6.37}$$

instead of the density of $\bigcup_{n\in\mathbb{N}} X_n$ in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$, here we use only the density of $(W^{1,2}(\Omega))^{2\times 2}$ in $(L^2(\Omega))^{2\times 2}$. In order to obtain the inequality

$$\limsup_{t\to 0+} \left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 + \|\mathbb{F}(t) - \mathbb{F}_0\|_2^2\right) \le 0, \tag{6.38}$$

we employ the estimate (5.55) with $\boldsymbol{v}_\varepsilon$ in the role of $\boldsymbol{v}$ and $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$, the convergences $\boldsymbol{v}_\varepsilon(t) \rightharpoonup \boldsymbol{v}$ weakly in $L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_\varepsilon \rightharpoonup \mathbb{F}$ weakly in $(L^2(\Omega))^{2\times 2}$ for a.a. $t \in (0,T)$ following from (6.13) and (6.16), the relations (6.22), (6.23) and the weak lower semicontinuity of all norms acting in (5.55). Employing (6.36) with $\boldsymbol{w} := \boldsymbol{v}_0$, (6.37) with $\mathbb{A} := \mathbb{F}_0$ and (6.38), we arrive at (4.11) and (4.12).

6.2. *Global in time continuity*

Let us recall that

$$\boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \tag{6.39}$$

$$\mathbb{F} \in C_{weak}([0,T]; (L^2(\Omega))^{2\times 2}) \cap (L^4(Q_T))^{2\times 2}. \tag{6.40}$$

Our aim is to prove that even

$$\mathbb{F} \in C([0,T]; (L^2(\Omega))^{2\times 2}). \tag{6.41}$$

Let $t_0, t_1 \in [0,T]$, $t_0 < t_1$. In (6.34) set $\boldsymbol{w} := 2\boldsymbol{v}$, use the integration by parts and (6.39), integrate the result over (t_0,t_1) to obtain

$$\|\boldsymbol{v}(t_1)\|_2^2 - \|\boldsymbol{v}(t_0)\|_2^2 + 2\int_{t_0}^{t_1} \|\mathbb{D}\|_2^2 + 2\int_{t_0}^{t_1}\int_\Omega \overline{\mathbb{F}\mathbb{F}^T} : \nabla\boldsymbol{v} = 0. \tag{6.42}$$

Now extend $\boldsymbol{v}$ and $\mathbb{F}$ by zero outside of Ω. Let $\delta > 0$ be arbitrary. Test (6.35) by $(\omega_\delta(\boldsymbol{x}-\,\cdot)\,\mathbb{A}(\boldsymbol{x}))$, where $\boldsymbol{x} \in \Omega$ is a fixed point, ω_δ is the standard space mollifying kernel, $h_\delta(\boldsymbol{x}) := \int_{\mathbb{R}^2} \omega_\delta(\boldsymbol{x}-\cdot)h(\cdot)$ for every $h \in L^1_{loc}(\mathbb{R}^2)$, $\mathbb{A} \in (C^\infty(\Omega))^{2\times 2}$ is arbitrary. After the multiplication by an arbitrary $\phi \in C^\infty_c((0,T))$, integration over Q_T, using (6.39), (6.40), standard properties of mollifying kernels and the Du Bois-Reymod lemma, we obtain

$$\partial_t \mathbb{F}_\delta \in L^{\frac{4}{3}}(0,T;(C^\infty(\Omega))^{2\times 2})$$

and

$$\partial_t \mathbb{F}_\delta = -\mathrm{Div}(\mathbb{F}\otimes\boldsymbol{v})_\delta + (\overline{\nabla \boldsymbol{v}\mathbb{F}})_\delta - (\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}})_\delta + \mathbb{F}_\delta \quad \text{a.e. in } Q_T. \tag{6.43}$$

Multiplying (6.43) scalarly by $2\mathbb{F}_\delta$ and integrating the result over $(t_0,t_1)\times\Omega$ leads to

$$\begin{aligned} \|\mathbb{F}_\delta(t_1)\|_2^2 - \|\mathbb{F}_\delta(t_0)\|_2^2 + 2\int_{t_0}^{t_1}\int_\Omega \big(\mathrm{Div}(\mathbb{F}_\delta\otimes\boldsymbol{v}) - (\overline{\nabla\boldsymbol{v}\mathbb{F}})_\delta\big):\mathbb{F}_\delta \\ + \int_{t_0}^{t_1}\int_\Omega \Big((\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}})_\delta - \mathbb{F}_\delta\Big):\mathbb{F}_\delta = 2\int_{t_0}^{t_1}\int_\Omega \mathbb{E}_\delta:\mathbb{F}_\delta, \end{aligned} \tag{6.44}$$

where

$$\mathbb{E}_\delta := \mathrm{Div}(\mathbb{F}_\delta\otimes\boldsymbol{v}) - \mathrm{Div}(\mathbb{F}\otimes\boldsymbol{v})_\delta.$$

In (6.44) pass $\delta \to 0+$. Employing (6.39) and (6.40), Lemma 3.1 implies

$$\mathbb{E}_\delta \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(\Omega))^{2\times 2} \text{ for a.a. } t \in (0,T) \tag{6.45}$$

and

$$\|\mathbb{E}_\delta\|_{\frac{4}{3}} \le \|\mathbb{F}\|_4\|\boldsymbol{v}\|_{1,2} \quad \text{for a.a. } t \in (0,T).$$

Applying Lebesgue's convergence theorem on (6.45) with majorant $\|\mathbb{F}\|_4^{\frac{4}{3}}\|\boldsymbol{v}\|_{1,2}^{\frac{4}{3}}$ integrable over $(0,T)$ (the integrability over $(0,T)$ follows from (6.39), (6.40) and Hölder's inequality) then leads to

$$\mathbb{E}_\delta \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}. \tag{6.46}$$

Using the integration by parts, (6.39), (6.40), (6.46) and standard properties of mollifying kernels, it follows from (6.44) by passing $\delta \to 0+$

$$\|\mathbb{F}(t_1)\|_2^2 - \|\mathbb{F}(t_0)\|_2^2 - 2\int_{t_0}^{t_1}\int_\Omega (\overline{\nabla\boldsymbol{v}\mathbb{F}}):\mathbb{F} + \int_{t_0}^{t_1}\int_\Omega \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}}:\mathbb{F} = \int_{t_0}^{t_1}\|\mathbb{F}\|_2^2. \tag{6.47}$$

By (6.39) and (6.40) the terms $\|\mathbb{D}\|_2^2$ and $\|\mathbb{F}\|_2^2$ are integrable over $(0,T)$, the terms $\overline{(\nabla\boldsymbol{v})\mathbb{F}}:\mathbb{F}$, $\overline{\mathbb{F}\mathbb{F}^T}:\nabla\boldsymbol{v}$ and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}}:\mathbb{F}$ are integrable over Q_T, hence we have for all $t_0 \in (0,T)$ (if $t_0 = 0$, resp. $t_0 = T$, then the following limit holds as $t_1 \to t_0+$, resp. as $t_1 \to t_0-$)

$$\lim_{t_1\to t_0}\int_{t_0}^{t_1}\big(2\|\mathbb{D}\|_2^2 - \|\mathbb{F}\|_2^2\big) + \int_{t_0}^{t_1}\int_\Omega\Big(2\overline{\mathbb{F}\mathbb{F}^T}:\nabla\boldsymbol{v} - 2\overline{(\nabla\boldsymbol{v})\mathbb{F}}:\mathbb{F} + \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}}:\mathbb{F}\Big) = 0. \tag{6.48}$$

Summing (6.42) and (6.47), using the property $\boldsymbol{v} \in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}})$ and (6.48), we conclude the following formulae equivalent to (6.41):

$$\lim_{t_1\to t_0}\|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(t_0)\|_2^2 \qquad \forall t_0 \in (0,T),$$

$$\lim_{t_1\to 0+}\|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(0)\|_2^2, \quad \lim_{t_1\to T-}\|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(T)\|_2^2.$$

To complete the proof of Theorem 4.2, except the property $\det \mathbb{F} > 0$ a.e. in Q_T, it remains to show $\overline{\mathbb{F}\mathbb{F}^T} = \mathbb{F}\mathbb{F}^T$, $\overline{\nabla \boldsymbol{v} \mathbb{F}} = \nabla \boldsymbol{v} \mathbb{F}$ and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} = \mathbb{F}\mathbb{F}^T\mathbb{F}$ in (6.34) and (6.35). As we already know that $\nabla \boldsymbol{v}_\varepsilon \rightharpoonup \nabla \boldsymbol{v}$ weakly in $(L^2(Q_T))^{2\times 2}$ by (6.14) and $\mathbb{F}_\varepsilon \rightharpoonup \mathbb{F}$ weakly in $(L^4(Q_T))^{2\times 2}$ by (6.17), it suffices to prove the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$.

6.3. ***Compactness of*** $\{\mathbb{F}_\varepsilon\}$ ***in*** $(L^2(Q_T))^{2\times 2}$

Let us start with the observation

$$\lim_{\varepsilon \to 0+} \|\mathbb{F}_\varepsilon - \mathbb{F}\|^2_{2,Q_T} = \lim_{\varepsilon \to 0+} \int_{Q_T} \left(|\mathbb{F}_\varepsilon|^2 - 2\mathbb{F}_\varepsilon : \mathbb{F} + |\mathbb{F}|^2 \right) = \int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right),$$

where the last equality follows from (6.17) and (6.29). This observation reduces the proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ to proving

$$\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2 \quad \text{a.e. in } Q_T. \tag{6.49}$$

Here we follow the concept by Masmoudi [1] and work with the difference between (5.3) (with $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$) formally multiplied scalarly by $\mathbb{F}_\varepsilon$ and (4.3) formally multiplied scalarly by $\mathbb{F}$. Let us note that Masmoudi does not consider the term $\varepsilon \Delta \mathbb{F}_\varepsilon$ in (5.3) as he deals only with the weak sequential stability of hypothetical weak solutions to the system (4.1)–(4.7). After the integration over $(0,T)$ and passing $\varepsilon \to 0+$, we arrive at the inequality formally written as

$$\partial_t (\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2) + \operatorname{div} \left((\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\, \boldsymbol{v} \right) \le L \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right), \tag{6.50}$$

where L is a sufficiently regular function. The inequality (6.50) may seem to be prepared (after the integration over time and space) for applying Gronwall's lemma and concluding $\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2$ a.e. in Q_T (the condition (6.49)). However, this conclusion is not straightforward unless $L \in L^\infty(Q_T)$, $\int_\Omega (\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)$ belongs to $C([0,T])$ and $\overline{|\mathbb{F}|^2}(0,\cdot) = |\mathbb{F}|^2(0,\cdot)$ a.e. in Ω, about which we have no information (we do not even know whether $\overline{|\mathbb{F}|^2}$ is weakly continuous with respect to time), hence some additional work is required. Moreover, deriving the inequality (6.50) by employing the concept described above, is also not a trivial task and requires some new techniques, for example, in order to avoid the obstacles connected with the presence of highly nonlinear terms $\overline{|\mathbb{F}\mathbb{F}^T|^2}$ comming from (5.3) (with $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$) formally multiplied scalarly by $\mathbb{F}_\varepsilon$ and limited as $\varepsilon \to 0+$, and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}$ comming from (4.3) formally multiplied scalarly by $\mathbb{F}$, we show that the difference $\overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}$ is nonnegative in $\mathcal{M}(Q_T)$ using the monotonicity of the matrix function $S(\mathbb{X}) = \mathbb{X}\mathbb{X}^T\mathbb{X}$ for all $\mathbb{X} \in \mathbb{R}^{2\times 2}$, introduced in Lemma 3.2. Last but not least, in order to obtain a version of the inequality (6.50), from which we will be capable of concluding the result (6.49), at certain point we need to use the balances of linear momenta (evolutionary equations for $\boldsymbol{v}_\varepsilon$ and $\boldsymbol{v}$) tested by functions that are not divergence free. This requires to reconstruct the pressures p_ε, p and show the convergence of p_ε to p in a suitable sense. As this is a kind of a more general tool, which might have further applications (as one may check, we can replace $\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T$ and $\overline{\mathbb{F}\mathbb{F}^T}$ acting in Proposition 6.1 by any $\mathbb{H}_\varepsilon$ converging weakly to $\mathbb{H}$ in $L^2(0,T;(L^2_{loc}(\Omega))^{2\times 2}))$, we introduce the result on the reconstruction of the pressures and their convergence before the own proof of the compactness.

6.3.1. ***Reconstruction of the pressures and their convergence***

We reconstruct the pressures by virtue of Wolf [24], then we show their convergence. For any $\tilde{\Omega} \in C^{0,1}$, $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$, let us define the spaces

$$\begin{aligned} W_0^{1,2}(\tilde{\Omega}) &:= \{u \in W^{1,2}(\tilde{\Omega}),\ \ u = 0 \text{ on } \partial\tilde{\Omega}\}, \\ \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}} &:= \{\boldsymbol{u} \in (W_0^{1,2}(\tilde{\Omega}))^2,\ \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \tilde{\Omega}\}. \end{aligned}$$

The resulting proposition reads as follows:

Proposition 6.1. *Let $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$, $\tilde{\Omega} \in C^\infty$. Then for every $\varepsilon > 0$ there exists p_ε of the form $p_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon}$, where*

$$p_{1,\varepsilon} \in L^2(0,T; W^{2,2}(\tilde{\Omega})), \tag{6.51}$$

$$p_{2,\varepsilon} \in L^2((0,T) \times \tilde{\Omega}), \tag{6.52}$$

$$\partial_t\left(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}\right) \in L^2\left(0,T; ((W_0^{1,2}(\tilde{\Omega}))^2)^*\right) \tag{6.53}$$

and for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and a.a. $t \in (0,T)$ it holds

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \boldsymbol{w}\rangle = \int_{\tilde{\Omega}} (\mathbb{G}_\varepsilon : \nabla \boldsymbol{w}) + \int_{\tilde{\Omega}} p_{2,\varepsilon} \operatorname{div} \boldsymbol{w}, \quad \mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T. \tag{6.54}$$

Next, there exists p of the form $p = p_1 + p_2$, where

$$p_1 \in L^2(0,T; W^{2,2}(\tilde{\Omega})), \tag{6.55}$$

$$p_2 \in L^2((0,T) \times \tilde{\Omega}), \tag{6.56}$$

$$\partial_t\left(\boldsymbol{v} + \nabla p_1\right) \in L^2\left(0,T; ((W_0^{1,2}(\tilde{\Omega}))^2)^*\right) \tag{6.57}$$

and for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and a.a. $t \in (0,T)$ it holds

$$\langle \partial_t(\boldsymbol{v} + \nabla p_1), \boldsymbol{w}\rangle = \int_{\tilde{\Omega}} (\mathbb{G} : \nabla \boldsymbol{w}) + \int_{\tilde{\Omega}} p_2 \operatorname{div} \boldsymbol{w}, \quad \mathbb{G} := (\boldsymbol{v} \otimes \boldsymbol{v}) - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}. \tag{6.58}$$

Moreover,

$$p_{1,\varepsilon} \to p_1 \text{ strongly in } L^2(0,T; W^{2,2}_{loc}(\tilde{\Omega})), \tag{6.59}$$

$$p_{2,\varepsilon} \rightharpoonup p_2 \text{ weakly in } L^2((0,T) \times \tilde{\Omega}). \tag{6.60}$$

The functions $\nabla p_{1,\varepsilon}$ and ∇p_1 belong to $C([0,T]; (L^2(\tilde{\Omega}))^2)$ and

$$\nabla p_{1,\varepsilon}(0,\cdot) = \nabla p_1(0,\cdot) \quad \text{a.e. in } \tilde{\Omega}. \tag{6.61}$$

Proof. Since the proof is very long and technical, we decided to move it to the Appendix. ∎

6.3.2. ***Own proof of the compactness of*** $\{\mathbb{F}_\varepsilon\}$ ***in*** $(L^2(Q_T))^{2\times 2}$

The own proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ consists of three steps.

Step 1: Deriving suitable forms of (5.3) (with $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$) multiplied scalarly by $\mathbb{F}_\varepsilon$ and limited as $\varepsilon \to 0+$ and of (4.3) multiplied scalarly by $\mathbb{F}$. More precisely, we show for all $\varphi \in C_c^\infty((-\infty, T)\times\Omega)$, $\varphi \geq 0$

$$-\int_{Q_T} \overline{|\mathbb{F}|^2}\,\partial_t\varphi - \int_\Omega |\mathbb{F}_0|^2\varphi(0) - \int_{Q_T}\left(\overline{|\mathbb{F}|^2\boldsymbol{v}}\right)\cdot\nabla\varphi - 2\left\langle \overline{\nabla\boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)}, \varphi\right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} + \left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2}, \varphi\right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} - \int_{Q_T}|\mathbb{F}|^2\varphi \leq 0 \quad (6.62)$$

and

$$-\int_{Q_T}|\mathbb{F}|^2\,\partial_t\varphi - \int_\Omega|\mathbb{F}_0|^2\varphi(0) - \int_{Q_T}\left(|\mathbb{F}|^2\boldsymbol{v}\right)\cdot\nabla\varphi - 2\int_{Q_T}\overline{\nabla\boldsymbol{v}\mathbb{F}} : (\varphi\mathbb{F}) + \int_{Q_T}\left(\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F} - |\mathbb{F}|^2\right)\varphi = 0. \quad (6.63)$$

Let us note that in (6.62) and (6.63) all differential operators act on the test functions φ. In further computations it enables us to extend all functions acting in (6.62) and (6.63) by zero in $(-\infty,0)\times(\mathbb{R}^2\setminus\Omega)$ and mollify the equations over time and space such that the terms $\int_\Omega(\overline{|\mathbb{F}|^2}(t,\cdot) - |\mathbb{F}|^2(t,\cdot))_\delta$ tend to zero as the mollification parameter δ tends to zero and t approaches zero from bellow. As we will see, this approach eliminates the obstacles connected with the lack of information on the time continuity of $\overline{|\mathbb{F}|^2}$. Moreover, in (6.62) there is no more the term containing $\varepsilon\Delta\mathbb{F}_\varepsilon : \mathbb{F}_\varepsilon$, and as a consequence, (6.62) does not hold true with equality, but only with inequality. However, the achieved inequality does not complicate further computations.

Step 2: Deriving the following form of (6.50):

$$-\int_{Q_T}\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\partial_t\varphi - \int_{Q_T}\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\boldsymbol{v}\cdot\nabla\varphi \leq \int_{Q_T} L\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi, \quad (6.64)$$

where $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \geq 0$, is arbitrary, and L is an $L^2(Q_T)$ function.

Step 3: Renormalisations of (6.64), passage to the test functions $\varphi \geq 0$ of the form $\varphi = \varphi(t,\boldsymbol{x}) = \Psi(t)\eta(\boldsymbol{x})$, where $\Psi \in C_c^\infty(-\infty,T)$ and $\eta \in C^\infty(\overline{\Omega})$, then by a suitable choice of the renormalisation function and of the test function concluding the result $\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2$ a.e. in Q_T, which is, as introduced above, equivalent to the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$.

Performing of Step 1. Let us extend $\mathbb{F}_\varepsilon$ continuously with respect to $(W^{1,2}(\mathbb{R}^2))^{2\times 2}$ norm and $\boldsymbol{v}_\varepsilon$ by zero outside of Ω. Let $\delta_0 > 0$, $\Omega_{\delta_0} := \{\boldsymbol{x}\in\Omega; \operatorname{dist}(\boldsymbol{x},\partial\Omega) \geq \delta_0\}$, let $\delta \in (0,\delta_0)$ be arbitrary. Let ω_δ denote the standard space mollifying kernel and $h_\delta(\boldsymbol{x}) := \int_{\mathbb{R}^2}\omega_\delta(\boldsymbol{x} - \cdot)h(\cdot)$ for any fixed point $\boldsymbol{x}\in\Omega$ and $h \in L^1_{loc}(\mathbb{R}^2)$. Let us note that $\partial_t\mathbb{F}_{\varepsilon_\delta} \in L^{\frac{4}{3}}(0,T;(C^\infty(\Omega))^{2\times 2})$ due to (6.6) tested by $(\omega_\delta(\boldsymbol{x} - \cdot)\ \widetilde{\mathbb{A}}(\boldsymbol{x}))$, where

$\widetilde{\mathbb{A}} \in (C_c^\infty(\Omega_{\delta_0}))^{2\times 2}$ is arbitrary, multiplied by an arbitrary $\phi \in C_c^\infty((0,T))$ and integrated over Q_T, and due to the facts $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v}_\varepsilon \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$. In (6.6) set $\mathbb{A} := 2(\omega_\delta(\boldsymbol{x}-\cdot)\,\varphi(\boldsymbol{x})\,\mathbb{F}_{\varepsilon_\delta}(\boldsymbol{x}))$, where $\boldsymbol{x}$ is a fixed point in Ω and $\varphi \in C_c^\infty((-\infty,T)\times\Omega_{\delta_0})$, $\varphi \geq 0$, is arbitrary, to obtain a.e. in Q_T

$$2\left(\partial_t\mathbb{F}_{\varepsilon_\delta} : (\varphi\mathbb{F}_{\varepsilon_\delta}) + \mathrm{Div}\,(\mathbb{F}_{\varepsilon_\delta}\otimes\boldsymbol{v}_\varepsilon) : (\varphi\mathbb{F}_{\varepsilon_\delta}) - (\nabla\boldsymbol{v}_\varepsilon\mathbb{F}_\varepsilon)_\delta : (\varphi\mathbb{F}_{\varepsilon_\delta})\right)$$
$$+\left(\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T\mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon\right)_\delta : (\varphi\mathbb{F}_{\varepsilon_\delta}) - 2\varepsilon\Delta\mathbb{F}_{\varepsilon_\delta} : (\varphi\mathbb{F}_{\varepsilon_\delta}) = 2\mathbb{E}_{\varepsilon_\delta} : (\varphi\mathbb{F}_{\varepsilon_\delta})$$

with

$$\mathbb{E}_{\varepsilon_\delta} := \mathrm{Div}(\mathbb{F}_{\varepsilon_\delta}\otimes\boldsymbol{v}_\varepsilon) - \mathrm{Div}(\mathbb{F}_\varepsilon\otimes\boldsymbol{v}_\varepsilon)_\delta.$$

Integrating over Q_T, using the integration by parts and the property $\mathrm{div}\,\boldsymbol{v}_\varepsilon = 0$, yields (let us note that $\mathbb{F}_{\varepsilon_\delta} \in C([0,T];(L^2(\Omega))^{2\times 2})$ since $\mathbb{F}_{\varepsilon_\delta} \in L^4(0,T;(C^\infty(\Omega))^{2\times 2})$ and as mentioned above, $\partial_t\mathbb{F}_{\varepsilon_\delta} \in L^{\frac{4}{3}}(0,T;(C^\infty(\Omega))^{2\times 2})$

$$-\int_{Q_T}|\mathbb{F}_{\varepsilon_\delta}|^2(\partial_t\varphi) - \int_\Omega|\mathbb{F}_{\varepsilon_\delta}(0)|^2\varphi(0) - \int_{Q_T}(|\mathbb{F}_{\varepsilon_\delta}|^2\boldsymbol{v}_\varepsilon)\cdot\nabla\varphi - 2\int_{Q_T}(\nabla\boldsymbol{v}_\varepsilon\mathbb{F}_\varepsilon)_\delta : (\varphi\mathbb{F}_{\varepsilon_\delta})$$
$$+\int_{Q_T}\left(\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T\mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon\right)_\delta : (\varphi\mathbb{F}_{\varepsilon_\delta}) + 2\varepsilon\left(\int_{Q_T}|\nabla\mathbb{F}_{\varepsilon_\delta}|^2\varphi + \int_{Q_T}\nabla\mathbb{F}_{\varepsilon_\delta} : (\mathbb{F}_{\varepsilon_\delta}\otimes\nabla\varphi)\right)$$
$$= 2\int_{Q_T}\mathbb{E}_{\varepsilon_\delta} : (\varphi\mathbb{F}_{\varepsilon_\delta}). \tag{6.65}$$

First let us pass $\delta \to 0+$. Since $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v}_\varepsilon \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, Lemma 3.1 implies

$$\varphi\mathbb{E}_{\varepsilon_\delta} \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(\Omega))^{2\times 2} \quad \text{for a.a. } t \in (0,T) \tag{6.66}$$

and

$$\|\varphi\mathbb{E}_{\varepsilon_\delta}\|_{\frac{4}{3}} \leq C\|\mathbb{F}_\varepsilon\|_4\|\boldsymbol{v}_\varepsilon\|_{1,2} \quad \text{for a.a. } t \in (0,T). \tag{6.67}$$

By Lebesgue's convergence theorem with majorant $\|\mathbb{F}_\varepsilon\|_4^{\frac{4}{3}}\|\boldsymbol{v}_\varepsilon\|_{1,2}^{\frac{4}{3}}$ integrable over $(0,T)$ (the integrability over $(0,T)$ holds true as $\mathbb{F} \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v}_\varepsilon \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, using the Hölder inequality), we obtain from (6.66) and (6.67) that $\varphi\mathbb{E}_{\varepsilon_\delta} \to \mathbb{O}$ strongly in $(L^{\frac{4}{3}}(Q_T))^{2\times 2}$, and since $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$, we arrive at

$$\lim_{\delta\to 0+}\int_{Q_T}\mathbb{E}_{\varepsilon_\delta} : (\varphi\mathbb{F}_{\varepsilon_\delta}) = 0. \tag{6.68}$$

Employing (6.68), the fact $\mathbb{F}_\varepsilon(0) = \mathbb{F}_0$ a.e. in Ω, which follows from (5.53), using also standard properties of mollifying kernels and nonnegativity of the term $\varepsilon\int_{Q_T}|\nabla\mathbb{F}_{\varepsilon_\delta}|^2\varphi$, taking $\limsup_{\delta\to 0+}$ in (6.65) leads to

$$-\int_{Q_T}|\mathbb{F}_\varepsilon|^2(\partial_t\varphi) - \int_\Omega|\mathbb{F}_0|^2\varphi(0) - \int_{Q_T}(|\mathbb{F}_\varepsilon|^2\boldsymbol{v}_\varepsilon)\cdot\nabla\varphi - 2\int_{Q_T}\nabla\boldsymbol{v}_\varepsilon : (\varphi\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T)$$
$$+\int_{Q_T}\left(|\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T|^2 - |\mathbb{F}_\varepsilon|^2\right)\varphi + 2\varepsilon\int_{Q_T}\nabla\mathbb{F}_\varepsilon : (\mathbb{F}_\varepsilon\otimes\nabla\varphi) \leq 0. \tag{6.69}$$

Now we pass $\varepsilon \to 0+$. The term $\varepsilon\int_{Q_T}\nabla\mathbb{F}_\varepsilon : (\mathbb{F}_\varepsilon\otimes\nabla\varphi)$ converges to zero by (6.17) and (6.18). Taking into account the convergences (6.24), (6.29), (6.31), (6.32) and the fact that $\delta_0 > 0$ (connected with φ) is arbitrary, passing $\varepsilon \to 0+$ in (6.69) implies the resulting inequality (6.62).

We proceed analogously in order to prove (6.63) (in fact it is more simple). Let us extend $\mathbb{F}$ and $\boldsymbol{v}$ by zero outside of Ω. Let $\delta > 0$. In (6.35) set $\mathbb{A} := 2(\omega_\delta(\boldsymbol{x} - \cdot)\, \varphi(\boldsymbol{x})\, \mathbb{F}_\delta(\boldsymbol{x}))$, where $\boldsymbol{x}$ is a fixed point in Ω, ω_δ is same as above, $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$ is arbitrary. Integrating the result over Q_T, using the integration by parts and the property $\operatorname{div} \boldsymbol{v} = 0$ in Q_T (let us note that $\mathbb{F}_\delta \in C([0,T]; (L^2(\Omega))^{2\times 2})$ as $\mathbb{F} \in C([0,T]; (L^2(\Omega))^{2\times 2})$ and $\mathbb{F}_\delta(0) = (\mathbb{F}_0)_\delta$ a.e. in Ω by (6.37)), we arrive at

$$-\int_{Q_T} |\mathbb{F}_\delta|^2 (\partial_t \varphi) - \int_\Omega |(\mathbb{F}_0)_\delta|^2 \varphi(0) - \int_{Q_T} (|\mathbb{F}_\delta|^2 \boldsymbol{v}) \cdot \nabla \varphi - 2\int_{Q_T} \left(\overline{\nabla \boldsymbol{v} \mathbb{F}}\right)_\delta : (\varphi \mathbb{F}_\delta)$$
$$+ \int_{Q_T} \Big((\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} - \mathbb{F})_\delta : \mathbb{F}_\delta \Big)\, \varphi = 2\int_{Q_T} \mathbb{E}_\delta : (\varphi \mathbb{F}_\delta) \tag{6.70}$$

with

$$\mathbb{E}_\delta := \operatorname{Div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - \operatorname{Div}(\mathbb{F} \otimes \boldsymbol{v})_\delta.$$

Using Lemma 3.1, Lebesgue's convergence theorem, the properties $\mathbb{F} \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}})$ and standard properties of molllifying kernels, passing $\delta \to 0+$ in (6.70) gives the result (6.63).

Performing of Step 2. We derive the inequality (6.64) from (6.62) and (6.63) attained in Step 1 by showing the following inequalities for all $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, $\varphi \geq 0$, and some $\tilde{L} \in L^2(Q_T)$:

$$\left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} \geq 0, \tag{6.71}$$

$$\left\langle \overline{\nabla \boldsymbol{v} : \mathbb{F}\mathbb{F}^T} - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} \leq \int_{Q_T} \tilde{L}(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\varphi. \tag{6.72}$$

For the proof of the inequality (6.71) we employ the monotonicity of the matrix function $S(\mathbb{X}) := \mathbb{X}\mathbb{X}^T\mathbb{X}$ for all $\mathbb{X} \in \mathbb{R}^{2\times 2}$, see Lemma 3.2. The convergences (6.17), (6.30) and (6.31) imply that the left handside of (6.71) is equal to

$$\lim_{\varepsilon \to 0+} \int_{Q_T} \big(|\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T|^2 - (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon) : \mathbb{F} - (\mathbb{F}\mathbb{F}^T\mathbb{F}) : (\mathbb{F}_\varepsilon - \mathbb{F}) \big)\, \varphi$$
$$= \lim_{\varepsilon \to 0+} \int_{Q_T} \big((\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}\mathbb{F}^T\mathbb{F}) : (\mathbb{F}_\varepsilon - \mathbb{F}) \big)\, \varphi \geq 0,$$

where the inequality follows from Lemma 3.2.

The proof of (6.72) requires to express its left handside as the sum $\sum_{j=1}^3 I_j$, where

$$I_1 := \left\langle \overline{\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)} - \nabla \boldsymbol{v} : \overline{\mathbb{F}\mathbb{F}^T}, \varphi \right\rangle_{\{M(\overline{Q_T}), C(\overline{Q_T})\}} \tag{6.73}$$

$$I_2 := \int_{Q_T} \Big(\nabla \boldsymbol{v} : \overline{\mathbb{F}\mathbb{F}^T} - \nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T) \Big)\, \varphi, \tag{6.74}$$

$$I_3 := \int_{Q_T} \big(\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T) - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F} \big)\, \varphi. \tag{6.75}$$

The term I_1 is treated by the following lemma.

Lemma 6.2. *For all* $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \geq 0$, *it holds*

$$I_1 = \left\langle \overline{|\mathbb{D}|^2} - |\mathbb{D}|^2, \varphi \right\rangle_{\{M(\overline{Q_T}),C(\overline{Q_T})\}} \leq 0. \tag{6.76}$$

Proof. The inequality in the relation (6.76) is obvious due to the weak convergence $\mathbb{D}_\varepsilon \rightharpoonup \mathbb{D}$ in $(L^2(Q_T))^{2\times 2}$ (see (6.14)) and the weak lower semicontinuity of $L^2(Q_T)$ norm. Let us show that

$$I_1 = \left\langle \overline{|\mathbb{D}|^2} - |\mathbb{D}|^2, \varphi \right\rangle_{\{M(\overline{Q_T}),C(\overline{Q_T})\}}$$

by employing Lemma 6.1 on the reconstruction of the pressures p_ε, p and their convergence and the convergence results from Subsection 6.1. Let $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \geq 0$, be arbitrary, let us recall that we assume Ω to be Lipschitz, hence for every $\delta > 0$ there exists a smooth set $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$ such that $|\Omega \setminus \tilde{\Omega}| < \delta$. For a fixed $t \in (0,T)$ subtract (6.58) tested by $(\boldsymbol{v} + \nabla p_1)\varphi$ from (6.54) tested by $(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\varphi$, integrate the result over $(0,T)$, use (6.33) and pass $\varepsilon \to 0+$ to obtain

$$\lim_{\varepsilon\to 0+} \int_{Q_T} \big(-\nabla \boldsymbol{v}_\varepsilon : (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) + \nabla \boldsymbol{v} : (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T)\big)\varphi = \lim_{\varepsilon\to 0+} \sum_{j=1}^{9} J_{j,\varepsilon},$$

where

$$\begin{aligned}
J_{1,\varepsilon} &:= \int_0^T \langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \varphi(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\rangle - \int_0^T \langle \partial_t(\boldsymbol{v} + \nabla p_1), \varphi(\boldsymbol{v} + \nabla p_1)\rangle,\\
J_{2,\varepsilon} &:= -\int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\varphi \nabla \boldsymbol{v}_\varepsilon) + \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\varphi \nabla \boldsymbol{v}),\\
J_{3,\varepsilon} &:= -\int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\boldsymbol{v}_\varepsilon \otimes \nabla\varphi) + \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\boldsymbol{v} \otimes \nabla\varphi),\\
J_{4,\varepsilon} &:= -\int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\nabla p_{1,\varepsilon} \otimes \nabla\varphi + \varphi \nabla^2 p_{1,\varepsilon}) + \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\nabla p_1 \otimes \nabla\varphi + \varphi \nabla^2 p_1),\\
J_{5,\varepsilon} &:= \int_{Q_T} \left(|\mathbb{D}_\varepsilon|^2 - |\mathbb{D}|^2\right)\varphi,\\
J_{6,\varepsilon} &:= \int_{Q_T} \mathbb{D}_\varepsilon : \big((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \otimes \nabla\varphi + \varphi \nabla^2 p_{1,\varepsilon}\big) - \int_{Q_T} \mathbb{D} : \big((\boldsymbol{v} + \nabla p_1) \otimes \nabla\varphi + \varphi \nabla^2 p_1\big),\\
J_{7,\varepsilon} &:= -\int_{Q_T} p_{2,\varepsilon}\, ((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \cdot \nabla\varphi + \varphi \Delta p_{1,\varepsilon}) + \int_{Q_T} p_2\, ((\boldsymbol{v} + \nabla p_1) \cdot \nabla\varphi + \varphi \Delta p_1),\\
J_{8,\varepsilon} &:= -\int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : ((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \otimes \nabla\varphi) + \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : ((\boldsymbol{v} + \nabla p_1) \otimes \nabla\varphi),\\
J_{9,\varepsilon} &:= -\int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : (\varphi \nabla^2 p_{1,\varepsilon}) + \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : (\varphi \nabla^2 p_1).
\end{aligned}$$

Our aim is to show that all terms $J_{j,\varepsilon}$, $j = 1,\dots,9$, except $J_{5,\varepsilon}$, converge to zero. The convergence of $J_{3,\varepsilon}$, $J_{4,\varepsilon}$ and $J_{j,\varepsilon}$ for $j = 6,\dots,9$ follows from (6.14), (6.24), (6.25), (6.28), (6.59) and (6.60). In order to treat $J_{1,\varepsilon}$, we use the integration by parts, i.e.

$$\int_0^T \langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \varphi(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\rangle = -\int_\Omega \frac{|\boldsymbol{v}_\varepsilon(0) + \nabla p_{1,\varepsilon}(0)|^2}{2}\varphi(0) - \int_{Q_T} \frac{|\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}|^2}{2}\partial_t\varphi$$

and

$$\int_0^T \langle -\partial_t(\boldsymbol{v}+\nabla p_1), \varphi(\boldsymbol{v}+\nabla p_1)\rangle = \int_\Omega \frac{|\boldsymbol{v}(0)+\nabla p_1(0)|^2}{2}\varphi(0) + \int_{Q_T} \frac{|\boldsymbol{v}+\nabla p_1|^2}{2}\partial_t\varphi.$$

Since $\boldsymbol{v}_\varepsilon$, $\boldsymbol{v}$ belong to $C([0,T];L^2_{\boldsymbol{n},\mathrm{div}})$, $\nabla p_{1,\varepsilon}$, ∇p_1 belong to $C([0,T];(L^2(\tilde\Omega))^2)$ for every smooth $\tilde\Omega \subset \overline{\tilde\Omega} \subset \Omega$ and for each $\varepsilon > 0$ it holds $\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_0 = \boldsymbol{v}(0)$ a.e. in Ω, $\nabla p_{1,\varepsilon}(0) = \nabla p_1(0)$ a.e. in $\tilde\Omega$, we have

$$-\int_\Omega \frac{|\boldsymbol{v}_\varepsilon(0)+\nabla p_{1,\varepsilon}(0)|^2}{2}\varphi(0) + \int_\Omega \frac{|\boldsymbol{v}(0)+\nabla p_1(0)|^2}{2}\varphi(0) = 0,$$

from (6.24) and (6.59) it follows

$$\lim_{\varepsilon\to 0+} -\int_{Q_T} \frac{|\boldsymbol{v}_\varepsilon+\nabla p_{1,\varepsilon}|^2}{2}\partial_t\varphi + \int_{Q_T} \frac{|\boldsymbol{v}+\nabla p_1|^2}{2}\partial_t\varphi = 0,$$

hence $J_{1,\varepsilon} \to 0$ as $\varepsilon \to 0+$. It remains to prove that $J_{2,\varepsilon} \to 0$ as $\varepsilon \to 0+$. Using the integration by parts and the property $\operatorname{div}\boldsymbol{v}_\varepsilon = \operatorname{div}\boldsymbol{v} = 0$ in Q_T, it holds

$$J_{2,\varepsilon} = -\frac{1}{2}\int_{Q_T} (\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) : (\boldsymbol{v}_\varepsilon\otimes\nabla\varphi) + \frac{1}{2}\int_{Q_T} (\boldsymbol{v}\otimes\boldsymbol{v}) : (\boldsymbol{v}\otimes\nabla\varphi),$$

which converges to zero by (6.24) and (6.25). Finally, as $J_{5,\varepsilon} \to I_1$ by (6.33), the lemma is proved. □

The term I_2 (see (6.74)) is estimated, using (6.17), (6.28), (6.29), the Cauchy-Schwartz inequality (together with the inequality $|\mathbb{X}\mathbb{Y}| \le |\mathbb{X}||\mathbb{Y}|$ for all $\mathbb{X}$, $\mathbb{Y} \in \mathbb{R}^{2\times 2}$) as follows:

$$\begin{aligned} I_2 = \lim_{\varepsilon\to 0+}\int_{Q_T} \nabla\boldsymbol{v} : \left((\mathbb{F}_\varepsilon-\mathbb{F})(\mathbb{F}_\varepsilon-\mathbb{F})^T\right)\varphi &\le \lim_{\varepsilon\to 0+}\int_{Q_T} |\nabla\boldsymbol{v}||\mathbb{F}_\varepsilon-\mathbb{F}|^2\varphi \\ &= \int_{Q_T} |\nabla\boldsymbol{v}|\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi. \end{aligned} \tag{6.77}$$

The term I_3 (see (6.75)) is estimated, employing (6.14), (6.17), (6.29), (6.33), the Cauchy-Schwartz inequality (together with the inequality $|\mathbb{X}\mathbb{Y}| \le |\mathbb{X}||\mathbb{Y}|$ for all $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{2\times 2}$) and employing Korn's and Yong's inequalities, as follows:

$$\begin{aligned} I_3 &= \lim_{\varepsilon\to 0+}\int_{Q_T} ((\nabla\boldsymbol{v}_\varepsilon - \nabla\boldsymbol{v})(\mathbb{F}-\mathbb{F}_\varepsilon)) : (\varphi\mathbb{F}) \\ &= \lim_{\varepsilon\to 0+}\int_{Q_T} |\nabla\boldsymbol{v}_\varepsilon - \nabla\boldsymbol{v}|\;|\mathbb{F}_\varepsilon - \mathbb{F}|\;|\mathbb{F}|\varphi \\ &\le \lim_{\varepsilon\to 0+}\int_{Q_T} \left(\tilde\varepsilon|\nabla\boldsymbol{v}_\varepsilon - \nabla\boldsymbol{v}|^2 + \tilde C|\mathbb{F}_\varepsilon-\mathbb{F}|^2|\mathbb{F}|^2\right)\varphi \\ &\le \lim_{\varepsilon\to 0+}\int_{Q_T} \left(\hat\varepsilon|\mathbb{D}_\varepsilon - \mathbb{D}|^2 + \tilde C|\mathbb{F}_\varepsilon-\mathbb{F}|^2|\mathbb{F}|^2\right)\varphi \\ &= \hat\varepsilon\left\langle \overline{|\mathbb{D}|^2} - |\mathbb{D}|^2, \varphi\right\rangle + \tilde C\int_{Q_T}\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)(|\mathbb{F}|^2\varphi), \end{aligned} \tag{6.78}$$

where $\tilde{\varepsilon} \in (0,1]$ is such small that we could set $\hat{\varepsilon} \le 1$ in the last inequality.
Summing (6.76), (6.77) and (6.78) and employing the definitions (6.73) (6.74), (6.75), we obtain for all $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \ge 0$

$$\left\langle \overline{\nabla \boldsymbol{v} : \mathbb{F}\mathbb{F}^T} - \overline{\nabla \boldsymbol{v}\mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{M(\overline{Q_T}),C(\overline{Q_T})\}} = \sum_{j=1}^{3} I_j \le \int_{Q_T} \tilde{L}(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\varphi$$

with

$$\tilde{L} := (|\nabla \boldsymbol{v}| + \tilde{C}|\mathbb{F}|^2) \in L^2(Q_T),$$

which is the inequality (6.72) completing Step 2, where in (6.64) we can set $L := 2\tilde{L} + 1$.

Performing of Step 3. As introduced above, the last Step 3 consists of renormalizing the inequality (6.64) achieved in Step 2, proving that the renormalized inequality is valid even for nonnegative smooth test functions φ supported up to the boundary of Ω and concluding by a suitable choice of such φ and of the renormalisation function the result $\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2$ a.e. in Q_T. The following Lemma 6.3 concerns renormalisations, in the next Lemma 6.4 the passage to the smooth test functions supported up to the boundary is treated.

Lemma 6.3. *Let $f := \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2$, $B \in C^1([0,\infty))$, $0 \le B'(s) \le K$ for all $s \in [0,\infty)$ and some $K \in (0,\infty)$. Then it holds for all $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \ge 0$*

$$-\int_{Q_T} B(f)\partial_t\varphi - \int_{\Omega} B(0)\varphi(0) - \int_{Q_T} B(f)\boldsymbol{v}\cdot\nabla\varphi \le \int_{Q_T} LfB'(f)\varphi \tag{6.79}$$

with $L := 1 + 2\left(|\nabla \boldsymbol{v}| + \tilde{C}|\mathbb{F}|^2\right)$.

Proof. First we need to show $f \ge 0$ a.e. in Q_T, so that the formulation of the lemma has sense. The relations (6.17) and (6.29) yield for all $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \ge 0$

$$\int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi = \lim_{\varepsilon\to 0+} \int_{Q_T} |\mathbb{F}_\varepsilon - \mathbb{F}|^2\varphi. \tag{6.80}$$

Since $\mathbb{F}_\varepsilon, \mathbb{F}$ definitely belong to $L^2(0,T;(L^2_{loc}(\Omega))^{2\times 2})$, we get from (6.80)

$$f := \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \ge 0 \quad \text{a.e. in } Q_T. \tag{6.81}$$

Now it follows the own proof of the lemma. As we proved in Step 2, it holds for all $\phi \in C_c^\infty((-\infty,T)\times\Omega)$, $\phi \ge 0$

$$-\int_{Q_T} f\partial_t\phi - \int_{Q_T} f\boldsymbol{v}\cdot\nabla\phi \le \int_{Q_T} Lf\phi. \tag{6.82}$$

Let us extend $\boldsymbol{v}$ and $\mathbb{F}$ by zero outside of Q_T, set $\phi := \omega_\delta(t - \cdot, \boldsymbol{x} - \bullet)$, where $[t,\boldsymbol{x}]$ is a fixed point from $(-\infty,T)\times\Omega$, $\delta > 0$ is arbitrary, ω_δ is the standard time-space mollifying kernel and $h_\delta(t,\boldsymbol{x}) = \int_{\mathbb{R}\times\mathbb{R}^2} \omega_\delta(t-\cdot,\boldsymbol{x}-\bullet)h(\cdot,\bullet)$ for any $h \in L^1_{loc}(\mathbb{R}\times\mathbb{R}^2)$, and multiply the result by $(B'(f_\delta)(t,\boldsymbol{x})\ \varphi(t,\boldsymbol{x}))$, where $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, $\varphi \ge 0$ is arbitrary. We obtain

$$\partial_t f_\delta B'(f_\delta)\varphi + \operatorname{div}(f\boldsymbol{v})_\delta B'(f_\delta)\varphi \le (Lf)_\delta B'(f_\delta)\varphi \quad \text{a.e. in } (-\infty,T)\times\Omega,$$

which can be rewritten into the form (use the property $\operatorname{div} \boldsymbol{v} = 0$ in $(-\infty, T) \times \Omega$)

$$\partial_t B(f_\delta)\varphi + \operatorname{div}\left(B(f_\delta)\boldsymbol{v}\right)\varphi \le (Lf)_\delta B'(f_\delta)\varphi + s_\delta B'(f_\delta)\varphi \quad \text{a.e. in } (-\infty, T) \times \Omega, \tag{6.83}$$

where

$$s_\delta := \operatorname{div}(f_\delta \boldsymbol{v}) - \operatorname{div}(f\boldsymbol{v})_\delta.$$

Let us note that $\operatorname{supp} f_\delta \subset (-\tilde{\delta}, T + \tilde{\delta}) \times \mathbb{R}^2$ and especially $f_\delta(-\tilde{\delta}) = 0$ whenever $\tilde{\delta} \ge \delta$. Integrate (6.83) over $(-\infty, T) \times \Omega$, use the integration by parts to obtain

$$\int_{-\tilde{\delta}}^{T}\int_\Omega \left(-B(f_\delta)\partial_t\varphi - B(f_\delta)\boldsymbol{v}\cdot\nabla\varphi\right) - \int_\Omega B(0)\varphi(-\tilde{\delta}) \\ \le \int_{-\tilde{\delta}}^{T}\int_\Omega \left((Lf)_\delta B'(f_\delta)\varphi + s_\delta B'(f_\delta)\varphi\right). \tag{6.84}$$

First we pass $\delta \to 0+$. Since $f \in L^2((-\infty, T) \times \Omega)$, $\boldsymbol{v} \in L^2(-\infty, T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, Lemma 3.1 implies that $s_\delta \to 0$ strongly in $L^1_{loc}((-\infty, T) \times \Omega)$. Since B' is bounded, it holds

$$\lim_{\delta\to 0+}\int_{-\tilde{\delta}}^{T}\int_\Omega s_\delta B'(f_\delta)\varphi = 0. \tag{6.85}$$

Now let us treat the first term on the right handside of (6.84). As $L \in L^2((-\infty, T) \times \Omega)$, $f \in L^2((-\infty, T) \times \Omega)$, it holds $(Lf)_\delta \to Lf$ strongly in $L^1_{loc}((-\infty, T) \times \Omega)$. For a suitable subsequence then $(Lf)_\delta \to Lf$ a.e. in $(-\infty, T) \times \Omega$, $f_\delta \to f$ a.e. in $(-\infty, T) \times \Omega$, and since $B \in C^1([0, \infty))$, it holds

$$(Lf)_\delta B'(f_\delta) - Lf B'(f) \to 0 \quad \text{a.e. in } (-\infty, T) \times \Omega. \tag{6.86}$$

Next, B' is bounded ($0 \le B' \le K$), from the standard properties of mollifying kernels it follows $\|(Lf)_\delta\|_{L^1((-\infty,T)\times\Omega)} \le \|Lf\|_{L^1((-\infty,T)\times\Omega)}$, hence the Lebesgue convergence theorem with the integrable majorant $2KLf$ applied on (6.86) implies

$$\lim_{\delta\to 0+}\int_{-\tilde{\delta}}^{T}\int_\Omega (Lf)_\delta B'(f_\delta)\varphi = \int_{-\tilde{\delta}}^{T}\int_\Omega Lf B'(f)\varphi. \tag{6.87}$$

Now the terms on the left handside of (6.84) will be treated. It holds $f_\delta \to f$ strongly in $L^2_{loc}((-\infty, T) \times \Omega)$ and B is Lipschitz ($B \in C^1([0, \infty))$ and the derivative is bounded), thus

$$B(f_\delta) \to B(f) \quad \text{strongly in } L^2_{loc}((-\infty, T) \times \Omega),$$

which together with the property $\boldsymbol{v} \in (L^2((-\infty, T) \times \Omega))^2$ implies

$$\lim_{\delta\to 0+}\int_{-\tilde{\delta}}^{T}\int_\Omega \left(-B(f_\delta)\partial_t\varphi - B(f_\delta)\boldsymbol{v}\cdot\nabla\varphi\right) = \int_{-\tilde{\delta}}^{T}\int_\Omega \left(-B(f)\partial_t\varphi - B(f)\boldsymbol{v}\cdot\nabla\varphi\right).$$

Taking the limit $\delta \to 0+$ in (6.84) and then $\tilde{\delta} \to 0+$, employing the last limit and the limits (6.85), (6.87), we conclude

$$-\int_{Q_T} B(f)\partial_t\varphi - \int_\Omega B(0)\varphi(0) - \int_{Q_T} B(f)\boldsymbol{v}\cdot\nabla\varphi \le \int_{Q_T} Lf B'(f)\varphi.$$

□

Lemma 6.4. *Let* $f := \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2$, $B \in C^1([0,\infty))$, $0 \le B'(s) \le K$ *for all* $s \in [0,\infty)$ *and some* $K \in (0,\infty)$. *Then it holds*

$$-\int_{Q_T} B(f)\partial_t\varphi - \int_\Omega B(0)\varphi(0) - \int_{Q_T} B(f)\boldsymbol{v}\cdot\nabla\varphi \le \int_{Q_T} |LfB'(f)\varphi| \tag{6.88}$$

with $L := 1 + 2\left(|\nabla \boldsymbol{v}| + \tilde{C}|\mathbb{F}|^2\right)$ *for all* $\varphi \ge 0$ *of the form*

$$\varphi(t,\boldsymbol{x}) = \psi(t)\eta(\boldsymbol{x}), \quad \psi \in C_c^\infty((-\infty,T)), \eta \in C^\infty(\overline{\Omega}).$$

Proof. Let us define

$$\xi_m(\boldsymbol{x}) := \chi_m(\mathrm{dist}(\boldsymbol{x},\partial\Omega)),$$

where

$$\chi_m(s) = \chi(ms),\ \chi \in C^\infty([0,\infty)),\ \chi(s) \in [0,1],\ \chi(s) = \begin{cases} 0 & \text{if } s \le \frac{1}{2} \\ 1 & \text{if } s \ge 1 \end{cases}.$$

From the definition of ξ_m one observes

$$\xi_m(\boldsymbol{x}) \in [0,1], \quad \xi_m(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \mathrm{dist}(\boldsymbol{x},\partial\Omega) \le \frac{1}{2m} \\ 1 & \text{if } \mathrm{dist}(\boldsymbol{x},\partial\Omega) \ge \frac{1}{m} \end{cases}. \tag{6.89}$$

Since Ω is a bounded Lipschitz domain, the function $\mathrm{dist}(\boldsymbol{x},\partial\Omega)$ is Lipschitz, and as a consequence

$$|\nabla\xi_m| \le Cm \quad \text{in } \Omega. \tag{6.90}$$

Notice that $1-\xi_m$ is not supported outside of the set $A_m := \left\{\boldsymbol{x} \in \Omega; \mathrm{dist}(\boldsymbol{x},\partial\Omega) \le \frac{1}{m}\right\}$ and $\nabla\xi_m$ is not supported outside of the set $\tilde{A}_m := \left\{\boldsymbol{x} \in \Omega; \frac{1}{2m} \le \mathrm{dist}(\boldsymbol{x},\partial\Omega) \le \frac{1}{m}\right\}$. Since Ω is a bounded Lipschitz domain, it holds $|A_m| \to 0$, $|\tilde{A}_m| \to 0$ as $m \to \infty$.

Let $\psi \in C_c^\infty((-\infty,T))$, $\eta \in C^\infty(\overline{\Omega})$, $\psi\eta \ge 0$, be arbitrary. Our goal is to prove

$$-\int_{Q_T} B(f)(\partial_t\psi)\eta - \int_\Omega B(0)\psi(0)\eta - \int_{Q_T} B(f)\ (\boldsymbol{v}\cdot\psi\nabla\eta) \le \int_{Q_T} |LfB'(f)\psi\eta|. \tag{6.91}$$

We write

$$\begin{aligned}
&-\int_{Q_T} B(f)(\partial_t\psi)\eta - \int_\Omega B(0)\psi(0)\eta - \int_{Q_T} B(f)\ (\boldsymbol{v}\cdot\psi\nabla\eta) \\
&\quad = -\int_{Q_T} B(f)\ \partial_t(\psi\eta\xi_m) - \int_\Omega B(0)\ (\psi\eta\xi_m)(0) - \int_{Q_T} B(f)\ (\boldsymbol{v}\cdot\nabla(\psi\eta\xi_m)) \\
&\qquad - \int_{Q_T} B(f)\partial_t\psi\ \eta(1-\xi_m) - \int_\Omega B(0)\psi(0)\ \eta(1-\xi_m) \\
&\qquad - \int_{Q_T} B(f)\ (\boldsymbol{v}\cdot\psi(1-\xi_m)\nabla\eta) + \int_{Q_T} B(f)\,(\boldsymbol{v}\cdot\psi\eta\nabla\xi_m)
\end{aligned} \tag{6.92}$$

For the first line of the right handside it holds

$$\lim_{m\to\infty}\left(-\int_{Q_T} B(f)\ \partial_t(\psi\eta\xi_m) - \int_\Omega B(0)\ (\psi\eta\xi_m)(0) - \int_{Q_T} B(f)\ \ (\boldsymbol{v}\cdot\nabla(\psi\eta\xi_m))\right)$$
$$\le \lim_{m\to\infty}\int_{Q_T} LfB'(f)\psi\eta\xi_m \le \int_{Q_T} |LfB'(f)\psi\eta|\,. \tag{6.93}$$

The first inequality follows from Lemma 6.3 and the second one from the fact $|\xi_m|\le 1$. For the first term of the second line of the right handside of (6.92) we have (use that B is Lipschitz, f is (quadratically) integrable over Q_T and $\lim_{m\to\infty}|A_m| = 0$)

$$\lim_{m\to\infty}\left|\int_{Q_T} B(f)\partial_t\psi\ \eta(1-\xi_m)\right| \le C\lim_{m\to\infty}\int_0^T\int_{A_m}|B(f)|$$
$$\le \tilde{C}\lim_{m\to\infty}\|f+B(0)\|_{L^1((0,T)\times A_m)} = 0.$$

The second term of the second line and the first term of the third line of the right handside of (6.92) are treated analogously, they converge to zero as $m\to\infty$. For the last term of (6.92) we have

$$\begin{aligned}\lim_{m\to\infty}\left|\int_{Q_T} B(f)\ \ (\boldsymbol{v}\cdot\psi\eta\nabla\xi_m)\right| &\le \tilde{C}\lim_{m\to\infty}\int_0^T\int_{\tilde{A}_m}|B(f)\ m\boldsymbol{v}| \\ &\le \tilde{C}\lim_{m\to\infty}\int_0^T\int_{\tilde{A}_m}\left|B(f)\ \frac{\boldsymbol{v}}{\operatorname{dist}(\cdot,\partial\Omega)}\right| \\ &\le \hat{C}\lim_{m\to\infty}\|f+B(0)\|_{L^2((0,T)\times\tilde{A}_m)}\ \|\nabla\boldsymbol{v}\|_{L^2(Q_T)} \\ &= 0,\end{aligned}$$

where in the last inequality we employed Hardy's inequality for $\boldsymbol{v}$ and at the end we used that $f\in L^2(Q_T)$, $\nabla\boldsymbol{v}\in(L^2(Q_T))^{2\times 2}$ and $\lim_{m\to\infty}|\tilde{A}_m| = 0$. Since the second and the third line of the right handside of (6.92) converge to zero (as we have just proved), the equality (6.92) together with the relation (6.93) yield (6.91), which ends the proof of the lemma. □

Now we are prepared to conclude $f := \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 = 0$ a.e. in Q_T. Let us employ Lemma 6.4 with $B(f) = \ln(f+\tilde{\varepsilon})$, where $\tilde{\varepsilon} > 0$ is a small number. Setting in (6.88) $\varphi(t,\boldsymbol{x}) = \psi(t)\eta(\boldsymbol{x})$ with $\partial_t\psi\le 0$, $\psi\not\equiv 0$ in $(0,T)$, $\eta\equiv 1$ in $\overline{\Omega}$ leads to

$$-\int_{Q_T}\ln(f+\tilde{\varepsilon})\ \partial_t\psi - \int_\Omega \ln\tilde{\varepsilon}\ \psi(0) \le \int_{Q_T}\left|\frac{Lf}{f+\tilde{\varepsilon}}\psi\right|,$$

where the left handside can be written as

$$-\int_{Q_T}\ln(f+\tilde{\varepsilon})\ \partial_t\psi + \int_{Q_T}\ln\tilde{\varepsilon}\ \partial_t\psi = -\int_{Q_T}\ln\left(1+\frac{f}{\tilde{\varepsilon}}\right)\ \partial_t\psi.$$

Hence it holds

$$-\int_{Q_T}\ln\left(1+\frac{f}{\tilde{\varepsilon}}\right)\ \partial_t\psi \le \int_{Q_T}\left|\frac{Lf}{f+\tilde{\varepsilon}}\psi\right| \le C(T,\Omega) \tag{6.94}$$

since $L \in L^2(Q_T)$. If it did not hold $f = 0$ a.e. in Q_T, then the left handside of (6.94) would blow up to $+\infty$ as $\tilde{\varepsilon} \to 0+$ since $f \geq 0$ a.e. in Q_T, $\partial_t \psi \leq 0$ and $\psi \not\equiv 0$, which would be a contradiction with the uniform bound of the right handside of (6.94).

Hence

$$f = \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 = 0 \quad \text{a.e. in } Q_T, \tag{6.95}$$

which is, as introduced above, equivalent to the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$.

6.4. ***Positivity of*** $\det \mathbb{F}$

The aim of this subsection is to prove $\det \mathbb{F} > 0$ a.e. in Q_T by showing the uniform estimate

$$\sup_{\tilde{\varepsilon}>0} \int_\Omega S_{\tilde{\varepsilon}}(\det \mathbb{F}(t)) > -\infty \quad \text{for a.a. } t \in (0,T), \tag{6.96}$$

where $S_{\tilde{\varepsilon}}(\cdot)$ are truncations of the function $\ln(\cdot)$ (specified later).

The evolutionary equation for the tensor $\mathbb{F}$ is formally written (see (4.3)) as

$$\partial_t \mathbb{F} + \mathrm{Div}(\mathbb{F} \otimes \boldsymbol{v}) - \nabla \boldsymbol{v} \mathbb{F} + \frac{1}{2}(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) = \mathbb{O}. \tag{6.97}$$

Formally, multiplying (6.97) scalarly by $\frac{(\det \mathbb{F} - \tilde{\varepsilon})^+ \mathbb{F}^{-T}}{\det \mathbb{F}}$ (with convention $\frac{(\det \mathbb{F} - \tilde{\varepsilon})^+ \mathbb{F}^{-T}}{\det \mathbb{F}} = \mathbb{O}$ if $\det \mathbb{F} = 0$), using $\partial \det \mathbb{F} = \det \mathbb{F} \, \mathrm{tr}(\mathbb{F}^{-1}\partial \mathbb{F})$, where ∂ represents either the partial time derivative ∂_t, either the partial space derivative ∂_{x_i}, $i = 1, 2$ (see e.g. [8]), we obtain

$$\partial_t S_{\tilde{\varepsilon}}(\det \mathbb{F}) + \mathrm{div}(S_{\tilde{\varepsilon}}(\det \mathbb{F})\boldsymbol{v}) + \frac{1}{2}\frac{(\det \mathbb{F} - \tilde{\varepsilon})^+}{\det \mathbb{F}} \, \mathrm{tr}\left(\mathbb{F}\mathbb{F}^T - \mathbb{I}\right) = 0, \tag{6.98}$$

where $S_{\tilde{\varepsilon}}(a)$ is a primitive function to $\frac{(a-\tilde{\varepsilon})^+}{a^2}$, i.e.

$$S_{\tilde{\varepsilon}}(a) := \begin{cases} \ln a + \frac{\tilde{\varepsilon}}{a} & \text{if } a > \tilde{\varepsilon}, \\ \ln \tilde{\varepsilon} + 1 & \text{if } a \leq \tilde{\varepsilon}. \end{cases}$$

Rigorously, extend $\boldsymbol{v}$ and $\mathbb{F}$ by zero outside of Ω. In (6.35) set

$$\mathbb{A} := 2\omega_\delta(\boldsymbol{x} - \boldsymbol{\cdot})\frac{(\det \mathbb{F}_\delta(\boldsymbol{x}) - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta(\boldsymbol{x})}\mathbb{F}_\delta^{-T}(\boldsymbol{x}),$$

where $\boldsymbol{x}$ is a fixed point in Ω, ω_δ denotes the standard space mollifying kernel and $h_\delta(\boldsymbol{x}) := \int_{\mathbb{R}^2} \omega_\delta(\boldsymbol{x} - \boldsymbol{\cdot})h(\boldsymbol{\cdot})$ for any function $h \in L^1_{loc}(\mathbb{R}^2)$, to obtain a.e. in Q_T

$$\begin{aligned} \partial_t S_{\tilde{\varepsilon}}(\det \mathbb{F}_\delta) + \mathrm{div}\left(S_{\tilde{\varepsilon}}(\det \mathbb{F}_\delta)\boldsymbol{v}\right) + \frac{2(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}(\nabla \boldsymbol{v}\mathbb{F})_\delta : \mathbb{F}_\delta^{-T} & \\ + \frac{(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}\left((\mathbb{F}\mathbb{F}^T\mathbb{F})_\delta : \mathbb{F}_\delta^{-T} - 2\right) = \frac{2(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}\mathbb{E}_\delta : \mathbb{F}_\delta^{-T}, & \end{aligned}$$

where

$$\mathbb{E}_\delta := \mathrm{Div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - \mathrm{Div}(\mathbb{F} \otimes \boldsymbol{v})_\delta.$$

Let $t \in (0,T)$. Integrating the last equation over $(0,t) \times \Omega$, using the integration by parts, the properties $\mathrm{div}\, \boldsymbol{v} = 0$ in Q_T, $\boldsymbol{v} = \boldsymbol{0}$ on Σ_T and $S_{\tilde{\varepsilon}}(\det \mathbb{F}_\delta) \in C([0,T]; L^1(\Omega))$,

$S_{\tilde{\varepsilon}}(\det \mathbb{F}_\delta(0)) = S_{\tilde{\varepsilon}}(\det(\mathbb{F}_0)_\delta)$ (we know that $\mathbb{F}_\delta \in C([0,T];(L^2(\Omega))^{2\times 2})$, $\mathbb{F}_\delta(0) = (\mathbb{F}_0)_\delta$ a.e. in Ω, see e.g. Subsection 6.3, Step 1, and $S_{\tilde{\varepsilon}}(\cdot)$ is Lipschitz continuous), we obtain

$$\int_\Omega S_{\tilde{\varepsilon}}(\det \mathbb{F}_\delta(t)) - \int_\Omega S_{\tilde{\varepsilon}}(\det(\mathbb{F}_0)_\delta) + 2\int_0^t\int_\Omega \frac{(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}(\nabla \boldsymbol{v}\mathbb{F})_\delta : \mathbb{F}_\delta^{-T}$$
$$+\int_0^t\int_\Omega \frac{(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}\big((\mathbb{F}\mathbb{F}^T\mathbb{F})_\delta : \mathbb{F}_\delta^{-T} - 2\big) = 2\int_0^t\int_\Omega \frac{(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}\mathbb{E}_\delta : \mathbb{F}_\delta^{-T}. \tag{6.99}$$

The right handside of (6.99) converges to zero, it follows from Lemma 3.1 and Lebesgue's convergence theorem (proceeding analogously as in Subsection 6.3, Step 1). Next, let us note that any nonsingular matrix $\mathbb{A}$ satisfy $\mathbb{A}^{-1} = \frac{\mathrm{adj}\,\mathbb{A}}{\det \mathbb{A}}$, thus in two spatial dimensions $\frac{(\det \mathbb{A} - \tilde{\varepsilon})^+}{\det \mathbb{A}}\mathbb{A}^{-1}$ has the same regularity as $\mathbb{A}$, and since $\mathbb{F} \in (L^4((0,T)\times \mathbb{R}^2))^{2\times 2}$ (thus for a subsequence also $\mathbb{F}_\delta \to \mathbb{F}$ a.e. in Q_T), we can state for a subsequence

$$\frac{(\det \mathbb{F}_\delta - \tilde{\varepsilon})^+}{\det \mathbb{F}_\delta}\mathbb{F}_\delta^{-1} \to \frac{(\det \mathbb{F} - \tilde{\varepsilon})^+}{\det \mathbb{F}}\mathbb{F}^{-1} \quad \text{a.e. in } Q_T, \text{ weakly in } (L^4(Q_T))^{2\times 2}. \tag{6.100}$$

Employing (6.99) with the right handside converging to zero, the property $\operatorname{div} \boldsymbol{v} = 0$, the convergence (6.100) and standard properties of mollifying kernels, we get

$$\int_\Omega S_{\tilde{\varepsilon}}(\det \mathbb{F}(t)) - \int_\Omega S_{\tilde{\varepsilon}}(\det \mathbb{F}_0) + \int_0^t\int_\Omega \left(\frac{(\det \mathbb{F} - \tilde{\varepsilon})^+}{\det \mathbb{F}}|\mathbb{F}|^2 - 2\right) = 0. \tag{6.101}$$

Since $\det \mathbb{F}_0 > 0$, $\ln \det \mathbb{F}_0 \in L^1(\Omega)$, $S_{\tilde{\varepsilon}}(\det \mathbb{F}_0) \geq \ln \det \mathbb{F}_0$ and $\mathbb{F} \in (L^2(Q_T))^{2\times 2}$, the relation (6.101) implies

$$\sup_{\tilde{\varepsilon}>0}\int_\Omega S_{\tilde{\varepsilon}}(\det \mathbb{F}(t)) > -\infty \quad \text{for a.a. } t \in (0,T), \tag{6.102}$$

which leads to the result

$$\det \mathbb{F} > 0 \quad \text{a. e. in } Q_T.$$

□

7. Proof of Theorem 2.3 with $G_1 = 1$, $\mathbb{B}_2 \equiv \mathbb{O}$

To complete the proof of Theorem 2.3 with restrictions $G_1 = 1$, $\mathbb{B} := \mathbb{B}_1$, $\mathbb{B}_2 \equiv \mathbb{O}$, except proving (2.13) and the continuity of $\mathbb{B}$ in time, it remains to proceed rigorously from (4.3) to (4.8). More precisely, it remains to derive from (4.10) the relations (2.9) and (2.10) with $G_1 = 1$, $\mathbb{B} := \mathbb{B}_1$, $\mathbb{B}_2 \equiv \mathbb{O}$. Let us extend $\mathbb{F}$ and $\boldsymbol{v}$ by zero outside of Ω. Test (4.10) by $(\omega_\delta(\boldsymbol{x} - \cdot)\ \mathbb{A}(\boldsymbol{x})\ \mathbb{F}_\delta(\boldsymbol{x}))$, where $\boldsymbol{x}$ is a fixed point in Ω, ω_δ is the standard space mollifying kernel, $h_\delta(\boldsymbol{x}) := \int_{\mathbb{R}^2}\omega_\delta(\boldsymbol{x} - \cdot)h(\cdot)$ for every $h \in L^1_{loc}(\mathbb{R}^2)$ and $\mathbb{A} \in (C^\infty(\Omega))^{2\times 2}$ is arbitrary. Then test transposed (4.10) by $(\omega_\delta(\boldsymbol{x} - \cdot)\ \mathbb{F}_\delta^T(\boldsymbol{x})\ \mathbb{A}(\boldsymbol{x}))$. Summing both acquired equations, multiplying by an arbitrary $\phi \in C_c^\infty((0,T))$, integrating over Q_T, using the integration by parts and the properties $\operatorname{div} \boldsymbol{v} = 0$ in Q_T, $\boldsymbol{v} = \mathbf{0}$ on Σ_T, we obtain

$$-\int_{Q_T}(\mathbb{F}_\delta\mathbb{F}_\delta^T):(\partial_t\phi)\mathbb{A} - \int_{Q_T}\big((\mathbb{F}_\delta\mathbb{F}_\delta^T)\otimes \boldsymbol{v}\big):\phi\nabla\mathbb{A} - \int_{Q_T}\big((\nabla \boldsymbol{v}\mathbb{F})_\delta\mathbb{F}_\delta^T + \mathbb{F}_\delta(\mathbb{F}^T(\nabla \boldsymbol{v})^T)_\delta\big) : \phi\mathbb{A}$$
$$+\frac{1}{2}\int_{Q_T}\big((\mathbb{F}\mathbb{F}^T\mathbb{F})_\delta\mathbb{F}_\delta^T + \mathbb{F}_\delta(\mathbb{F}^T\mathbb{F}\mathbb{F}^T)_\delta\big) : \phi\mathbb{A} - \int_{Q_T}(\mathbb{F}_\delta\mathbb{F}_\delta^T):\phi\mathbb{A} = \int_{Q_T}\Big(\mathbb{E}_\delta\mathbb{F}_\delta^T + \mathbb{F}_\delta{\mathbb{E}_\delta}^T\Big) : \phi\mathbb{A},$$

where

$$\mathbb{E}_\delta := \operatorname{Div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - \operatorname{Div}(\mathbb{F} \otimes \boldsymbol{v})_\delta.$$

Employing $\mathbb{F} \in (L^4((0,T) \times \mathbb{R}^2))^{2\times 2}$, $\boldsymbol{v} \in L^2(0,T;(W^{1,2}(\mathbb{R}^2))^2)$ and Lemma 3.1 together with Lebesgue's convergence theorem, we obtain $\mathbb{E}_\delta \to \mathbb{O}$ strongly in $(L^{\frac{4}{3}}(Q_T))^{2\times 2}$ and due to standard properties of mollifying kernels then

$$-\int_{Q_T} \big(\mathbb{F}\mathbb{F}^T\big) : (\partial_t \phi)\mathbb{A} - \int_{Q_T} \big((\mathbb{F}\mathbb{F}^T) \otimes \boldsymbol{v}\big) : \phi \nabla \mathbb{A} + \int_{Q_T} \big(-\nabla \boldsymbol{v}(\mathbb{F}\mathbb{F}^T) - (\mathbb{F}\mathbb{F}^T)\nabla \boldsymbol{v}^T\big) : \phi \mathbb{A}$$
$$+ \int_{Q_T} \big((\mathbb{F}\mathbb{F}^T)^2 - \mathbb{F}\mathbb{F}^T\big) : \phi \mathbb{A} = 0. \quad (7.1)$$

Setting $\mathbb{B} := \mathbb{F}\mathbb{F}^T$, we get $\mathbb{B} \in (L^2(Q_T))^{2\times 2}$, and as also $\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}}) \cap (L^4(Q_T))^2$, $C^\infty(\Omega)$ is dense in $W^{1,4}(\Omega)$ and $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, from (7.1) it follows

$$\partial_t \mathbb{B} \in L^1(0,T;((W^{1,4}(\Omega))^{2\times 2})^*)$$

and, using also the Du Bois-Reymond lemma, from (7.1) we conclude (2.9) with $\mathbb{B} := \mathbb{B}_1$ of the form (2.10).

It remains to show

$$\mathbb{B} \in C([0,T];(L^1(\Omega))^{2\times 2})$$

and the attainment of the initial condition (2.13). We write for all t_0, $t_1 \in [0,T]$ (the second inequality follows from the Hölder inequality)

$$\begin{aligned}
\int_\Omega \big|\mathbb{F}(t_1)\mathbb{F}(t_1)^T - \mathbb{F}(t_0)\mathbb{F}(t_0)^T\big| &= \int_\Omega \big|\mathbb{F}(t_1)(\mathbb{F}(t_1) - \mathbb{F}(t_0))^T + (\mathbb{F}(t_1) - \mathbb{F}(t_0))\mathbb{F}(t_0)^T\big|, \\
&\le \int_\Omega |\mathbb{F}(t_1)|\, |\mathbb{F}(t_1) - \mathbb{F}(t_0)| + |\mathbb{F}(t_1) - \mathbb{F}(t_0)|\, |\mathbb{F}(t_0)| \\
&\le \|\mathbb{F}(t_1)\|_2 \|\mathbb{F}(t_1) - \mathbb{F}(t_0)\|_2 + \|\mathbb{F}(t_1) - \mathbb{F}(t_0)\|_2 \|\mathbb{F}(t_0)\|_2,
\end{aligned}$$

which converges to zero as $t_1 \to t_0$ if $t_0 \in (0,T)$, as $t_1 \to t_0+$ if $t_0 = 0$, as $t_1 \to t_0-$ if $t_0 = T$, since $\mathbb{F} \in C([0,T];(L^2(\Omega))^{2\times 2})$. Hence $\mathbb{B} = \mathbb{F}\mathbb{F}^T \in C([0,T];(L^1(\Omega))^{2\times 2})$ and the property $\mathbb{F}(0,\cdot) = \mathbb{F}_0$ a.e. in Ω (which follows from (6.37)) then implies fulfilling of the condition(2.13).

□

8. Proof of Theorem 2.3 with G_1, $G_2 > 0$ arbitrary

Let us follow step by step the proof of Theorem 2.3 with $G_1 = 1$, $\mathbb{B}_2 \equiv \mathbb{O}$.

8.1. ***System with equations for*** $\mathbb{F}_1$, $\mathbb{F}_2$

The system with evolutionary equations for $\boldsymbol{v}$, p, $\mathbb{F}_1$, $\mathbb{F}_2$ in Q_T reads

$$\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0, && (8.1) \\
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \operatorname{div} \mathbb{D} - \operatorname{div}\big(G_1(\mathbb{F}_1\mathbb{F}_1^T) + G_2(\mathbb{F}_2\mathbb{F}_2^T)\big) &= \mathbf{0},\ G_1, G_2 > 0 && (8.2) \\
\partial_t \mathbb{F}_i + \operatorname{Div}(\mathbb{F}_i \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{F}_i + \frac{1}{2}(\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i - \mathbb{F}_i) &= \mathbb{O},\ i = 1,2 && (8.3) \\
\det \mathbb{F}_i &> 0,\ \ i = 1,2 && (8.4)
\end{aligned}$$

completed with the boundary condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T \tag{8.5}$$

and with the initial conditions

$$\begin{aligned} \boldsymbol{v}(0,\cdot) &= \boldsymbol{v}_0 \quad \text{in } \Omega, & (8.6)\\ \mathbb{F}_i(0,\cdot) &= \mathbb{F}_{i_0} \quad \text{in } \Omega, \quad i=1,2. & (8.7) \end{aligned}$$

We prove the existence of weak solutions to (8.1)–(8.7), i.e. the existence of $\boldsymbol{v}$, $\mathbb{F}_1$, $\mathbb{F}_2$ fulfilling for $i = 1, 2$

$$\begin{aligned} \boldsymbol{v} &\in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}}),\\ \partial_t \boldsymbol{v} &\in L^2(0,T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*),\\ \mathbb{F}_i &\in C([0,T]; (L^2(\Omega))^{2\times 2}) \cap (L^4(Q_T))^{2\times 2},\\ \partial_t \mathbb{F}_i &\in L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right),\\ \det \mathbb{F}_i &> 0 \quad \text{a.e. in } Q_T \end{aligned}$$

and satisfying for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t \in (0,T)$

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i(\mathbb{F}_i\mathbb{F}_i^T) : \nabla \boldsymbol{w} = 0, \tag{8.8}$$

$$\langle \partial_t \mathbb{F}_i, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v})\mathbb{F}_i) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i - \mathbb{F}_i) : \mathbb{A} = 0 \tag{8.9}$$

with initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_{i_0}$ fulfilled in the sense

$$\begin{aligned} \lim_{t\to 0+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 &= 0, & (8.10)\\ \lim_{t\to 0+} \|\mathbb{F}_i(t) - \mathbb{F}_{i_0}\|_2 &= 0. & (8.11) \end{aligned}$$

8.1.1. ***Approximations and their existence***

We start with approximations, where on the left handside of (8.9) the term representing the stress diffusion is added, the term reads $\varepsilon \int_\Omega \nabla \mathbb{F}_i : \nabla \mathbb{A}$. The local in time existence of Galerkin's approximations to the corresponding system follows from the Carathéodory theory for ordinary differential equations, simmilarly as in Subsection 5.1. Properly written, let $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ form a basis of $W^{1,2}_{\mathbf{0},\mathrm{div}}$ composed of eigenfunctions of the Stokes operator subject to the boundary condition $\boldsymbol{w} = \mathbf{0}$ on $\partial\Omega$, orthogonal in $W^{1,2}_{\mathbf{0},\mathrm{div}}$, orthonormal in $L^2_{\boldsymbol{n},\mathrm{div}}$, let $\{\mathbb{A}_j\}_{j\in\mathbb{N}}$ form a basis of $(W^{1,2}(\Omega))^{2\times 2}$ composed of eigenfunctions of the Laplace operator subject to the boundary condition $\nabla \mathbb{A} \cdot \boldsymbol{n} := \{\nabla A_{kl} \cdot \boldsymbol{n}\}_{k,l=1}^2 = \mathbb{O}$ on $\partial\Omega$, orthogonal in $(W^{1,2}(\Omega))^{2\times 2}$, orthonormal in $(L^2(\Omega))^{2\times 2}$, then for every $n \in \mathbb{N}$ there exist $\alpha_{1,n}(t), ..., \alpha_{n,n}(t)$, $\beta_{i,1,n}(t), ..., \beta_{i,n,n}(t)$ (but we write only $\alpha_1, ..., \alpha_n$, $\beta_{i,1}$,..., $\beta_{i,n}$), $i = 1, 2$, such that

$$\boldsymbol{v}_n = \sum_{j=1}^{n} \alpha_j \boldsymbol{w}_j \quad \text{and} \quad \mathbb{F}_{i,n} = \sum_{j=1}^{n} \beta_{i,j}\mathbb{A}_j \tag{8.12}$$

solves for all $j \in \{1,...,n\}$, for all $t \in (0,\tilde{t})$, where $\tilde{t}$ is certain positive number, the following system:

$$\partial_t \left(\int_\Omega \boldsymbol{v}_n \cdot \boldsymbol{w}_j\right) - \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla \boldsymbol{w}_j + \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{w}_j + \sum_{i=1}^{2} \int_\Omega G_i(\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T) : \nabla \boldsymbol{w}_j = 0, \tag{8.13}$$

$$\begin{aligned} \partial_t \left(\int_\Omega \mathbb{F}_{i,n} : \mathbb{A}_j\right) - \int_\Omega (\mathbb{F}_{i,n} \otimes \boldsymbol{v}_n) : \nabla \mathbb{A}_j - \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_{i,n}) : \mathbb{A}_j \\ + \frac{1}{2} \int_\Omega \big(\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T\mathbb{F}_{i,n} - \mathbb{F}_{i,n}\big) : \mathbb{A}_j + \varepsilon \int_\Omega \nabla \mathbb{F}_{i,n} : \nabla \mathbb{A}_j = 0. \end{aligned} \tag{8.14}$$

Let P_n denote the orthogonal projection from $W^{1,2}_{\mathbf{0},\mathrm{div}}$ to $W_n := \mathrm{span}\{\boldsymbol{w}_1, ..., \boldsymbol{w}_n\}$ and let Q_n denote the orthogonal projection from $(W^{1,2}(\Omega))^{2\times 2}$ to $X_n := \mathrm{span}\{\mathbb{A}_1, ..., \mathbb{A}_n\}$. Let us note that P_n is continuous in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\mathbf{0},\mathrm{div}}$, Q_n is continuous in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$. The functions $\boldsymbol{v}_n$ are absolutely continuous in $[0,\tilde{t})$ with values in W_n, the functions $\mathbb{F}_{i,n}$ are absolutely continuous in $[0,\tilde{t})$ with values in X_n and they satisfy the initial conditions

$$\boldsymbol{v}_n(0,\cdot) = P_n(\boldsymbol{v}_0), \quad \mathbb{F}_{i,n}(0,\cdot) = Q_n(\mathbb{F}_{i_0}) \;\text{ in } \Omega. \tag{8.15}$$

The fact that $\tilde{t} = T$ is an easy consequence of the uniform estimates that follow.

In order to obtain the uniform estimates for $\boldsymbol{v}_n$, $\mathbb{F}_{1,n}$ and $\mathbb{F}_{2,n}$, let us multiply (8.13) by α_j, (8.14) by $\beta_{1,j}$, resp. by $\beta_{2,j}$, and take the sum over $j = 1, ..., n$ to obtain for $i = 1,2$ and for all $t \in (0,T)$ (use also the symmetry of $\mathbb{D}_n$, the symmetry of $\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T$ and the equality $\nabla \boldsymbol{v}_n \mathbb{F}_{i,n} : \mathbb{F}_{i,n} = (\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T) : \nabla \boldsymbol{v}_n$)

$$\frac{\|\boldsymbol{v}_n(t)\|_2^2}{2} + \int_0^t \|\mathbb{D}_n\|_2^2 + \sum_{i=1}^{2} \int_0^t \int_\Omega G_i(\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T) : \nabla \boldsymbol{v}_n = \frac{\|\boldsymbol{v}_n(0)\|_2^2}{2}, \tag{8.16}$$

$$\begin{aligned} \frac{\|\mathbb{F}_{i,n}(t)\|_2^2}{2} - \int_0^t \int_\Omega (\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T) : \nabla \boldsymbol{v}_n + \frac{1}{2} \int_0^t \big(\|\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T\|_2^2 - \|\mathbb{F}_{i,n}\|_2^2\big) \\ + \varepsilon \int_0^t \|\nabla \mathbb{F}_{i,n}\|_2^2 = \frac{\|\mathbb{F}_{i,n}(0)\|_2^2}{2}. \end{aligned} \tag{8.17}$$

Summing (8.16), (8.17) for $i = 1$ multiplied by G_1 and (8.17) for $i = 2$ multiplied by G_2, multiplying the result by 2, we get for all $t \in (0,T)$

$$\begin{aligned} \|\boldsymbol{v}_n(t)\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_{i,n}(t)\|_2^2 + \int_0^t \left(2\|\mathbb{D}_n\|_2^2 + \sum_{i=1}^{2} G_i (\|\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T\|_2^2 + 2\varepsilon \|\nabla \mathbb{F}_{i,n}\|^2) \right) \\ \le \|\boldsymbol{v}_n(0)\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_{i,n}(0)\|_2^2 + \int_0^t \sum_{i=1}^{2} G_i \|\mathbb{F}_{i,n}\|_2^2. \end{aligned} \tag{8.18}$$

Since $\|\boldsymbol{v}_n(t)\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_{i,n}(t)\|_2^2$ is estimated by the right handside of (8.18), the Gronwall lemma applied on (8.18) (the functions $\|\boldsymbol{v}_n(\cdot)\|_2$ and $\|\mathbb{F}_{i,n}(\cdot)\|_2$ are continuous in $[0,T)$) together with the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$, of Q_n in $(L^2(\Omega))^{2\times 2}$, and with the conditions (8.15) implies for all $t \in (0,T)$ the inequality

$$\|\boldsymbol{v}_n(t)\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_{i,n}(t)\|_2^2 \le e^t \left(\|\boldsymbol{v}_0\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_{i_0}\|_2^2 \right). \tag{8.19}$$

Recall that G_1, $G_2 > 0$, employ the fact $\|\mathbb{F}_{i,n}\|_4^4 \le 2\|\mathbb{F}_{i,n}\mathbb{F}_{i,n}^T\|_2^2$ in $(0,T)$ (see the argumentation in Subsection 5.2, here we only replace $\mathbb{F}_n$ by $\mathbb{F}_{i,n}$). Then from (8.18), (8.19) and the Korn inequality it follows

$$\begin{aligned}\sup_{t\in(0,T)} \|\boldsymbol{v}_n(t)\|_2^2 + \sum_{i=1}^{2}\left(\sup_{t\in(0,T)} \|\mathbb{F}_{i,n}(t)\|_2^2\right) + \|\nabla \boldsymbol{v}_n\|_{2,Q_T}^2 + \sum_{i=1}^{2} \|\mathbb{F}_{i,n}\|_{4,Q_T}^4 \\ + \varepsilon \sum_{i=1}^{2} \|\nabla \mathbb{F}_{i,n}\|_{2,Q_T}^2 \le C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_{1_0}\|_2, \|\mathbb{F}_{2_0}\|_2).\end{aligned} \tag{8.20}$$

In (8.13) we can, obviously, replace the base functions $\boldsymbol{w}_j$ by any function of the form $P_n(\boldsymbol{w})$, where $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ is arbitrary, and in (8.14) we can replace $\mathbb{A}_j$ by any function of the form $Q_n(\mathbb{A})$, where $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ is arbitrary. Repeating the procedure from Subsection 5.2, employing (8.20), the orthogonality and the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, the orthogonality and the continuity of Q_n in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$, we achieve the estimates for the time derivatives of $\boldsymbol{v}_n$ and $\mathbb{F}_{i,n}$ that read

$$\begin{aligned}\|\partial_t \boldsymbol{v}_n\|_{L^2\left(0,T;(W^{1,2}_{0,\mathrm{div}})^*\right)} + \sum_{i=1}^{2} \|\partial_t \mathbb{F}_{i,n}\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \\ \le \tilde{C}(T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_{1_0}\|_2, \|\mathbb{F}_{2_0}\|_2).\end{aligned} \tag{8.21}$$

The estimates (8.20) and (8.21) suffice to acquire the existence of $\boldsymbol{v}$, $\mathbb{F}_1$, $\mathbb{F}_2$ such that for $i = 1,2$ it holds

$$\boldsymbol{v}_n \rightharpoonup^* \boldsymbol{v} \quad \text{weakly-* in } L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}), \tag{8.22}$$

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \tag{8.23}$$

$$\partial_t \boldsymbol{v}_n \rightharpoonup \partial_t \boldsymbol{v} \quad \text{weakly in } L^2\left(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), \tag{8.24}$$

$$\mathbb{F}_{i,n} \rightharpoonup^* \mathbb{F}_i \quad \text{weakly-* in } L^\infty\left(0,T;(L^2(\Omega))^{2\times 2}\right), \tag{8.25}$$

$$\mathbb{F}_{i,n} \rightharpoonup \mathbb{F}_i \quad \text{weakly in } L^2\left(0,T;(W^{1,2}(\Omega))^{2\times 2}\right) \cap (L^4(Q_T))^{2\times 2}, \tag{8.26}$$

$$\partial_t \mathbb{F}_{i,n} \rightharpoonup \partial_t \mathbb{F}_i \quad \text{weakly in } L^{\frac{4}{3}}\left(0,T;((W^{1,2}(\Omega))^{2\times 2})^*\right), \tag{8.27}$$

by the Aubin-Lions compactness lemma also

$$\boldsymbol{v}_n \to \boldsymbol{v} \quad \text{strongly in } (L^q(Q_T))^2 \text{ for all } q \in [1,4), \tag{8.28}$$

$$\mathbb{F}_{i,n} \to \mathbb{F}_{i,n} \text{ strongly in } (L^q(Q_T))^{2\times 2} \text{ for all } q \in [1,4). \tag{8.29}$$

Let us note that thanks to the properties $\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^2\left(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right)$, $\mathbb{F}_i \in L^\infty\left(0,T;(L^2(\Omega))^{2\times 2}\right)$ and $\partial_t \mathbb{F}_i \in L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)$, $i = 1,2$, together with the density of $(W^{1,2}(\Omega))^{2\times2}$ in $(L^2(\Omega))^{2\times2}$, the functions $\boldsymbol{v}$, $\mathbb{F}_i$ after a possible change on a zero-measure subset of $(0,T)$ enjoy

$$\begin{aligned} \boldsymbol{v} &\in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}}), && (8.30)\\ \mathbb{F}_i &\in C_{weak}\left([0,T];(L^2(\Omega))^{2\times2}\right), \quad i=1,2, && (8.31)\end{aligned}$$

and thus (use also the weak lower semicontinuity of $L^2(\Omega)$ norm)

$$\begin{aligned} \sup_{t\in(t_0,t_1)}\|\boldsymbol{v}(t)\|_2^2 &= \mathrm{esssup}_{t\in(t_0,t_1)}\|\boldsymbol{v}(t)\|_2^2, \quad 0\le t_0<t_1\le T, && (8.32)\\ \sup_{t\in(t_0,t_1)}\|\mathbb{F}_i(t)\|_2^2 &\le \mathrm{esssup}_{t\in(t_0,t_1)}\|\mathbb{F}_i(t)\|_2^2, \quad 0\le t_0<t_1\le T, \quad i=1,2. && (8.33)\end{aligned}$$

Due to (8.13), (8.14), both multiplied by any $\phi\in C_c^\infty((0,T))$, due to (8.22)–(8.29) and the density of $\bigcup_{n\in\mathbb{N}}W_n$ in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, of $\bigcup_{n\in\mathbb{N}}X_n$ in $(W^{1,2}(\Omega))^{2\times2}$, the functions $\boldsymbol{v}$, $\mathbb{F}_i$, $i=1,2$, satisfy for all $\boldsymbol{w}\in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, for all $\mathbb{A}\in(W^{1,2}(\Omega))^{2\times2}$ and a.a. $t\in(0,T)$

$$\langle\partial_t\boldsymbol{v},\boldsymbol{w}\rangle-\int_\Omega(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{w}+\int_\Omega\mathbb{D}:\nabla\boldsymbol{w}+\sum_{i=1}^2\int_\Omega G_i(\mathbb{F}_i\mathbb{F}_i^T):\nabla\boldsymbol{w}=0, \qquad (8.34)$$

$$\langle\partial_t\mathbb{F}_i,\mathbb{A}\rangle-\int_\Omega(\mathbb{F}_i\otimes\boldsymbol{v}):\nabla\mathbb{A}-\int_\Omega((\nabla\boldsymbol{v})\mathbb{F}_i):\mathbb{A}+\frac{1}{2}\int_\Omega(\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i-\mathbb{F}_i):\mathbb{A}$$
$$+\varepsilon\int_\Omega\nabla\mathbb{F}_i:\nabla\mathbb{A}=0. \qquad (8.35)$$

The attainment of the initial conditions (8.10), (8.11), where $\boldsymbol{v}$, $\mathbb{F}_i$, $i=1,2$ is a solution to (8.34)–(8.35), is proved following step by step the procedure from Subsection 5.4: We derive (5.51) and (5.53) with $\mathbb{F}$ replaced by $\mathbb{F}_i$ and $\mathbb{F}_0$ replaced by $\mathbb{F}_{i_0}$, i.e.

$$\begin{aligned} \lim_{t\to0+}\int_\Omega\boldsymbol{v}(t):\boldsymbol{w} &= \int_\Omega\boldsymbol{v}_0:\boldsymbol{w} \quad \forall\boldsymbol{w}\in L^2_{\boldsymbol{n},\mathrm{div}}, && (8.36)\\ \lim_{t\to0+}\int_\Omega\mathbb{F}_i(t):\mathbb{A} &= \int_\Omega\mathbb{F}_{i_0}:\mathbb{A} \quad \forall\mathbb{A}\in(L^2(\Omega))^{2\times2}, \quad i=1,2. && (8.37)\end{aligned}$$

Then, working with $\sum_{i=1}^2 G_i\mathbb{F}_i$ instead of $\mathbb{F}$, we derive the inequality

$$\limsup_{t\to0+}\left(\|\boldsymbol{v}(t)\|_2^2+\sum_{i=1}^2G_i\|\mathbb{F}_i(t)\|_2^2\right)\le\|\boldsymbol{v}_0\|_2^2+\sum_{i=1}^2G_i\|\mathbb{F}_{i_0}\|_2^2,$$

which together with (8.36), where $\boldsymbol{w}:=\boldsymbol{v}_0$, (8.37), where $\mathbb{A}:=\mathbb{F}_{1_0}$ if $i=1$, $\mathbb{A}:=\mathbb{F}_{2_0}$ if $i=2$ implies

$$\limsup_{t\to0+}\left(\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2+\sum_{i=1}^2G_i\|\mathbb{F}_i(t)-\mathbb{F}_{i_0}\|_2^2\right)\le0, \qquad (8.38)$$

which due to the positivity of G_1, G_2 implies fulfilling of (8.10) and (8.11).

8.1.2. ***Limit in approximations***

Let us consider the sequence $\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_{i,\varepsilon}\}$, $i = 1, 2$, of solutions to (8.34), (8.35). Employing (8.20), the convergences (8.22), (8.23), (8.25), (8.26), the properties (8.32), (8.33) and weak lower semicontinuity of all norms acting in (8.20), we deduce that in (8.20) we can replace $\boldsymbol{v}_n$ by $\boldsymbol{v}_\varepsilon$ and $\mathbb{F}_{i,n}$ by $\mathbb{F}_{i,\varepsilon}$, $i = 1, 2$. And as one can check, using the same argumentation as in Subsection 6.1, we gain (8.21) with $\partial_t \boldsymbol{v}_\varepsilon$ instead of $\partial_t \boldsymbol{v}_n$ and $\partial_t \mathbb{F}_{i,\varepsilon}$ instead of $\partial_t \mathbb{F}_{i,n}$. Hence there exist $\boldsymbol{v}$, $\mathbb{F}_i$, $i = 1, 2$, satisfying the convergence relations (8.22)–(8.25), (8.27), (8.28) with $\boldsymbol{v}_n$ replaced by $\boldsymbol{v}_\varepsilon$ and $\mathbb{F}_{i,n}$ replaced by $\mathbb{F}_{i,\varepsilon}$, instead of (8.26) we have the convergences

$$\begin{aligned} \mathbb{F}_{i,\varepsilon} &\rightharpoonup \mathbb{F}_i \quad \text{weakly in } (L^4(Q_T))^{2\times 2}, && (8.39)\\ \varepsilon \nabla \mathbb{F}_{i,\varepsilon} &\to \mathbb{O} \quad \text{strongly in } (L^2(Q_T))^{2\times 2\times 2}. && (8.40)\end{aligned}$$

Taking the limit $\varepsilon \to 0+$ in (8.34) with $\boldsymbol{v} := \boldsymbol{v}_\varepsilon$ and in (8.35) with $\mathbb{F}_i := \mathbb{F}_{i,\varepsilon}$, $i = 1, 2$ then leads for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and a.a. $t \in (0, T)$ to

$$\begin{aligned} \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \overline{\mathbb{D}} : \nabla \boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i \overline{\mathbb{F}_i \mathbb{F}_i^T} : \nabla \boldsymbol{w} &= 0, && (8.41)\\ \langle \partial_t \mathbb{F}_i, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega \overline{(\nabla \boldsymbol{v})\mathbb{F}_i} : \mathbb{A} + \frac{1}{2}\int_\Omega (\overline{\mathbb{F}_i \mathbb{F}_i^T \mathbb{F}_i} - \mathbb{F}_i) : \mathbb{A} &= 0, && (8.42)\end{aligned}$$

where the notation $\overline{a}$ stands for the weak limit of a weakly convergent subsequence of $\{a_\varepsilon\}$. Since $\boldsymbol{v} \in L^2(0, T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^2(0, T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, $\mathbb{F}_i \in L^\infty(0, T; (L^2(\Omega))^{2\times 2})$, $\partial_t \mathbb{F}_i \in L^{\frac{4}{3}}(0, T; (W^{1,2}(\Omega))^{2\times 2})^*)$, $i = 1, 2$, and $(W^{1,2}(\Omega))^{2\times 2}$ is dense in $(L^2(\Omega))^{2\times 2}$, the functions $\boldsymbol{v}$, $\mathbb{F}_i$ after a possible change on a zero-measure subset of $(0, T)$ satisfy

$$\begin{aligned} \boldsymbol{v} &\in C([0, T]; L^2_{\boldsymbol{n},\mathrm{div}}), && (8.43)\\ \mathbb{F}_i &\in C_{weak}\left([0, T]; (L^2(\Omega))^{2\times 2}\right), \quad i = 1, 2, && (8.44)\end{aligned}$$

and thus (use also the weak lower semicontinuity of $L^2(\Omega)$ norm)

$$\begin{aligned} \sup_{t\in(t_0,t_1)} \|\boldsymbol{v}(t)\|_2^2 &= \mathrm{esssup}_{t\in(t_0,t_1)} \|\boldsymbol{v}(t)\|_2^2, \quad 0 \le t_0 < t_1 \le T, && (8.45)\\ \sup_{t\in(t_0,t_1)} \|\mathbb{F}_i(t)\|_2^2 &\le \mathrm{esssup}_{t\in(t_0,t_1)} \|\mathbb{F}_i(t)\|_2^2, \quad 0 \le t_0 < t_1 \le T, \quad i = 1, 2. && (8.46)\end{aligned}$$

In order to obtain the attainment of the initial conditions (8.10) and (8.11), where $\boldsymbol{v}$, $\mathbb{F}_i$, $i = 1, 2$ is a solution to (8.41)–(8.42), we follow the procedure from the end of Subsection 6.1: For $\boldsymbol{v}$, $\mathbb{F}_i$, $i = 1, 2$ solving (8.41)–(8.42) we derive (8.36) and (8.37). Then we take the limit in (8.38) with $\boldsymbol{v} := \boldsymbol{v}_\varepsilon$, $\mathbb{F}_i := \mathbb{F}_{i,\varepsilon}$, $i = 1, 2$ solving (8.34)–(8.35), employing (8.22) with $\boldsymbol{v}_n$ replaced by $\boldsymbol{v}_\varepsilon$, (8.25) with $\mathbb{F}_{i,n}$ replaced by $\mathbb{F}_{i,\varepsilon}$, (8.45), (8.46) and weak lower semicontinuity of $L^2(\Omega)$ norm, we obtain (8.38) for $\boldsymbol{v}$, $\mathbb{F}_i$, $i = 1, 2$ solving (8.41)–(8.42). Setting $\boldsymbol{w} := \boldsymbol{v}_0$ in (8.36), $\mathbb{A} := \mathbb{F}_0$ in (8.37) and employing (8.38), we arrive at (8.10) and (8.11).

In order to obtain the property

$$\mathbb{F}_i \in C([0, T]; (L^2(\Omega))^{2\times 2}), \quad i = 1, 2, \tag{8.47}$$

we repeat the procedure from Subsection 6.2, only instead of (6.34) and (6.35) we work with (8.41) and (8.42), setting $i = 1$ in order to prove (8.47) with $i = 1$, setting $i = 2$ for the proof of (8.47) with $i = 2$.

As $\mathbb{F}_{i,\varepsilon} \rightharpoonup \mathbb{F}_i$, $i = 1, 2$ weakly in $(L^4(Q_T))^{2\times 2}$ and $\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v}$ weakly in $L^2(0, T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, in order to conclude (8.8)–(8.9) from (8.41)–(8.42) it suffices to prove the compactness of $\{\mathbb{F}_{i,\varepsilon}\}$ in $(L^2(Q_T))^{2\times 2}$.

8.1.3. *Compactness of* $\mathbb{F}_{1,\varepsilon}, \mathbb{F}_{2,\varepsilon}$ *in* $(L^2(Q_T))^{2\times 2}$

As $\mathbb{F}_{i,\varepsilon} \rightharpoonup \mathbb{F}_i$ weakly in $(L^4(Q_T))^{2\times 2}$, $i = 1, 2$, the compactness of $\mathbb{F}_{i,\varepsilon}$ in $(L^2(Q_T))^{2\times 2}$ is equivalent to the condition

$$f_i := \overline{|\mathbb{F}_i|^2} - |\mathbb{F}_i|^2 = 0 \quad \text{a.e. in } Q_T,\ i = 1, 2. \tag{8.48}$$

Proceeding exactly in the same way as in Subsection 6.3, Step 1, we prove (6.62) and (6.63) with $\mathbb{F}_{i,\varepsilon}$ in the role of $\mathbb{F}_\varepsilon$ and $\mathbb{F}_i$ in the role of $\mathbb{F}$ ($i = 1, 2$). Next, following the computations from Subsection 6.3, Step 2, we derive from the difference between (6.62) and (6.63) (with $\mathbb{F}_{i,\varepsilon}$ in the role of $\mathbb{F}_\varepsilon$ and $\mathbb{F}_i$ in the role of $\mathbb{F}$) multiplied by G_i and summed over $i = 1, 2$, for all $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, $\varphi \geq 0$, the inequality

$$-\int_{Q_T} \sum_{i=1}^{2} (G_i f_i)\partial_t \varphi - \int_{Q_T} \sum_{i=1}^{2} (G_i f_i)\boldsymbol{v} \cdot \nabla\varphi \leq \int_{Q_T} L \sum_{i=1}^{2} (G_i f_i)\varphi \tag{8.49}$$

with $L := \max_{i\in\{1,2\}} \left(1 + 2(|\nabla \boldsymbol{v}| + \hat{C}|\mathbb{F}_i|^2)\right)$. The only difference from the computations in Step 2 in Subsection 6.3 is that instead of $\mathbb{F}$ we work with the sum $\sum_{i=1}^2 G_i \mathbb{F}_i$. After that, following the argumentation in Step 3 in Subsection 6.3, we show

$$f_i \geq 0 \quad \text{a.e. in } Q_T,\ i = 1, 2, \tag{8.50}$$

and then, working with $\sum_{i=1}^2 G_i f_i$ instead of f, we conclude

$$\sum_{i=1}^{2} G_i f_i = 0 \quad \text{a.e. in } Q_T. \tag{8.51}$$

Since $G_i > 0$, $i = 1, 2$, the relations (8.50) and (8.51) imply fulfilling of (8.48), which is equivalent to the compactness of $\{\mathbb{F}_{i,\varepsilon}\}$, $i = 1, 2$, in $(L^2(Q_T))^{2\times 2}$.

8.2. *Concluding the result*

To complete the proof of Theorem 2.3, it suffices to conclude (2.9), (2.10) and the initial condition (2.13). However, in order to conclude (2.10), we repeat the procedure from Subsection 6.4 with $\mathbb{F}_i$ ($i = 1, 2$) in the role of $\mathbb{F}$, and in order to conclude (2.9) and (2.13), we repeat the procedure from Section 7 with $\mathbb{F}_i$ in the role of $\mathbb{F}$ and $\mathbb{B}_i$ in the role of $\mathbb{B}$, $i = 1, 2$.

□

9. Appendix

Proof of Proposition 6.1

Before making the proof of Proposition 6.1 let us quote from [25] the following lemma concerning stationary Stokes problems.

Lemma 9.1. *Let $d \geq 2$, $m \geq -1$, $q \in (1,\infty)$, $\tilde{\Omega} \subset \mathbb{R}^d$, $\tilde{\Omega} \in C^{\max\{m+2,2\}}$, $\boldsymbol{g} \in (W^{m,q}(\tilde{\Omega}))^d$, $\boldsymbol{w}^* \in (W^{m+2-\frac{1}{q},q}(\partial\tilde{\Omega}))^d$, $\int_{\partial\tilde{\Omega}} \boldsymbol{w}^* \cdot \mathbf{n} = 0$. Then there exists unique weak solution $[\boldsymbol{w}, \tilde{p}]$, $\int_{\tilde{\Omega}} \tilde{p} = 0$, to the Stokes problem*

$$\begin{aligned} -\Delta \boldsymbol{w} + \nabla \tilde{p} &= \boldsymbol{g} && \text{in } \tilde{\Omega}, \\ \operatorname{div} \boldsymbol{w} &= 0 && \text{in } \tilde{\Omega}, \\ \boldsymbol{w} &= \boldsymbol{w}^* && \text{on } \partial\tilde{\Omega}, \end{aligned}$$

more specifically there exists unique couple $[\boldsymbol{w}, \tilde{p}]$ fulfilling

$$\boldsymbol{w} \in (W^{m+2,q}(\tilde{\Omega}))^d, \quad \boldsymbol{w} - \boldsymbol{w}^* \in (W_0^{1,q}(\tilde{\Omega}))^d, \quad \tilde{p} \in W^{m+1,q}(\tilde{\Omega}), \quad \int_{\tilde{\Omega}} \tilde{p} = 0$$

and

$$\begin{aligned} \int_{\tilde{\Omega}} \nabla \boldsymbol{w} : \nabla \boldsymbol{\Phi} - \int_{\tilde{\Omega}} \tilde{p} \operatorname{div} \boldsymbol{\Phi} &= \langle \boldsymbol{g}, \boldsymbol{\Phi} \rangle && \text{for all } \boldsymbol{\Phi} \in (W_0^{1,q'}(\tilde{\Omega}))^d, \\ \int_{\tilde{\Omega}} \operatorname{div} \boldsymbol{w} \; \phi &= 0 && \text{for all } \phi \in L^{q'}(\tilde{\Omega}), \end{aligned}$$

which is equivalent to the existence of unique

$$\boldsymbol{w} \in (W^{m+2,q}(\tilde{\Omega}))^d, \quad \boldsymbol{w} - \boldsymbol{w}^* \in (W_0^{1,q}(\tilde{\Omega}))^d, \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \tilde{\Omega}$$

fulfilling

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w} : \nabla \boldsymbol{\Phi} = \langle \boldsymbol{g}, \boldsymbol{\Phi} \rangle \quad \text{for all } \boldsymbol{\Phi} \in \widetilde{W}^{1,q'}_{\mathbf{0},\mathrm{div}}.$$

Moreover, the solution satisfies the estimate

$$\|\boldsymbol{w}\|_{(W^{m+2,q}(\tilde{\Omega}))^d} + \|\tilde{p}\|_{W^{m+1,q}(\tilde{\Omega})} \leq \|\boldsymbol{f}\|_{(W^{m,q}(\tilde{\Omega}))^d} + \|\boldsymbol{w}^*\|_{(W^{m+2-\frac{1}{q},q}(\partial\tilde{\Omega}))^d}$$

with the convention $W^{-1,q}(\tilde{\Omega}) = (W_0^{1,q}(\tilde{\Omega}))^*$, $W^{0,q}(\tilde{\Omega}) = L^q(\tilde{\Omega})$.

For lucidity let us repeat the formulation of Proposition 6.1.

Proposition 9.2. *Let $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$, $\tilde{\Omega} \in C^\infty$. Then for every $\varepsilon > 0$ there exists p_ε of the form $p_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon}$, where*

$$p_{1,\varepsilon} \in L^2(0,T; W^{2,2}(\tilde{\Omega})), \tag{9.1}$$

$$p_{2,\varepsilon} \in L^2((0,T) \times \tilde{\Omega}), \tag{9.2}$$

$$\partial_t (\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \in L^2\left(0,T; ((W_0^{1,2}(\tilde{\Omega}))^2)^*\right) \tag{9.3}$$

and for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde\Omega))^2$ *and a.a.* $t \in (0,T)$ *it holds*

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \boldsymbol{w}\rangle = \int_{\tilde\Omega} (\mathbb{G}_\varepsilon : \nabla \boldsymbol{w}) + \int_{\tilde\Omega} p_{2,\varepsilon} \operatorname{div} \boldsymbol{w}, \quad \mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T. \tag{9.4}$$

Next, there exists p *of the form* $p = p_1 + p_2$*, where*

$$p_1 \in L^2(0,T; W^{2,2}(\tilde\Omega)), \tag{9.5}$$
$$p_2 \in L^2((0,T)\times\tilde\Omega), \tag{9.6}$$
$$\partial_t(\boldsymbol{v} + \nabla p_1) \in L^2\left(0,T; ((W_0^{1,2}(\tilde\Omega))^2)^*\right) \tag{9.7}$$

and for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde\Omega))^2$ *and a.a.* $t \in (0,T)$ *it holds*

$$\langle \partial_t(\boldsymbol{v} + \nabla p_1), \boldsymbol{w}\rangle = \int_{\Omega} (\mathbb{G} : \nabla \boldsymbol{w}) + \int_{\Omega} p_2 \operatorname{div} \boldsymbol{w}, \quad \mathbb{G} := (\boldsymbol{v}\otimes\boldsymbol{v}) - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}. \tag{9.8}$$

Moreover,

$$p_{1,\varepsilon} \to p_1 \text{ strongly in } L^2(0,T; W^{2,2}_{loc}(\tilde\Omega)), \tag{9.9}$$
$$p_{2,\varepsilon} \rightharpoonup p_2 \text{ weakly in } L^2((0,T)\times\tilde\Omega). \tag{9.10}$$

The functions $\nabla p_{1,\varepsilon}$ *and* ∇p_1 *belong to* $C([0,T];(L^2(\tilde\Omega))^2)$ *and*

$$\nabla p_{1,\varepsilon}(0,\cdot) = \nabla p_1(0,\cdot) \quad \text{a.e. in } \tilde\Omega. \tag{9.11}$$

Proof. Let $\tilde\Omega$ be an arbitrary smooth domain fulfilling $\tilde\Omega \subset \overline{\tilde\Omega} \subset \Omega$. For every time $t \in [0,T]$ let us introduce the Stokes problems

$$-\Delta \boldsymbol{w}_{1,\varepsilon} + \nabla p_{1,\varepsilon} = \boldsymbol{v}_\varepsilon \text{ in } \tilde\Omega, \tag{9.12}$$
$$\operatorname{div} \boldsymbol{w}_{1,\varepsilon} = 0 \text{ in } \tilde\Omega, \tag{9.13}$$
$$\boldsymbol{w}_{1,\varepsilon} = \boldsymbol{0} \text{ on } \partial\tilde\Omega, \tag{9.14}$$

$$-\Delta \boldsymbol{w}_{2,\varepsilon} + \nabla p_{2,\varepsilon} = \operatorname{div} \mathbb{G}_\varepsilon \text{ in } \tilde\Omega, \tag{9.15}$$
$$\operatorname{div} \boldsymbol{w}_{2,\varepsilon} = 0 \text{ in } \tilde\Omega, \tag{9.16}$$
$$\boldsymbol{w}_{2,\varepsilon} = \boldsymbol{0} \text{ on } \partial\tilde\Omega. \tag{9.17}$$

Since $\boldsymbol{v}_\varepsilon \in L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$, Lemma 9.1 implies for all $t \in [0,T]$ the existence of unique weak solution $[\boldsymbol{w}_{1,\varepsilon}, p_{1,\varepsilon}]$, $\int_{\tilde\Omega} p_{1,\varepsilon} = 0$, to the system (9.12)–(9.14), more precisely $[\boldsymbol{w}_{1,\varepsilon}, p_{1,\varepsilon}]$ satisfy for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde\Omega))^2$, $\phi \in L^2(\tilde\Omega)$ and all $t \in [0,T]$

$$\int_{\tilde\Omega} \nabla \boldsymbol{w}_{1,\varepsilon} : \nabla \boldsymbol{\Phi} - \int_{\tilde\Omega} p_{1,\varepsilon} \operatorname{div} \boldsymbol{\Phi} = \int_{\tilde\Omega} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\Phi}, \tag{9.18}$$
$$\int_{\tilde\Omega} \operatorname{div} \boldsymbol{w}_{1,\varepsilon}\ \phi = 0 \tag{9.19}$$

and the estimate

$$\|\boldsymbol{w}_{1,\varepsilon}\|_{(W^{m+2,2}(\tilde\Omega))^2} + \|p_{1,\varepsilon}\|_{W^{m+1,2}(\tilde\Omega)} \le \|\boldsymbol{v}_\varepsilon\|_{(W^{m,2}(\Omega))^2}, \quad m \in \{-1,0,1\}. \tag{9.20}$$

Let us note that (9.19) and (9.20) imply the condition

$$\operatorname{div} \boldsymbol{w}_{1,\varepsilon} = 0 \quad \text{a.e. in } \tilde{\Omega}. \tag{9.21}$$

Next, $\mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) \in (L^2(Q_T))^{2\times 2}$, hence $\operatorname{div} \mathbb{G}_\varepsilon \in L^2(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*)$, and by Lemma 9.1 there exists for a.a. $t \in (0,T)$ unique weak solution $[\boldsymbol{w}_{2,\varepsilon}, p_{2,\varepsilon}]$, $\int_{\tilde{\Omega}} p_{2,\varepsilon} = 0$, to the Stokes problem (9.15)–(9.17), more precisely, $[\boldsymbol{w}_{2,\varepsilon}, p_{2,\varepsilon}]$ satisfy for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$ and a.a. $t \in (0,T)$

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_{2,\varepsilon} : \nabla \boldsymbol{\Phi} - \int_{\tilde{\Omega}} p_{2,\varepsilon} \operatorname{div} \boldsymbol{\Phi} \;\; = \;\; -\int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla \boldsymbol{\Phi}, \tag{9.22}$$

$$\int_{\tilde{\Omega}} \operatorname{div} \boldsymbol{w}_{2,\varepsilon} \, \phi \;\; = \;\; 0 \tag{9.23}$$

and the estimate

$$\|\boldsymbol{w}_{2,\varepsilon}\|_{(W^{1,2}(\tilde{\Omega}))^2} + \|p_{2,\varepsilon}\|_{L^2(\tilde{\Omega})} \le \|\operatorname{div} \mathbb{G}_\varepsilon\|_{((W_0^{1,2}(\tilde{\Omega}))^2)^*}. \tag{9.24}$$

Let us note that (9.23) and (9.24) imply the condition

$$\operatorname{div} \boldsymbol{w}_{2,\varepsilon} = 0 \quad \text{a.e. in } \tilde{\Omega}. \tag{9.25}$$

Let $\theta \in C_c^\infty((0,T))$, $\boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ be arbitrary. In (9.18) set $\boldsymbol{\Phi} := \boldsymbol{\Phi}_0$, multiply the result by $\partial_t \theta$ and integrate over $(0,T)$ to obtain (use also the estimate (9.20))

$$\int_0^T \int_{\tilde{\Omega}} (\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\Phi}_0) \, \partial_t \theta \;\; = \;\; -\int_0^T \int_{\tilde{\Omega}} (\Delta \boldsymbol{w}_{1,\varepsilon} \cdot \boldsymbol{\Phi}_0) \, \partial_t \theta. \tag{9.26}$$

Since $\partial_t \boldsymbol{v}_\varepsilon \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, from (9.26) it follows $\partial_t \Delta \boldsymbol{w}_{1,\varepsilon} \in L^2(0,T;(\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})^*)$ and

$$\langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{\Phi}_0 \rangle = -\langle \partial_t \Delta \boldsymbol{w}_{1,\varepsilon}, \boldsymbol{\Phi}_0 \rangle = \int_\Omega (\partial_t \nabla \boldsymbol{w}_{1,\varepsilon} : \nabla \boldsymbol{\Phi}_0) \quad \text{a.e. in } (0,T). \tag{9.27}$$

Next, setting in (9.22) $\boldsymbol{\Phi} := \boldsymbol{\Phi}_0$ yields

$$\int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla \boldsymbol{\Phi}_0 = -\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_{2,\varepsilon} : \nabla \boldsymbol{\Phi}_0 \quad \text{a.e. in } (0,T). \tag{9.28}$$

Summing (9.27) with (9.28) leads to

$$0 = \langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{\Phi}_0 \rangle - \int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla \boldsymbol{\Phi}_0 = \int_{\tilde{\Omega}} \nabla (\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon}) : \nabla \boldsymbol{\Phi}_0 \quad \text{a.e. in } (0,T), \tag{9.29}$$

where the first equality follows from (6.5). Since $\boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ is arbitrary, the relations (9.19), (9.23) and (9.29) implies that $\boldsymbol{w}_\varepsilon := \partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ solves for a.a. $t \in (0,T)$ the Stokes problem

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_\varepsilon : \nabla \boldsymbol{\Phi}_0 \;\; = \;\; 0 \quad \forall \boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}. \tag{9.30}$$

Lemma 9.1 guarantees the existence of unique solution $\boldsymbol{w}_\varepsilon \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ to the Stokes problem (9.30), hence

$$\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}) \tag{9.31}$$

and

$$\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} = \mathbf{0} \quad \text{a.e. in } (0,T)\times\tilde{\Omega}. \tag{9.32}$$

Now we can write for all $\mathbf{\Phi} \in (C_c^\infty(\tilde{\Omega}))^2$ and $\theta \in C_c^\infty((0,T))$ (in the first equality we use (9.18) and (9.22), the second and the third equality is just the integration by parts, the last equality follows from (9.32))

$$\begin{aligned}\int_0^T \int_{\tilde{\Omega}} -(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\cdot(\partial_t\theta)\mathbf{\Phi} - \int_0^T \int_{\Omega} (\mathbb{G}_\varepsilon : \nabla\mathbf{\Phi})\theta - \int_0^T \int_{\tilde{\Omega}} (p_{2,\varepsilon}\operatorname{div}\mathbf{\Phi})\theta \\ = \int_0^T \int_{\tilde{\Omega}} \Delta \boldsymbol{w}_{1,\varepsilon}\cdot(\partial_t\theta)\mathbf{\Phi} + \int_0^T \int_{\tilde{\Omega}} (\nabla \boldsymbol{w}_{2,\varepsilon} : \nabla\mathbf{\Phi})\theta \\ = \int_0^T \int_{\tilde{\Omega}} \boldsymbol{w}_{1,\varepsilon}\cdot(\partial_t\theta)\Delta\mathbf{\Phi} - \int_0^T \int_{\tilde{\Omega}} (\boldsymbol{w}_{2,\varepsilon}\cdot\Delta\mathbf{\Phi})\theta \\ = \int_0^T \int_{\tilde{\Omega}} -(\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon})\cdot(\theta\Delta\mathbf{\Phi}) = 0.\end{aligned}$$

Since $\mathbb{G}_\varepsilon \in (L^2(Q_T))^{2\times 2}$ and $p_{2,\varepsilon} \in L^2((0,T)\times\tilde{\Omega})$ (which follows from the estimate (9.24) and the fact that $\mathbb{G}_\varepsilon$ is quadratically integrable over time), the last chain yields

$$\partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \in L^2(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*),$$

which is (9.3), and (recall that $C_c^\infty(\tilde{\Omega})$ is dense in $W_0^{1,2}(\tilde{\Omega})$) for all $\mathbf{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and a.a. $t \in (0,T)$ it holds

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + p_{1,\varepsilon}), \mathbf{\Phi}\rangle = \int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla\mathbf{\Phi} + \int_{\tilde{\Omega}} p_{2,\varepsilon}\operatorname{div}\mathbf{\Phi},$$

which is the relation (9.4) that we wanted to prove.

Next, let us prove (9.7) and (9.8). Since $\boldsymbol{v}_\varepsilon$ is uniformly bounded in $L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, the estimate (9.20) implies that $\boldsymbol{w}_{1,\varepsilon}$ is uniformly bounded in $L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2)$, $p_{1,\varepsilon}$ is uniformly bounded in $L^2(0,T;W^{2,2}(\tilde{\Omega}))$, and since $\boldsymbol{w}_{1,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\int_{\tilde{\Omega}} p_{1,\varepsilon} = 0$ for every $\varepsilon > 0$, there exists $\boldsymbol{w}_1 \in L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2 \cap \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $p_1 \in L^2(0,T;W^{2,2}(\tilde{\Omega}))$, $\int_{\tilde{\Omega}} p_1 = 0$, such that (for suitable subsequences of $\{\boldsymbol{w}_{1,\varepsilon}\}, \{p_{1,\varepsilon}\}$, which we do not relabel)

$$\boldsymbol{w}_{1,\varepsilon} \rightharpoonup \boldsymbol{w}_1 \quad \text{weakly in } L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2 \cap \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}), \tag{9.33}$$

$$p_{1,\varepsilon} \rightharpoonup p_1 \quad \text{weakly in } L^2(0,T;W^{2,2}(\tilde{\Omega})). \tag{9.34}$$

Let $t \in (0,T)$ be fixed. As $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$ strongly in $(L^2(Q_T))^2$ (see (6.24)), taking the limit $\varepsilon \to 0+$ in (9.18) and (9.19), employing the convergences (9.33) and (9.34), we observe that $[\boldsymbol{w}_1, p_1]$ satisfy for all $\mathbf{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_1 : \nabla\mathbf{\Phi} - \int_{\tilde{\Omega}} p_1 \operatorname{div}\mathbf{\Phi} = \int_{\tilde{\Omega}} \boldsymbol{v}\cdot\mathbf{\Phi}, \tag{9.35}$$

$$\int_{\tilde{\Omega}} \operatorname{div}\boldsymbol{w}_1\, \phi = 0. \tag{9.36}$$

Next, from the estimate (9.24) and the facts that $\mathbb{G}_\varepsilon$ is uniformly bounded in $(L^2(Q_T))^{2\times 2}$, $\boldsymbol{w}_{2,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\int_{\tilde{\Omega}} p_{2,\varepsilon} = 0$ for every $\varepsilon > 0$, there exist $\boldsymbol{w}_2 \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$,

$p_2 \in L^2((0,T)\times\tilde\Omega)$, $\int_{\tilde\Omega} p_2 = 0$, such that (for suitable subsequences)

$$\begin{aligned}
\boldsymbol{w}_{2,\varepsilon} &\rightharpoonup \boldsymbol{w}_2 \text{ weakly in } L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}), && (9.37)\\
p_{2,\varepsilon} &\rightharpoonup p_2 \text{ weakly in } L^2((0,T)\times\tilde\Omega), && (9.38)\\
\mathbb{G}_\varepsilon &\rightharpoonup \mathbb{G} \text{ weakly in } (L^2(Q_T))^{2\times 2}, && (9.39)
\end{aligned}$$

where $\mathbb{G} := \boldsymbol{v}\otimes\boldsymbol{v} - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}$. Taking the limit $\varepsilon \to 0+$ in (9.22) and (9.23), we observe that $\boldsymbol{w}_2$, p_2 satisfy for all $\boldsymbol{\Phi} \in (W^{1,2}_0(\tilde\Omega))^2$, $\phi \in L^2(\tilde\Omega)$

$$\begin{aligned}
\int_{\tilde\Omega} \nabla\boldsymbol{w}_2 : \nabla\boldsymbol{\Phi} - \int_{\tilde\Omega} p_2 \operatorname{div}\boldsymbol{\Phi} &= -\int_{\tilde\Omega} \mathbb{G} : \nabla\boldsymbol{\Phi}, && (9.40)\\
\int_{\tilde\Omega} \operatorname{div}\boldsymbol{w}_2\ \phi &= 0. && (9.41)
\end{aligned}$$

Lemma 9.1 guarantees the uniqueness of the solution to problems (9.35)–(9.36) and (9.40)–(9.41) and the estimates

$$\begin{aligned}
\|\boldsymbol{w}_1\|_{(W^{m+2,2}(\tilde\Omega))^2} + \|p_1\|_{W^{m+1,2}(\tilde\Omega)} &\le \|\boldsymbol{v}\|_{(W^{m,2}(\Omega))^2}, \quad m \in \{-1,0,1\},\\
\|\boldsymbol{w}_2\|_{(W^{1,2}(\tilde\Omega))^2} + \|p_2\|_{L^2(\tilde\Omega)} &\le \|\operatorname{div}\mathbb{G}\|_{((W^{1,2}_0(\Omega))^2)^*}.
\end{aligned}$$

Now proceeding in the same way as on the approximate level, we conclude

$$\partial_t(\boldsymbol{v} + \nabla p_1) \in L^2(0,T;((W^{1,2}_0(\tilde\Omega))^2)^*),$$

which is (9.7), and for all $\boldsymbol{\Phi} \in (W^{1,2}_0(\tilde\Omega))^2$ and a.a. $t \in (0,T)$

$$\langle \partial_t(\boldsymbol{v} + p_1), \boldsymbol{\Phi}\rangle = \int_{\tilde\Omega} \mathbb{G} : \nabla\boldsymbol{\Phi} + \int_{\tilde\Omega} p_2 \operatorname{div}\boldsymbol{\Phi},$$

which is the relation (9.8) that we wanted to prove.

Concerning the convergence results – the weak convergence (9.10) follows immediately from (9.38). Let us show the strong convergence (9.9). By subtracting (9.35) from (9.18) and (9.36) from (9.19) it is obvious that $\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1$, $p_{1,\varepsilon} - p_1$ solve the Stokes problem with the right handside $\boldsymbol{v}_\varepsilon - \boldsymbol{v}$. By Lemma 9.1 the solution is unique and satisfies the estimate

$$\|\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1\|_{(W^{m+2,2}(\tilde\Omega))^2} + \|p_{1,\varepsilon} - p_1\|_{W^{m+1,2}(\tilde\Omega)} \le \|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{(W^{m,2}(\Omega))^2} \quad m \in \{-1,0,1\}. \tag{9.42}$$

From (9.42) with $m = 0$ it follows

$$\lim_{\varepsilon\to 0+} \int_0^T \|p_{1,\varepsilon} - p_1\|^2_{W^{1,2}(\tilde\Omega)} \le \lim_{\varepsilon\to 0+} \int_0^T \|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|^2_{(L^2(\Omega))^2} = 0, \tag{9.43}$$

where the second equality is achieved by the strong convergence (6.24). The relation (9.43) yields

$$p_{1,\varepsilon} \to p_1 \quad \text{strongly in } L^2(0,T;W^{1,2}(\tilde\Omega)). \tag{9.44}$$

We need to show that also

$$\nabla^2 p_{1,\varepsilon} \to \nabla^2 p_1 \quad \text{strongly in } L^2(0,T;(L^2_{loc}(\tilde\Omega))^{2\times 2}). \tag{9.45}$$

Let us denote

$$\tilde{p}_\varepsilon := p_{1,\varepsilon} - p_1.$$

To prove (9.45) we will use the property

$$\Delta\tilde{p}_\varepsilon = 0 \quad \text{a.e. in } (0,T) \times \tilde{\Omega}, \tag{9.46}$$

which follows from the equality

$$-\Delta(\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1) + \nabla\tilde{p}_\varepsilon = \boldsymbol{v}_\varepsilon - \boldsymbol{v} \quad \text{a.e. in } (0,T) \times \tilde{\Omega}$$

($\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1$, $\tilde{p}_\varepsilon$ is the solution to the Stokes problem with the right handside $\boldsymbol{v}_\varepsilon - \boldsymbol{v}$ and $\Delta(\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1)$, $\nabla\tilde{p}_\varepsilon$ are integrable over Q_T) and from the fact $\operatorname{div}\boldsymbol{v}_\varepsilon = \operatorname{div}\boldsymbol{v} = 0 = \operatorname{div}\boldsymbol{w}_{1,\varepsilon} = \operatorname{div}\boldsymbol{w}_1$ a.e. in $(0,T) \times \tilde{\Omega}$. The equation (9.46) implies

$$\int_{\tilde{\Omega}} \nabla\tilde{p}_\varepsilon \cdot \nabla\varphi = 0 \quad \forall\, \varphi \in W_0^{1,2}(\tilde{\Omega}). \tag{9.47}$$

Take $\varphi := \phi\xi^2$, where $\phi \in W^{1,2}(\tilde{\Omega})$, $\xi \in C_c^\infty(\tilde{\Omega})$. We can rewrite (9.47) into the form

$$\int_{\tilde{\Omega}} \nabla(\tilde{p}_\varepsilon\xi) \cdot \nabla(\phi\xi) = -\int_{\tilde{\Omega}} \operatorname{div}(\tilde{p}_\varepsilon\nabla\xi)\ \phi\xi - \int_{\tilde{\Omega}} (\nabla\tilde{p}_\varepsilon \cdot \nabla\xi)\ \phi\xi. \tag{9.48}$$

Denote $\psi := \phi\xi$, $g_\varepsilon := -\operatorname{div}(\tilde{p}_\varepsilon\nabla\xi) - (\nabla\tilde{p}_\varepsilon \cdot \nabla\xi)$. Since every $\psi \in W_0^{1,2}(\tilde{\Omega})$ can be written in the form $\phi\xi$, where the functions ϕ, ξ have the properties described above, (9.48) gives

$$\int_{\tilde{\Omega}} \nabla(\tilde{p}_\varepsilon\xi) \cdot \nabla\psi = \int_{\tilde{\Omega}} g_\varepsilon\psi \quad \forall\psi \in W_0^{1,2}(\tilde{\Omega}). \tag{9.49}$$

Employing the local regularity of weak solutions to elliptic problems, we have

$$\|\tilde{p}_\varepsilon\xi\|_{W^{2,2}(\tilde{\Omega})} \le \|g_\varepsilon\|_{L^2(\tilde{\Omega})}. \tag{9.50}$$

Since $\xi \in C_c^\infty(\tilde{\Omega})$ is arbitrary, the inequality (9.50) together with the definitions of $\tilde{p}_\varepsilon$ and g_ε implies for every $\tilde{\tilde{\Omega}} \subset \overline{\tilde{\tilde{\Omega}}} \subset \tilde{\Omega}$

$$\int_0^T \|\nabla^2 p_{1,\varepsilon} - \nabla^2 p_1\|_{(L^2(\tilde{\tilde{\Omega}}))^{2\times 2}} \le C \int_0^T \|p_{1,\varepsilon} - p_1\|_{W^{1,2}(\tilde{\Omega})}, \tag{9.51}$$

and since the right handside of (9.51) converges to zero by (9.44), we obtain (9.45), which together with (9.44) yields the strong convergence (9.9).

To finish the proof of the lemma, it remains to show the continuity of $p_{1,\varepsilon}$, p_1 with respect to time and the convergence of the initial conditions $p_{1,\varepsilon}(0)$, $p_1(0)$. Let t_1 and t_2 from $[0,T]$ be arbitrary. The functions $\boldsymbol{w}_{1,\varepsilon}(t_1) - \boldsymbol{w}_{1,\varepsilon}(t_2)$, $p_{1,\varepsilon}(t_1) - p_{1,\varepsilon}(t_2)$ solve the Stokes problem with the right handside $\boldsymbol{v}_\varepsilon(t_1) - \boldsymbol{v}_\varepsilon(t_2)$. By Lemma 9.1 this solution is unique and it satisfies the estimate

$$\|\boldsymbol{w}_{1,\varepsilon}(t_1) - \boldsymbol{w}_{1,\varepsilon}(t_2)\|_{(W^{2,2}(\tilde{\Omega}))^2} + \|p_{1,\varepsilon}(t_1) - p_{1,\varepsilon}(t_2)\|_{W^{1,2}(\tilde{\Omega})} \le \|\boldsymbol{v}_\varepsilon(t_1) - \boldsymbol{v}_\varepsilon(t_2)\|_{(L^2(\Omega))^2}.$$

Since the right handside converges to zero as $t_2 \to t_1$ if $t_1 \in (0,T)$, as $t_2 \to t_1+$ if $t_1 = 0$, as $t_2 \to t_1-$ if $t_1 = T$ (since $\boldsymbol{v}_\varepsilon \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$), we conclude that $\nabla p_{1,\varepsilon}$ belongs to $C([0,T]; (L^2(\tilde{\Omega}))^2)$. The fact $\nabla p_1 \in C([0,T]; (L^2(\tilde{\Omega}))^2)$ is proved in the same way.

Finally, from the relation (9.42) we know that

$$\|\nabla p_{1,\varepsilon}(t) - \nabla p_1(t)\|_{(L^2(\tilde{\Omega}))^2} \leq \|\boldsymbol{v}_\varepsilon(t) - \boldsymbol{v}(t)\|_{(L^2(\Omega))^2} \tag{9.52}$$

for all $t \in [0,T]$ (recall that $\boldsymbol{v}_\varepsilon, \boldsymbol{v}$ belong to $C([0,T];(L^2(\Omega))^2)$ and $\nabla p_{1,\varepsilon}, \nabla p_1$ belong to $C([0,T];(L^2(\tilde{\Omega}))^2)$). And since $\boldsymbol{v}_\varepsilon(0)=\boldsymbol{v}(0)=\boldsymbol{v}_0$ a.e. in Ω, we conclude $\nabla p_{1,\varepsilon}(0)=\nabla p_1(0)$ a.e. in $\tilde{\Omega}$, which is the relation (9.11) completing the proof of the proposition. □

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