Nečas Center for Mathematical Modeling

Regularity of planar flows for shear-thickening fluids under perfect slip boundary conditions

J. Tichý

Preprint no. 2013-007

http://ncmm.karlin.mff.cuni.cz/

Regularity of planar flows for shear-thickening fluids under perfect slip boundary conditions

Jakub Tichý*

Abstract: For evolutionary planar flows of shear-thickening fluids in a bounded domain we prove the existence of a solution with the Hölder continuous velocity gradients and pressure. The problem is equipped with perfect slip boundary conditions. We also show L^q theory result for Stokes system under perfect slip boundary conditions.

Keywords: Generalized Newtonian fluid; Regularity; Perfect Slip Boundary Conditions.
Mathematics Subject Classification (2000): 35B65, 35K51, 35Q35, 76D03.

1 Introduction

We investigate systems describing motions of incompressible shear-thickening fluids, which in evolutionary case are governed by following initial value problem:

$$\begin{gathered} \partial_t u - \operatorname{div} \mathcal{S}(Du) + (u \cdot \nabla) u + \nabla \pi = f, \quad \operatorname{div} u = 0 \text{ in } Q, \\ u(0, \cdot) = u_0 \text{ in } \Omega, \end{gathered} \tag{1.1}$$

where u is the velocity, π represents the pressure, f stands for the density of volume forces and $\mathcal{S}(Du)$ denotes the extra stress tensor. Du is the symmetric part of the velocity gradient, i.e. $Du = \frac{1}{2}[\nabla u + (\nabla u)^\top]$, $\Omega \subset \mathbb{R}^2$ is a bounded domain, $I = (0, T)$ denotes a finite time interval and $Q = I \times \Omega$. We are interested in the case, when (1.1) is equipped with the perfect slip boundary conditions

$$u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau = 0 \quad \text{on } I \times \partial\Omega, \tag{1.2}$$

where τ is the tangent vector and ν is the outward normal to $\partial\Omega$. The constitutive relation for $\mathcal{S}(Du)$ is given via the generalized viscosity μ and is of the form

$$\mathcal{S}(Du) := \mu(|Du|)Du.$$

The extra stress tensor $\mathcal{S}(Du)$ is assumed to possess $p-$potential structure with $p \geq 2$. More precisely, we can construct scalar potential $\Phi : [0, \infty) \mapsto [0, \infty)$ to the stress tensor $\mathcal{S}$, i.e.

$$\mathcal{S}(A) = \partial_A \Phi(|A|) = \Phi'(|A|) \frac{A}{|A|} \quad \forall A \in \mathbb{R}^{2\times 2}_{sym},$$

such that $\Phi \in \mathcal{C}^{1,1}((0, \infty)) \cap \mathcal{C}^1([0, \infty))$, $\Phi(0) = 0$ and there exist $p \in [2, \infty)$ and $0 < C_1 \leq C_2$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$C_1(1 + |A|^2)^{\frac{p-2}{2}} |B|^2 \leq \partial_A^2 \Phi(|A|) : B \otimes B \leq C_2(1 + |A|^2)^{\frac{p-2}{2}} |B|^2. \tag{1.3}$$

This paper closely follows [10], where P. Kaplický shows Hölder continuity of velocity gradients and pressure for (1.1) with $p \in [2, 4)$ under no slip boundary conditions. Based on the same structure of the proof and using the results from [14] we extend the result for perfect slip boundary conditions in the case $p \in [2, \infty)$.

The idea of the proof goes back to [17], where the authors show that every weak solution u of $\partial_t u - \operatorname{div}(\mathcal{S}(\nabla u)) = 0$ in Q has locally Hölder continuous gradient in case that $\Omega \subset \mathbb{R}^2$ and $p = 2$. This results was extended in [9] to the case $p \in (1, 2)$. Regularity of $\partial_t u$ is shown first and after moving $\partial_t u$ to the right hand side the stationary L^q theory is applied.

In the case of generalized Newtonian fluids this method was modified in [13], where the authors consider the shear-thinning fluid model with periodic boundary conditions. In contrary to [17] the regularity of $\partial_t u$ and ∇u had to be obtained at once. The authors showed that velocity gradients are Hölder continuous for $p \in (4/3, 2]$. These results were extended to electro-rheological fluids and non-zero initial condition in [7].

*Charles University in Prague, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic and Czech Technical University in Prague, Faculty of Information Technology, Thákurova 9, 160 00, Praha 6, Czech Republic, e-mail: kuba.tichy@gmail.com

Among many works concerning regularity theory for generalized Newtonian fluids we would like to mention two papers dealing with the stationary case. In [12] the stationary version of (1.1) under homogeneous Dirichlet boundary conditions are considered. The same authors later in [11] studied the problem equipped with non-homogeneous Dirichlet boundary conditions with two types of restriction on boundary data and perfect slip boundary conditions.

Let E be a Banach space and $\alpha \in (0,1)$, $q \in (1,\infty)$, $s \in \mathbb{R}$. In this paper we use standard notation[1] for Lebesgue spaces $L^q(\Omega)$, Sobolev-Slobodeckiĭ spaces $W^{s,q}(\Omega)$, Bochner spaces $L^q(I,E)$ and $W^{\alpha,q}(I,E)$. By $H^s_q(\Omega)$ we mean Bessel potential spaces and $B^s_{p,q}(\Omega)$ are Besov spaces. We set $B^s_q(\Omega) := B^s_{q,q}(\Omega)$ and $B^s_{q,\sigma}(\Omega) = \{u \in B^s_q(\Omega);\, \operatorname{div} u = 0\}$. By BUC we mean bounded and uniformly continuous functions.

Let $\mathcal{C}^\infty_{0,\sigma}(\Omega) = \{u \in \mathcal{C}^\infty_0(\Omega),\, \operatorname{div} u = 0 \,\text{in}\, \Omega\}$ and $L^q_\sigma(\Omega)$ resp. $W^{1,q}_\sigma(\Omega)$ denote the closure of $\mathcal{C}^\infty_{0,\sigma}(\Omega)$ in L^q norm resp. $W^{1,q}$ norm. Since the domain Ω is in our case at least $\mathcal{C}^{2,1}$, $L^q_\sigma(\Omega)$, resp. $W^{1,q}_\sigma(\Omega)$ is characterized as

$$L^q_\sigma(\Omega) = \{\varphi \in L^q(\Omega),\, \operatorname{div}\varphi = 0 \,\text{in}\, \Omega,\, \varphi\cdot\nu = 0 \,\text{on}\, \partial\Omega\},\ \text{resp.}$$
$$W^{1,q}_\sigma(\Omega) = \{\varphi \in W^{1,q}(\Omega),\, \operatorname{div}\varphi = 0,\, \text{in}\, \Omega,\ \varphi\cdot\nu = 0 \,\text{on}\, \partial\Omega\}.$$

The duality between Banach space E and its dual E' is denoted by $\langle\cdot,\cdot\rangle$. Set $W^{-1,p'}_\sigma(\Omega) := (W^{1,p}_\sigma(\Omega))'$.

We begin with the definition of the weak solution to the problem (1.1) with (1.2).

Definition 1.1. *Let $f \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $p \in [2,\infty)$ and $u_0 \in L^2(\Omega)$. We say that the function $u : Q \mapsto \mathbb{R}^2$ is a weak solution to the problem (1.1) with (1.2), if $u \in L^\infty(I, L^2(\Omega)) \cap L^p(I, W^{1,p}_\sigma(\Omega))$, $\partial_t u \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $u(0,\cdot) = u_0$ in $L^2(\Omega)$ and weak formulation*

$$\int_I \langle \partial_t u, \varphi\rangle \,\mathrm{d}t + \int_Q \mathcal{S}(Du) : D\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q (u\cdot\nabla)u\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I \langle f, \varphi\rangle \,\mathrm{d}t$$

holds for all $\varphi \in L^p(I, W^{1,p}_\sigma(\Omega))$.

If we considered also the case $p \in (1,2)$, we would have to consider only test functions from the space of smooth functions. It is well known that the weak solution exists and is unique. It could be easily proven using the monotone operator theory. See for example [15, Chapter 5] for periodic boundary conditions.

Now we formulate the main results of this paper.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded $\mathcal{C}^3$ domain and (1.3) holds for some $p \in [2,\infty)$. Let $u_0 \in W^{2+\beta,2}(\Omega)$ for $\beta \in (0,1/4)$, $f \in L^\infty(I, L^{q_0}(\Omega))$ and $\partial_t f \in L^{q_0}(I, W^{-1,q_0'}_\sigma(\Omega))$ for some $q_0 > 2$. Then there exists the unique solution (u,π) of (1.1) with (1.2), such that for some $\alpha > 0$*

$$\nabla u, \pi \in \mathcal{C}^{0,\alpha}(Q).$$

In Section 3 we gather L^q theory results for the classical Stokes system. Further we extend these results to generalized Stokes system where the Laplace operator is replaced by a general elliptic operator in divergence form with bounded measurable coefficients.

Section 4 is devoted to the proof of the main theorem in the case of quadratic growth, i.e. $p = 2$. In Section 5 we introduce the quadratic approximation of the stress tensor $\mathcal{S}(Du)$ which is done by the truncation of the generalized viscosity from above, i.e. $\mu^\varepsilon(|Du^\varepsilon|) := \min\{\mu(|Du|), 1/\varepsilon\}$ for $\varepsilon \in (0,1)$. We prove the main result for the approximated problem and pass from the approximated problem to the original one at the end.

2 Preliminary general material

2.1 Function spaces

Let E, F, be reflexive Banach spaces. Although it is not necessary to have reflexive spaces in all definitions, for convenience we assume it. By $\mathcal{L}(E,F)$ we mean the Banach space of all bounded linear operators from E to F and $\mathcal{L}(E) := \mathcal{L}(E,E)$. If E is a linear subspace of F and the natural injection $i : x \mapsto x$ belongs to $\mathcal{L}(E,F)$, we write $E \hookrightarrow F$. In the case E is also dense in F, it will be denoted by $E \overset{d}{\hookrightarrow} F$. Furthermore, $\mathcal{L}is(E,F)$ consists of all topological linear isomorphisms from E onto F. We also write $E \doteq F$ if $E \hookrightarrow F$ and $F \hookrightarrow E$, i.e. E equals F with equivalent norms.

A Banach space E is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R}, E)$ for some (and then for all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, E)$, the Schwartz space of rapidly decreasing $E-$valued functions, is defined by $Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f$. These spaces are often also called UMD Banach spaces, where the UMD stands for the property of unconditional martingale differences. It is well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. Note that all closed subspaces of $L^q(\Omega)$ are UMD spaces.

[1] We won't use different notation for scalar, vector-valued or tensor valued functions.

2.2 Semigroups and interpolation-extrapolation scales

For a linear operator A on E_0 we denote the domain of A by $\mathcal{D}(A)$. $A \in \mathcal{H}(E_1, E_0)$ means that A is the negative infinitesimal generator of a bounded analytic semigroup on E_0 and $E_1 \doteq \mathcal{D}(A)$. It holds

$$\mathcal{H}(E_1, E_0) = \bigcup_{\kappa \geq 1,\, \omega > 0} \mathcal{H}(E_1, E_0, \kappa, \omega),$$

where $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ if $\omega + A \in \mathcal{L}is(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(A-\lambda)^{-1}x\|_{E_0}}{|\lambda|\|x\|_{E_0} + \|x\|_{E_1}} \leq \kappa, \quad Re(\lambda) \leq \omega, \quad x \in E_1.$$

By $\sigma(A)$ we mean the spectrum of A and $\varrho(A)$ denotes the resolvent set. A linear operator A in E is said to be of positive type if it belongs to $\mathcal{P}(E) := \bigcup_{K \geq 1} P_K(E)$. $A \in P_K(E)$ if it is closed, densely defined, $\mathbb{R}^+ \subset \varrho(-A)$ and $(1+s)\|(s+A)^{-1}\|_{\mathcal{L}(E)} \leq K$ for $s \in \mathbb{R}^+$, where $K \geq 1$.

We say that a linear operator A in E is of type (E, K, ϑ), denoted by $A \in \mathcal{P}(E, K, \vartheta)$, if it is densely defined and if

$$\Sigma_\vartheta := \{|\arg z| \leq \vartheta\} \cup \{0\} \subset \varrho(-A) \quad \text{and} \quad 1 + |\lambda|\|(\lambda + A)^{-1}\|_{\mathcal{L}(E)} \leq K, \quad \lambda \in \Sigma_\vartheta.$$

Put $\mathcal{P}(E, \vartheta) := \bigcup_{K \geq 1} \mathcal{P}(E, K, \vartheta)$.

A linear operator A on E is said to have bounded imaginary powers, in symbols,

$$A \in \mathcal{BIP}(E) := \bigcup_{K \geq 1,\, \theta \geq 0} \mathcal{BIP}(E, K, \theta),$$

provided $A \in \mathcal{P}(E)$ and there exist $\theta \geq 0$ and $K \geq 1$ such that $A^{is} \in \mathcal{L}(E)$ and $\|A^{is}\|_{\mathcal{L}(E)} \leq K e^{\theta|s|}$ for $s \in \mathbb{R}$.

We introduce an interpolation-extrapolation scale which is essential in the proof of Theorem 3.5. Let $p, q \in (1, \infty)$, $\theta \in (0,1)$ and $[\cdot,\cdot]_\theta$ denotes the complex and $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor. Let $A \in \mathcal{H}(E_1, E_0)$. Then we denote by $[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$ the interpolation-extrapolation scale generated by (E, A) and $[\cdot,\cdot]_\theta$ or $(\cdot,\cdot)_{\theta,q}$, where we set $E_k := \mathcal{D}(A^k)$ for $k \in \mathbb{N}$ with $k \geq 2$. Also set $E^\sharp := E'$ and $A^\sharp := A'$, where A' is the dual of A in E in the sense of unbounded linear operators. Finally let $E^\sharp_k := \mathcal{D}((A^\sharp)^k)$ for $k \in \mathbb{N}$. Then we define E_{-k} for $k \in \mathbb{N}$ by $E_{-k} := (E^\sharp_k)'$. We put $E_{k+\theta} := [E_k, E_{k+1}]_\theta$ (and similarly for the real interpolation functor). If $\alpha \geq 0$ we denote by A_α the maximal restriction of A to E_α whose domain equals $\{u \in E_\alpha \cap E_1;\, Au \in E_\alpha\}$. If $\alpha < 0$ then A_α is the closure of A in E_α.

For the dual interpolation functor $(\cdot,\cdot)^\sharp_\theta$ (which is equal to $[\cdot,\cdot]_\theta$ for the complex interpolation and $(\cdot,\cdot)_{\theta,q'}$ for real interpolation) we abbreviate the interpolation-extrapolation scale generated by $(E^\sharp, A^\sharp)$ and $(\cdot,\cdot)^\sharp_\theta$, by $[(E^\sharp_\alpha, A^\sharp_\alpha); \alpha \in \mathbb{R}]$ and call it interpolation-extrapolation scale dual to $[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$. It holds $(E_{-\alpha})' \doteq E^\sharp_\alpha$ and $(A_{-\alpha})' = A^\sharp_\alpha$. For more details see [2, Section V.2].

2.3 Description of the boundary

In order to discuss boundary regularity, we will need a suitable description of the boundary $\partial\Omega$. Let us denote $x = (x_1, x_2)$. We suppose that $\Omega \in \mathcal{C}^3$, therefore there exists $c_0 > 0$ such that for all $a_0 > 0$ there exists n_0 points $P \in \partial\Omega$, $r > 0$ and open smooth set $\Omega_0 \subset\subset \Omega$ that we have

$$\Omega \subset \Omega_0 \cup \bigcup_P B_r(P)$$

and for each point $P \in \partial\Omega$ there exists local system of coordinates for which $P = 0$ and the boundary $\partial\Omega$ is locally described by $\mathcal{C}^3$ mapping a_P that for $x_1 \in (-3r, 3r)$ fulfils

$$x \in \partial\Omega \Leftrightarrow x_2 = a_P(x_1), \quad B_{3r}(P) \cap \Omega = \{x \in B_r(P) \,\text{and}\, x_2 > a_P(x_1)\} =: \Omega^P_{3r},$$
$$\partial_1 a_P(0) = 0, \quad |\partial_1 a_P(x_1)| \leq a_0, \quad |\partial_1^2 a_P(x_1)| + |\partial_1^3 a_P(x_1)| \leq c_0.$$

Points P can be divided into k groups such that in each group Ω^P_{3r} are disjoint and k depends only on dimension n. Let the cut-off function $\xi_P(x) \in \mathcal{C}^\infty(B_{3r}(P))$ and reaches values

$$\xi_P(x) \begin{cases} = 1 & x \in B_r(P), \\ \in (0,1) & x \in B_{2r}(P) \setminus B_r(P), \\ = 0 \in \mathbb{R}^2 \setminus B_{2r}(P). \end{cases}$$

Next, we assume that we work in the coordinate system corresponding to P. Particularly, $P = 0$. Let us fix P and drop for simplicity the index P. The tangent vector and the outer normal vector to $\partial\Omega$ are defined as

$$\tau = \big(1, \partial_1 a(x_1)\big), \quad \nu = \big(\partial_1 a(x_1), -1\big),$$

tangent and normal derivatives as

$$\partial_\tau = \partial_1 + \partial_1 a(x_1)\partial_2, \quad \partial_\nu = -\partial_2 + \partial_1 a(x_1)\partial_1.$$

3 L^q theory for Stokes system

In this section we collect facts about L^q theory for the Stokes system

$$\begin{aligned} \partial_t u - \Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in}\, Q, \\ u(0,\cdot) = u_0 \quad \text{on}\, \Omega, \end{aligned} \tag{3.1}$$

equipped with the perfect slip boundary conditions

$$u \cdot \nu = 0, \quad [(Du)\nu] \cdot \tau = 0 \quad \text{on}\, I \times \partial\Omega. \tag{3.2}$$

Let P denote the projection operator from $L^q(\Omega)$ to $L^q_\sigma(\Omega) := \{\varphi \in L^q(\Omega),\, \operatorname{div}\varphi = 0\,\text{in}\,\Omega,\, \varphi\cdot\nu = 0\,\text{on}\,\partial\Omega\}$ associated with the Helmholtz decomposition. Denoting $Bu := [(Du)\nu]_\tau$ in the sense of traces and using the projection P we shall define the Stokes operator A by $Au = -P\Delta u$ for $u \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), \quad H^2_{q,B}(\Omega) := \{u \in H^2_q(\Omega),\ Bu = 0,\ \text{on}\ \partial\Omega\}.$$

Applying the Helmholz projection P to (3.1) with (3.2), we eliminate the pressure from equations and with the help of the newly established notation the Stokes system reduces to

$$\begin{aligned} \partial_t u + Au = Pf, \quad \operatorname{div} u = 0 \quad \text{in}\, Q, \\ u(0,\cdot) = u_0 \quad \text{on}\, \Omega, \quad Bu = 0 \quad \text{on}\, I \times \partial\Omega. \end{aligned} \tag{3.3}$$

At first we mention some basic properties of the Stokes operator A. From [19] we know that $A \in \mathcal{H}(L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), L^q_\sigma(\Omega))$. This also tells us that $A \in \mathcal{P}(\omega)$ for $\omega \in [0, \pi/2)$ (see [8, Theorem II.4.6]). R. Shimada later showed in [18] the L^q-maximal regularity for A. From [1, Theorem 1] we know

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $\mathcal{C}^{1,1}$ domain and $\vartheta \in (0,\pi)$. Then for every $\omega \in \mathbb{R}$ and $\vartheta' \in (0,\vartheta)$ such that $\omega + \Sigma_{\vartheta'} \subset \varrho(-A)$ the shifted Stokes operator $\omega + A$ admits a bounded $H^\infty-$calculus.*

For the definition and properties of a bounded $H^\infty-$calculus we refer for example to [6, Section 2, Subsection 2.4]. For us it is important to know that the class of operators with a bounded $H^\infty-$calculus is a subclass of the operators which have $\mathcal{BIP}$, therefore these operators admit all important properties which has operators with bounded imaginary powers. From the work of R. Shimada and Proposition 3.1 follows that $A \in \mathcal{BIP}$.

For the Stokes operator A we have realizations A_α on E_α for some α (see Subsection 2.2 for details). Concretely, from [20, Section 2.2] we know that $A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha)$ for $\alpha \geq -1$. Set $s_\alpha := \{-2+1/q, -1+1/q, 1/q, 1+1/q\}$. Steiger in [20, Corollary 2.6] shows that $E_\alpha \doteq H^{2\alpha}_{q,B,\sigma}(\Omega)$ for $2\alpha \in [-2,2] \setminus s_\alpha$ and complex interpolation functor, where

$$H^s_{q,B,\sigma}(\Omega) := \begin{cases} \{u \in H^s_q(\Omega),\, \operatorname{div} u = 0,\, Bu = 0\,\text{on}\,\partial\Omega\}, & s \in (1+1/q, 2], \\ \{u \in H^s_q(\Omega),\, \operatorname{div} u = 0,\, u\cdot\nu = 0\,\text{on}\,\partial\Omega\}, & s \in (1/q, 1+1/q), \\ \{u \in H^s_q(\Omega),\, \operatorname{div} u = 0\}, & s \in [0, 1/q), \\ \left(H^{-s}_{q',B,\sigma}(\Omega)\right)', & s \in [-2,0) \setminus \{-2+1/q, -1+1/q\}, \end{cases} \tag{3.4}$$

This gives us

$$A_\alpha \in \mathcal{H}(H^{2\alpha+2}_{q,B,\sigma}(\Omega), H^{2\alpha}_{q,B,\sigma}(\Omega)), \quad 2\alpha \in [-2,2] \setminus s_\alpha.$$

We will use the fact, that the property of bounded imaginary powers can be carried over the interpolation-extrapolation scales:

Proposition 3.2. *[2, Proposition V.1.5.5] Let $A \in \mathcal{P}(E)$ and let $[(E_\alpha, A_\alpha); \alpha \in (-n,\infty)]$ be the interpolation-extrapolation scale generated by (E, A) and an exact functor. If $A \in \mathcal{BIP}(E, M, \sigma)$ then $A_\alpha \in \mathcal{BIP}(E_\alpha, M, \sigma)$.*

Let us define the maximal L^q-regularity for the operator A (compare [2, Section III.1, Subsection 1.5 and Section III.4, Remark 4.10.9.c])

Definition 3.3. *Let $A \in \mathcal{H}(E_1, E_0)$ and $q \in (1,\infty)$. We say that $\left(L^q(I,E_0), L^q(I,E_1) \cap W^{1,q}(I,E_0)\right)$ is a pair of maximal regularity for A (or A has maximal regularity), if for $u_0 \in E_{1-1/q,q} := (E_0, E_1)_{1-1/q,q}$ and $f \in L^q(I,E_0)$ there exists the unique solution $u \in L^q(I,E_1) \cap W^{1,q}(I,E_0)$ of (3.3), and*

$$\|\partial_t u\|_{L^q(I,E_0)} + \|u\|_{L^q(I,E_0)} + \|Au\|_{L^q(I,E_0)} \leq C\Big(\|f\|_{L^q(I,E_0)} + \|u_0\|_{E_{1-1/q,q}}\Big).$$

Further we mention the relation between maximal regularity and negative infinitesimal generators of a bounded analytic semigroup.

Proposition 3.4. *[2, Theorem III.4.10.8] Suppose that E_0 is a UMD space and $\kappa, N \geq 1$, $\omega > 0$, $\vartheta \in [0, \pi/2)$ and $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ satisfies $\omega + A \in \mathcal{BIP}(E_0, N, \vartheta)$. Then A has maximal regularity.*

The main result of this section is the following

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^{1,1}$ domain, $q \in (1,\infty)$, $f \in L^q(I, W_\sigma^{-1,q'}(\Omega))$, $u_0 \in B_{q,\sigma}^{1-2/q}(\Omega)$ then there exists a constant $C > 0$ and the unique weak solution of* (3.3) *satisfying*

$$\|\nabla u\|_{L^q(Q)} + \|u\|_{BUC(I, B_{q,\sigma}^{1-2/q}(\Omega))} \leq C\Big(\|f\|_{L^q(I, W_\sigma^{-1,q'}(\Omega))} + \|u_0\|_{B_{q,\sigma}^{1-2/q}(\Omega)}\Big).$$

The constant C is independent of T, u, f and u_0.

Proof. We consider the system (3.3) instead of (3.1) with (3.2). Since for UMD space E, E' is one as well and for an interpolation couple of UMD spaces the interpolation spaces are also UMD (see [2, Theorem III.4.5.2]), $E_{-1/2}$ is a UMD space. Proposition 3.2 gives us $A_{-1/2}$ has $\mathcal{BIP}$. Therefore we can use Proposition 3.4 for $\alpha = -1/2$ and obtain

$$\|\partial_t u\|_{L^q(I,E_{-1/2})} + \|u\|_{L^q(I,E_{-1/2})} + \|A_{-1/2}u\|_{L^q(I,E_{-1/2})} \leq C\Big(\|f\|_{L^q(I,E_{-1/2})} + \|u_0\|_{(E_{-1/2},E_{1/2})_{1-1/q,q}}\Big). \quad (3.5)$$

In order to determine the correct spaces in (3.5) we use interpolation-extrapolation scales defined in [20, Section 2.2].

$$u_0 \in (E_{-1/2}, E_{1/2})_{1-1/q,q} = (H_{q,B,\sigma}^{-1}(\Omega), H_{q,B,\sigma}^{1}(\Omega))_{1-1/q,q} = B_{q,\sigma}^{1-2/q}(\Omega),$$

where we used [20, Corollary 2.6], (3.4), the theorem about the interpolation of Bessel potential spaces on domains ([21, Subsection 4.3.1, Theorem 1]) together with the theorem about interpolation of closed subspaces (in our case of solenoidal functions see [3, Lemma 3.2]). From the embedding [2, Theorem V.4.10.2]

$$L^q(I, E_1) \cap W^{1,q}(I, E_0) \hookrightarrow BUC(I, (E_0, E_1)_{1-1/q,q}),$$

we obtain $u \in BUC(I, B_{q,\sigma}^{1-2/q}(\Omega))$. Due to $\|A_{-1/2}u\|_{E_{-1/2}} = \|u\|_{E_{1/2}}$ and $E_{1/2} = W_\sigma^{1,q}(\Omega)$ we obtain boundedness of ∇u in $L^q(Q)$. It remains to find the space for f. From (3.4) we can see that

$$f \in L^q(I, W_\sigma^{-1,q'}(\Omega)),$$

since $H_q^s(\Omega) \doteq W^{s,q}(\Omega)$ for $s \in \mathbb{Z}$.

□

Without loss of generality we may assume that there exists a symmetric tensor $G \in L^q(Q)$, such that the weak formulation of the right hand side of (3.1) can be written in the form

$$\int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I \langle f, \varphi\rangle \,\mathrm{d}t \quad \forall \varphi \in L^q(I, W_\sigma^{1,q}(\Omega)). \quad (3.6)$$

To prove it, we proceed in the same way like in [13, Proof of Proposition 2.1, Step 1] where the authors are dealing with periodic boundary conditions. Consider the Stokes system which can be formulated in the weak form for a. a. t as follows

$$\int_\Omega Dw(t) : D\varphi \,\mathrm{d}x = \langle f(t), \varphi\rangle \quad \forall \varphi \in W_\sigma^{1,q}(\Omega). \quad (3.7)$$

As $f \in L^q(I, W_\sigma^{-1,q}(\Omega))$, there exists a solution $w(t) \in W_\sigma^{1,q}(\Omega)$ of (3.7) enjoying the estimate

$$\|w(t)\|_{W^{1,q}(\Omega)} \leq C\|f\|_{W_\sigma^{-1,q}(\Omega)}$$

with the constant C independent of t. Consequently, $w \in L^q(I, W_\sigma^{1,q}(\Omega))$ and

$$\|w\|_{L^q(I,W^{1,q}(\Omega))} \leq C\|f\|_{L^q(I,W_\sigma^{-1,q}(\Omega))}.$$

Defining $G = Dw$ we conclude (3.6) from (3.7) by density arguments. Therefore for all $f \in L^q(I, W_\sigma^{-1,q}(\Omega))$ there exists $G \in L^q(Q)$ such that (3.6) and following estimate

$$\|G\|_{L^q(Q)} \leq C\|f\|_{L^q(I,W_\sigma^{-1,q}(\Omega))}$$

holds. We would like to point out that the perfect slip boundary conditions are hidden in the weak formulation. If G is smooth enough then it holds

$$\int_I \langle f, \varphi \rangle \,\mathrm{d}t = -\int_Q \operatorname{div} G \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\partial\Omega} (G\nu)\tau(\varphi\cdot\tau)\,\mathrm{d}\sigma\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W^{1,q}_\sigma(\Omega)).$$

The Stokes system (3.1) with (3.2) can be formulated in the weak form as follows

$$\int_I \langle \partial_t u, \varphi \rangle \,\mathrm{d}t + \int_Q Du : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W^{1,q}_\sigma(\Omega)). \tag{3.8}$$

Introducing the solution operator $S : (G, u_0) \mapsto Du$, we conclude first from the existence theory, that S is continuous from $L^2(Q) \times L^2_\sigma(\Omega)$ to $L^2(Q)$ with the norm less or equal to 1. By Lemma 3.5 we know that S is continuous from $L^{q_1}(Q) \times B^{1-2/q_1}_{q_1,\sigma}(\Omega)$ to $L^{q_1}(Q)$ with norm estimated by $C_q > 1$. Since $S(G, u_0) = S(G, 0) + S(0, u_0)$, Riesz-Thorin theorem and the real interpolation method implies following assertion, see for example [5, Theorem 5.2.1 and Theorem 6.4.5].

Lemma 3.6. *Let Ω be bounded $\mathcal{C}^{2,1}$ domain and $q_1 > 2$. There exist constant $C > 0$ and $K := C_{q_1}^{q_1/(q_1-2)}$ such that for every $q \in (2, q_1)$, arbitrary $G \in L^q(I, L^q_\sigma(\Omega))$, $u_0 \in B^{1-2/q)}_{q,\sigma}(\Omega)$ there exists the unique solution u of* (3.8) *satisfying*

$$\|Du\|_{L^q(Q)} \le K^{1-\frac{2}{q}} \Big(\|G\|_{L^q(Q)} + C\|u_0\|_{B^{1-2/q}_q(\Omega)} \Big).$$

For $q > 2$ small enough Lemma 3.6 allows us to prove the L^q theory for generalized Stokes system, where the Laplace operator is replaced by a general elliptic operator with bounded measurable coefficients. More precisely, let $0 < \gamma_1 \le \gamma_2$ and suppose that the coefficient matrix $\mathbb{A} \in L^\infty(Q)$ is symmetric in the sense $A^{kl}_{ij} = A^{ij}_{kl} = A^{ji}_{kl}$ for $i, j, k, l = 1, 2$ and fulfills for all $B \in \mathbb{R}^{2\times 2}$, $x \in \Omega$ and $t \in I$

$$\gamma_1 |B|^2 \le \mathbb{A}(t,x) : B \otimes B \le \gamma_2 |B|^2.$$

We consider the following system

$$\int_I \langle \partial_t u, \varphi \rangle \,\mathrm{d}t + \int_Q \mathbb{A} : Du \otimes D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_Q G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall \varphi \in L^q(I, W^{1,q}_\sigma(\Omega)). \tag{3.9}$$

Following lemma states the L^q theory result.

Lemma 3.7. *Let Ω be bounded $\mathcal{C}^{2,1}$ domain and $q > 2$. There exist constant $K, L > 0$ such that if $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$, $G \in L^q(Q)$ and $u_0 \in B^{1-2/q}_{q,\sigma}(\Omega)$ then the unique weak solution $u \in L^q(I, W^{1,q}_\sigma(\Omega))$ of* (3.9) *satisfies*

$$\|Du\|_{L^q(Q)} + \gamma_2^{-\frac{1}{q}} \|u\|_{BUC(I, B^{1-2/q}_{q,\sigma}(\Omega))} \le \frac{K}{\gamma_1} \Big(\|G\|_{L^q(Q)} + \gamma_2^{1-\frac{1}{q}} \|u_0\|_{B^{1-2/q}_{q,q}(\Omega)} \Big).$$

Proof. We omit the proof. It can be found in [12, Proposition 2.1] for periodic boundary conditions or in [10, Proposition 2.1] for homogeneous Dirichlet boundary conditions. The only generalization consists of including perfect slip boundary conditions. L^q theory result for classical Stokes system with perfect slip boundary conditions is needed, but it is shown in Lemma 3.6. □

We also use the L^q theory for stationary variant of the system (3.9). For symmetric coefficient matrix $\mathbb{A} \in L^\infty(\Omega)$ fulfilling for all $B \in \mathbb{R}^{2\times 2}$ and $x \in \Omega$ $\gamma_1|B|^2 \le \mathbb{A}(x) : B \otimes B \le \gamma_2 |B|^2$, $0 < \gamma_1 \le \gamma_2$ we investigate the problem

$$\int_\Omega \mathbb{A} : Du \otimes D\varphi \,\mathrm{d}x = \int_\Omega G : D\varphi \,\mathrm{d}x \quad \forall \varphi \in W^{1,q}_\sigma(\Omega). \tag{3.10}$$

It holds

Lemma 3.8. *Let Ω be a bounded $\mathcal{C}^{2,1}$ domain. Then there are constants $K, L > 0$ such that if $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$ and $G \in L^q(\Omega)$, then the unique weak solution of* (3.10) *satisfies*

$$\|Du\|_{L^q(\Omega)} \le \frac{K}{\gamma_1} \|G\|_{L^q(\Omega)}.$$

Proof. See [12, Lemma 2.6] for no slip boundary conditions. For perfect slip boundary conditions we would proceed analogically. □

4 Proof of the main results for the quadratic potential

In this section we prove Theorem 1.2 for $p = 2$.

Step 1. *recalls apriori estimates from the existence theory.*

For $f \in W^{1,2}(I, W^{-1,2}_\sigma(\Omega))$ with $f(0) \in L^2(\Omega)$ and $u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_\sigma(\Omega)$ we know the existence of a unique weak solution of (1.1) with (1.2) fulfilling

$$u \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}_\sigma(\Omega)), \quad \partial_t u \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}_\sigma(\Omega)), \quad \pi \in L^\infty(I, L^2(\Omega)).$$

It can be shown using Galerkin approximation. See for example [15, Section 5.3], where the computation is done for periodic boundary conditions. Since $\partial_t u$, $\operatorname{div} \mathcal{S}(Du)$, $\operatorname{div}(u \otimes u)$ and f lie in $L^2(I, W^{-1,2}_\sigma(\Omega))$, we can reconstruct the pressure π at almost every time level via De Rham's theorem and Nečas' theorem on negative norms and obtain $\pi \in L^2(\Omega)$ for almost every $t \in I$.

Step 2. *improves the regularity in space.*

If we additionally assume $f \in L^\infty(I, L^2(\Omega))$ we are able to show that

$$u \in L^\infty(I, W^{2,2}(\Omega)), \quad \pi \in L^\infty(I, W^{1,2}(\Omega)).$$

From Step 1 we know that $\partial_t u$ is regular enough in order to move it to the right hand side of $(1.1)_1$. At almost every time level $t \in I$ we can use the stationary theory. Boundary regularity in tangent direction is based on the difference quotient technique. In normal direction near the boundary the main tools are the operator curl and Nečas' theorem on negative norms. See for example [16, Section 3] for homogeneous Dirichlet boundary conditions. The information about the pressure comes from the fact that the right hand side of $\nabla \pi = f + \operatorname{div} \mathcal{S}(Du) - \operatorname{div}(u \otimes u)$ is in $L^2(\Omega)$. Adding the assumption $\int_\Omega \pi \, dx = 0$ we get by Poincaré inequality the existence of $\pi \in W^{1,2}(\Omega)$ at almost every time level $t \in I$ together with a bound independent of t.

Step 3. *improves the regularity in time using L^p theory for Stokes system.*

If we moreover suppose that $f \in L^{q_1}(I, W^{-1,q_1'}_\sigma(\Omega))$ for some $q_1 > 2$ and $u_0 \in W^{2+\beta,2}$ for $\beta \in (0, 1/4)$ we are able to prove the existence of $q_2 > 2$ such that unique weak solution satisfies for all $q \in (2, q_2)$ and $s \in (0, \frac{1}{2})$

$$\partial_t u \in L^q(I, W^{1,q}_\sigma(\Omega)) \cap W^{s,q}(I, L^q(\Omega)). \tag{4.1}$$

Denoting $w := \partial_t u$ and $\tau := \partial_t \pi$ in the sense of distributions, we observe from (1.1) that (w, τ) solves

$$\int_I \langle \partial_t w, \varphi \rangle \, dt + \int_Q \partial^2_{Du} \Phi(|Du|) : Dw \otimes D\varphi \, dx \, dt = \int_I \langle \partial_t (f - (u \cdot \nabla) u), \varphi \rangle \, dt, \quad \forall \varphi \in L^q(I, W^{1,q}_\sigma(\Omega)). \tag{4.2}$$

It is easy to see that $\partial_t (u \cdot \nabla u) \in L^s(W^{-1,s}_\sigma(\Omega))$ for all $s \in [1, 4]$.

In order to obtain (4.1) as a result of application of Lemma 3.7 for the system (4.2) we need to ensure that $\|\partial_t u(0)\|_{B^{1-2/q}_{q,q}(\Omega)}$ is bounded. Let P be the projection onto the solenoidal functions. Let $\beta \in (0, 1/4)$ and $\varphi \in W^{-\beta,2}(\Omega)$ with $\|\varphi\|_{W^{-\beta,2}(\Omega)} \leq 1$ be arbitrary. Thus

$$\begin{aligned} |\langle \partial_t u(0), \varphi \rangle| = |\langle \partial_t u(0), P\varphi \rangle| &\leq |\langle \operatorname{div} S(Du_0) + (u_0 \cdot \nabla) u_0 - f(0), P\varphi \rangle| \leq \\ &\leq C(\|u_0\|_{W^{2+\beta,2}(\Omega)} + \|u_0\|^2_{W^{2,2}(\Omega)} + \|f(0)\|_{W^{\beta,2}(\Omega)}) \leq C, \end{aligned} \tag{4.3}$$

where we used the continuity of the projection P (see [7, Chapter 4, Section 4] for periodic boundary conditions). Since $W^{\beta,2}(\Omega) \hookrightarrow B^{1-2/q}_{q,q}(\Omega)$ if q is close enough to 2 we obtain $\|\partial_t u(0)\|_{B^{1-2/q}_{q,q}(\Omega)} \leq C$ for all $q \in (2, q_2)$ where q_2 is sufficiently close to 2.

Step 4. *gives $u \in L^\infty(I, W^{2,q}(\Omega))$ due to the stationary theory.*

Previous step shows us that $\partial_t u \in L^\infty(I, L^q(\Omega))$ for some $q > 2$. Therefore we are able to move $\partial_t u$ to the right hand side of $(1.1)_1$ and apply the result [11, Theorem 3] for $p = 2$ which tells us that there exists a positive ε, such that $u \in W^{2,2+\varepsilon}$ and $\pi \in W^{1,2+\varepsilon}$ for (1.1) with perfect slip boundary conditions.

Step 5. *improves the regularity of π in time.*

There exists a $q > 2$ such that for all $s \in (0, \frac{1}{2})$

$$\pi \in W^{s,q}(I, L^q(\Omega)).$$

See [10, Lemma 3.4] for the proof. The idea is based on subtracting the equation $(1.1)_1$ in the time t' from the same equation in time t and constructing special test function via the Bogovskiĭ Lemma.

Step 6. *summarizes the result of this section and uses imbedding theorems to complete the proof.*

Up to now we have shown

$$u \in L^\infty(I, W^{2,q}(\Omega)) \cap W^{1,q}(I, L^q(\Omega)), \quad \pi \in L^\infty(I, W^{1,q}(\Omega)) \cap W^{s,q}(I, L^q(\Omega)).$$

As we are in two dimensions, $q > 2$, $s \in (\frac{1}{q}, \frac{1}{2})$, following imbeddings holds

$$L^\infty(I, W^{1,q}(\Omega)) \hookrightarrow L^\infty(I, \mathcal{C}^{0,1-\frac{2}{q}}(\overline{\Omega})), \tag{4.4}$$
$$W^{1,q}(I, L^q(\Omega)) \hookrightarrow \mathcal{C}^{1-\frac{1}{q}}(\overline{I}, L^q(\Omega)), \tag{4.5}$$
$$W^{s,q}(I, L^q(\Omega)) \hookrightarrow \mathcal{C}^{s-\frac{1}{q}}(\overline{I}, L^q(\Omega)). \tag{4.6}$$

Now we are ready to apply

Lemma 4.1. *[10, Lemma 2.6] Let $\Omega \subset \mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain. Let $f \in L^\infty(I, \mathcal{C}^{0,\alpha}(\overline{\Omega}))$ and $f \in \mathcal{C}^{0,\beta}(\overline{I}, L^s(\Omega))$ for some $\alpha, \beta \in (0,1)$ and $s > 1$. Then $f \in \mathcal{C}^{0,\gamma}(\overline{Q})$ with $\gamma = \min\{\alpha, \frac{\alpha\beta s}{\alpha s + 2}\}$.*

Using (4.4) and (4.5) together with Lemma 4.1 we obtain $\nabla u \in \mathcal{C}^{0,\alpha}$ for certain $\alpha > 0$. (4.4), (4.6) with Lemma 4.1 gives us $\pi \in \mathcal{C}^{0,\alpha}$ for some $\alpha > 0$, which concludes proof of main results for $p = 2$. □

5 Proof of the main results for the super-quadratic potential

In this section we prove Theorem 1.2 for $p > 2$. The proof consists of several steps.

Step 1. *introduces quadratic approximations.*

In a similar way like in [14] we are concerned with the regularized problem

$$\begin{aligned} \partial_t u^\varepsilon - \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla \pi^\varepsilon = f, \quad \operatorname{div} u^\varepsilon = 0 \text{ in } Q, \\ u^\varepsilon(0,\cdot) = u_0 \text{ in } \Omega, \end{aligned} \tag{5.1}$$

where we consider quadratic approximation $\mathcal{S}^\varepsilon$ of $\mathcal{S}$ defined for $\varepsilon \in (0,1)$ by the truncation of the viscosity μ from above:

$$\mu^\varepsilon(|Du^\varepsilon|) := \min\Big\{\mu(|Du|), \frac{1}{\varepsilon}\Big\}, \quad \mathcal{S}^\varepsilon(Du^\varepsilon) := \mu^\varepsilon(|Du^\varepsilon|)Du^\varepsilon. \tag{5.2}$$

Scalar potential Φ^ε to $\mathcal{S}^\varepsilon(Du^\varepsilon)$ can be constructed in the following way

$$\Phi^\varepsilon(s) := \int_0^s \mu^\varepsilon(t)t \, \mathrm{d}t$$

and satisfies growth conditions (1.3) for $p = 2$, i.e. there exists $C_1 > 0$ and $C(\varepsilon)$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$C_1|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C(\varepsilon)|B|^2. \tag{5.3}$$

The approximation (5.2) guarantees that for a fixed $\varepsilon \in (0,1)$ the results of the previous section holds for u^ε and π^ε solving (5.1) equipped with the perfect slip boundary conditions.

Step 2. *gives growth conditions dependent on ε.*

Due to the results of the previous section we are able to use techniques which enable us to gain uniform estimates with respect to ε. At first we need a growth estimates of Φ^ε with precise dependence on ε. In other words, the constant $C(\varepsilon)$ in the estimate (5.3) needs to be specified. To this purpose we define the function ϑ_ε by $\vartheta_\varepsilon(s) := \min\{(1+s^2)^{\frac{1}{2}}, \frac{1}{\varepsilon}\}$. Now, there exist constants $0 < C_3 \le C_4$ such that for all $\varepsilon \in (0,1)$ an $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$C_3\vartheta_\varepsilon(|A|)^{p-2}|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C_4\vartheta_\varepsilon(|A|)^{p-2}|B|^2. \tag{5.4}$$

As a corollary of (5.4) following estimates can be derived (see [16, Lemma 2.22] for the proof.)

$$C\vartheta_\varepsilon(|A|)^{p-2}|A|^2 \le \mathcal{S}^\varepsilon(A) : A, \tag{5.5}$$
$$C|\mathcal{S}^\varepsilon(A)| \le \vartheta_\varepsilon(|A|)^{p-2}|A|. \tag{5.6}$$

The lower estimate in (5.5) can be done independent of ε, since (5.3) holds:

$$C_5|A|^2 \le \mathcal{S}^\varepsilon(A) : A. \tag{5.7}$$

At this point we would like to emphasize that from now all constants in following steps are independent of ε.

Step 3. *provides $L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$ estimates of u^ε and $\partial_t u^\varepsilon$.*

We recall estimates from the previous section which hold also for the approximated problem since the lower bound in (5.7) is independent on ε.

$$\|u^\varepsilon\|_{L^\infty(I,L^2(\Omega))} + \|\nabla u^\varepsilon\|_{L^2(Q)} \le C, \tag{5.8}$$

$$\|\partial_t u^\varepsilon\|^2_{L^\infty(I,L^2(\Omega))} + \|\nabla \partial_t u^\varepsilon\|_{L^2(Q)} \le C. \tag{5.9}$$

The relation (5.8) is apriori estimate obtained by taking solution as a test function (at the level of Galerkin approximation). Roughly speaking, the estimate (5.9) is performed by taking time derivative of the equation (5.1) and testing by time derivative of u^ε. More precisely, it is done not directly to the equation (5.1), but still to the Galerkin system. In order to estimate the time derivative of the Galerkin approximation of u^ε at the time $t = 0$ we proceed in the same way like in (4.3).

Note that (5.8) and (5.9) gives us $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$:

$$\|\nabla u^\varepsilon(s,\cdot)\|_2^2 - \|\nabla u^\varepsilon(0,\cdot)\|_2^2 = \int_\Omega \int_0^s \partial_t |\nabla u^\varepsilon(t,\cdot)|^2 \,\mathrm{d}t\,\mathrm{d}x \le 2\|\nabla u^\varepsilon\|_{L^2(Q)} \|\partial_t \nabla u^\varepsilon\|_{L^2(Q)} \le C.$$

Step 4. *gives $u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega))$ uniformly in $\varepsilon \in (0,1)$.*

From Step 3 we obtained that $\partial_t u^\varepsilon \in L^\infty(I, L^2(\Omega))$, therefore we can fix $t \in I$, move $\partial_t u^\varepsilon$ to the right hand side of (5.1) and at almost every time level consider the stationary problem

$$\begin{aligned} -\operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla \pi^\varepsilon = h, \quad \operatorname{div} u^\varepsilon = 0 \text{ in } \Omega, \\ u^\varepsilon \cdot \nu = 0, \ [\mathcal{S}^\varepsilon(Du^\varepsilon)\nu] \cdot \tau = 0 \text{ at } \partial\Omega, \end{aligned} \tag{5.10}$$

where $h := f - \partial_t u^\varepsilon \in L^2(\Omega)$. Previous section provides $u^\varepsilon \in W^{2,2}(\Omega)$, $\mathcal{S}^\varepsilon(Du^\varepsilon) \in W^{1,2}(\Omega)$ and $\pi^\varepsilon \in W^{1,2}(\Omega)$. Thus we can multiply (5.10) by a suitable test function which is at least in $L^2(\Omega)$ and integrate over Ω. We focus only on the boundary regularity and work in the local system of coordinates. Following [14, Lemma 4.2, Remark 4.9] we choose as a test function $\varphi = (\varphi_1, \varphi_2)$

$$\varphi = (\partial_2[\Theta - \partial_\tau(u^\varepsilon \cdot \nu)\xi^2], \partial_1[-\Theta + \partial_\tau(u^\varepsilon \cdot \nu)\xi^2]), \quad \Theta := \partial_\nu(u^\varepsilon \cdot \tau)\xi^2 - u^\varepsilon \cdot (\partial_\nu \tau + \partial_\tau \nu)\xi^2.$$

This test function is constructed in order to get rid of the pressure π^ε and to obtain optimal information from the elliptic term. These most difficult estimates, in which we extract from $-\int_\Omega \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) \cdot \varphi \,\mathrm{d}x$ boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \,\mathrm{d}x$, are done in [14, Proof of Theorem 1.7], therefore we omit the calculations. It remains to estimate the convective term and the right hand side of (5.10). After long, but elementary calculations we are able to show

$$\left|\int_\Omega (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot \varphi \,\mathrm{d}x\right| \le C \int_\Omega (|u^\varepsilon||\nabla u^\varepsilon|^2 + |u^\varepsilon|^2 |\nabla u^\varepsilon|) \,\mathrm{d}x, \tag{5.11}$$

where we used the divergence-free constraint and the properties of the test function φ. Using Hölder and Young inequalities, $\|\cdot\|_4^2 \le C\|\cdot\|_{1,2}\|\cdot\|_2$ and the information $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$ we continue estimating (5.11):

$$C(\|u^\varepsilon\|_2 \|\nabla u^\varepsilon\|_4^2 + \|u^\varepsilon\|_4^2 \|\nabla u^\varepsilon\|_2) \le \varepsilon \|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2 + C\|\nabla u^\varepsilon\|_2^2 \|u^\varepsilon\|_2^2.$$

The last estimate is easy.

$$\left|\int_\Omega h \cdot \varphi \,\mathrm{d}x\right| \le \int_\Omega |h|(|\nabla^2 u^\varepsilon| + |\nabla u^\varepsilon| + |u^\varepsilon|) \,\mathrm{d}x \le C\|h\|_2^2 + \varepsilon \|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2.$$

Since $\mu^\varepsilon(|Du^\varepsilon|) > 1$ and $\varepsilon > 0$ can be chosen arbitrarily small, we obtain

$$\|\nabla^2 u^\varepsilon\|_2^2 \le \int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \,\mathrm{d}x \le C, \tag{5.12}$$

where C doesn't depend on ε and $t \in I$, therefore we have

$$u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega)). \tag{5.13}$$

Step 5. *improves information about $\partial_t u^\varepsilon$.*

In the same spirit as in Step 3 from the previous section we denote $w := \partial_t u$ and $\tau := \partial_t \pi$ in the sense of distributions, which solves (4.2) where Φ is replaced by Φ^ε. The right hand side of (4.2) is bounded uniformly with respect to $\varepsilon \in (0,1)$ in $L^{q_0}(I, W_\sigma^{-1,q_0'}(\Omega))$ for some $q_0 > 2$, since from (5.8), (5.9) and (5.13) (resp. [4, Theorem 1.2]) we have $\partial_t[(u^\varepsilon \cdot \nabla)u^\varepsilon] \in L^s(I, W_\sigma^{-1,s}(\Omega))$ for all $s \in [1,4]$.

Set $V_\varepsilon := \sup_Q |\vartheta_\varepsilon(|Du^\varepsilon|)|$. From (5.4) we have for all $t \in I$, $x \in \Omega$, for all $\varepsilon \in (0,1)$ and $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$c|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C V_\varepsilon^{p-2}(|A|)|B|^2.$$

From Lemma 3.7 we have the existence of positive constants K and L such that for all $q \in (2, q_2]$, where $q_2 := 2 + L/V_\varepsilon^{p-2}$ holds

$$\|\nabla w\|_{L^q(Q)} + V_\varepsilon^{\frac{2-p}{q}} \|w\|_{BUC(I, B_q^{1-2/q}(\Omega))} \le K\Big(\|f\|_{L^q(I, W^{-1,q'}(\Omega))} + V_\varepsilon^{(p-2)(1-1/q)} \|\partial_t u_0\|_{B_q^{1-2/q}(\Omega)}\Big). \tag{5.14}$$

Without loss of generality we may assume that $q_2 < q_0$. Thus, after estimating last norm on the right hand side of (5.14) in the same way like in Step 3 in Section 3 we have

$$\|\partial_t u^\varepsilon\|_{BUC(I, B_{q_2}^{1-2/q_2}(\Omega))} \le C\Big(V_\varepsilon^{\frac{p-2}{q_2}} + V_\varepsilon^{p-2}\Big) \le C V_\varepsilon^{p-2}. \tag{5.15}$$

Step 6. *improves information about* $\nabla^2 u^\varepsilon$.

In this step we obtain better space regularity. Up to now we have $\vartheta_\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$. We are going to show that $\vartheta_\varepsilon \in L^\infty(I, W^{1,q}(\Omega))$ for some $q > 2$.

We omit estimates of $\nabla^2 u^\varepsilon$ in the interior of Ω and we focus on estimates near the boundary. We start with the tangential direction. Localizing the problem, we work in Ω_{3r}^P, where the boundary is locally described by the $\mathcal{C}^3$ mapping a_p (see Subsection 2.3). For simplicity we drop the index P.

We multiply (5.10) by $-\partial_\tau \varphi \xi$, integrate over Ω_{3r} and after similar steps as in [14, Lemma 4.6] we derive the identity

$$\begin{aligned}
\int_{\Omega_{3r}} \partial_\tau \mathcal{S}^\varepsilon(Du^\varepsilon) : D\varphi\xi \,\mathrm{d}x = &-\int_{\Omega_{3r}} h \cdot \partial_\tau(\varphi\xi)\,\mathrm{d}x + \int_\Omega (u^\varepsilon \cdot \nabla)u^\varepsilon \partial_\tau \varphi \xi \,\mathrm{d}x + \\
&+ \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^\varepsilon) : \Big[\partial_\tau \varphi \otimes \nabla\xi - \nabla\varphi \partial_\tau \xi + (\partial_1^2 a, 0) \otimes \partial_2 \varphi \xi + \nabla\Big(\varphi \cdot \partial_\tau \nu \frac{\nu}{|\nu|^2}\xi\Big)\Big]\,\mathrm{d}x + \\
+ \int_{\Omega_{3r}} \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) \cdot \Big[(\varphi \cdot \partial_\tau \nu)\frac{\nu}{|\nu|^2}\xi - \varphi \partial_\tau \xi\Big]\,\mathrm{d}x &+ \int_{\Omega_{3r}} \partial_1^2 a[h_2 + (\operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon))_2 - (u^\varepsilon \cdot \nabla u^\varepsilon)_2]\varphi_1 \xi \,\mathrm{d}x + \\
+ \int_{\Omega_{3r}} [h_1 + \partial_1 a h_2 + \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon)_1 &+ \partial_1 a \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon)_2 + (u^\varepsilon \cdot \nabla u^\varepsilon)_1 + \partial_1 a (u^\varepsilon \cdot \nabla u^\varepsilon)_2]\varphi \nabla\xi \,\mathrm{d}x
\end{aligned} \tag{5.16}$$

for all $\varphi \in W_\sigma^{1,q'}(\Omega)$, $\operatorname{supp}\varphi \subset \overline{\Omega_{3r}}$. Terms on the right hand side of (5.16) comes at first from the fact that we add subtract some lower order terms in order to let the boundary term vanish while integrating by parts. Second, tangent derivative doesn't commute with the gradient and we use $\nabla\partial_\tau \varphi = \partial_\tau \nabla\varphi + (\partial_1^2 a, 0) \otimes \partial_2 \varphi$. Third, we use the equation (5.10) and replace $\partial_2 \pi^\varepsilon$ by $h_2 + [\operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon)]_2 + (u^\varepsilon \cdot \nabla u^\varepsilon)_2$ and similarly for $\partial_\tau \pi^\varepsilon$.

We denote $w := \partial_\tau u^\varepsilon \xi - (0, \partial_1^2 a u_1^\varepsilon)\xi + z$, where z is the solution of

$$\operatorname{div} z = -\partial_\tau u^\varepsilon \cdot \nabla\xi - \partial_1^2 a u_1^\varepsilon \partial_2 \xi \qquad \text{in } \Omega_{3r}, \tag{5.17}$$

$$z = 0 \qquad \text{on } \partial\Omega_{3r}. \tag{5.18}$$

The right hand side of (5.17) was obtained from $\operatorname{div}(-\partial_\tau u^\varepsilon \xi + (0, \partial_1^2 a u_1^\varepsilon)\xi)$ using the fact that $\operatorname{div} u^\varepsilon = 0$. The role of z is to ensure that $\operatorname{div} w = 0$. On $\partial\Omega$ it holds $w \cdot \nu = 0$ since

$$w \cdot \nu = [\partial_\tau u^\varepsilon \cdot \nu + \partial_1^2 a u_1^\varepsilon]\xi + z \cdot \nu = \partial_\tau(u^\varepsilon \cdot \nu)\xi = 0.$$

Thus the compatibility condition holds

$$\int_{\partial\Omega} z \cdot \nu \,\mathrm{d}\sigma = \int_\Omega \operatorname{div} z \,\mathrm{d}x = \int_\Omega \operatorname{div}(-\partial_\tau u^\varepsilon \xi + (0, \partial_1^2 a u_1^\varepsilon)\xi)\,\mathrm{d}x = -\int_{\partial\Omega} \partial_\tau(u^\varepsilon \cdot \nu)\xi \,\mathrm{d}\sigma = 0$$

and z solving (5.17) and (5.18) exists by Bogovskiĭ's Lemma and enjoys the estimate $\|z\|_{1,q} \le C\|\nabla u^\varepsilon\|_q$ for some $C > 0$.

Using the definition of w we get from (5.16)

$$\int_\Omega \partial_{Du^\varepsilon} \mathcal{S}^\varepsilon(Du^\varepsilon) : Dw \otimes D\varphi \,\mathrm{d}x = \langle g, \varphi\rangle \quad \forall \varphi \in W_\sigma^{1,q'}(\Omega),$$

with

$$\langle g, \varphi \rangle = \text{RHS of (5.16)} + \int_\Omega \partial_{Du^\varepsilon} \mathcal{S}^\varepsilon(Du^\varepsilon) : [Dz + \partial_\tau u^\varepsilon \otimes \nabla\xi + (\partial_1^2 a, 0) \otimes_S \partial_2 u^\varepsilon \xi - D\big((0, \partial_1^2 a, 0)\xi\big)] D\varphi \, \mathrm{d}x.$$

Due to the assumption on f and results from Step 4 we have $\|g\|_{-1,q_2'} \le CV_\varepsilon^{p-2}$ and after application of Lemma 3.8 we obtain

$$\|\nabla \partial_\tau u^\varepsilon \xi\|_{L^q(\Omega)} \le CV_\varepsilon^{p-2}. \tag{5.19}$$

We recall that q depends on ε by the relation $q \in (2, 2 + L/V_\varepsilon^{p-2}]$. In order to control whole $\nabla^2 u^\varepsilon$ we need an estimate of type (5.19) in the normal direction which is locally x_2. Since $\partial_2^2 u_2^\varepsilon$ can be expressed from the condition $\operatorname{div} u^\varepsilon = 0$, we focus on $\partial_2^2 u_1^\varepsilon$. Following [12, Theorem 3.19] we can extract the desired estimate from the equation (5.10) after employment of the operator curl. Let us shorten $\mathcal{S}^\varepsilon(Du^\varepsilon)$ to $\mathcal{S}^\varepsilon$ and $\vartheta_\varepsilon(|Du^\varepsilon|)$ to ϑ_ε. Denoting $G := \partial_2 \mathcal{S}_{12}^\varepsilon$ we have due to (5.6) and (5.4)

$$\|\xi G\|_{-1,q} \le \|\mathcal{S}_{12}^\varepsilon\|_q \le \|{\vartheta_\varepsilon}^{p-2} Du^\varepsilon\|_q,$$
$$\|\partial_1(\xi G)\|_{-1,q} \le C\|{\vartheta_\varepsilon}^{p-2} Du^\varepsilon\|_q + C'\|{\vartheta_\varepsilon}^{p-2}\partial_1 \nabla u^\varepsilon\|_q.$$

From the equation (5.10) after application of curl we have

$$\|\partial_2(\xi G)\|_{-1,q} \le C(\|\partial_1(\mathcal{S}_{21}^\varepsilon + \mathcal{S}_{22}^\varepsilon - \mathcal{S}_{11}^\varepsilon)\|_q + \|f\|_q + \|u^\varepsilon \cdot \nabla u^\varepsilon\|_q + \|\partial_t u^\varepsilon\|_q) \le$$
$$\le C\big\{\|{\vartheta_\varepsilon}^{p-2} Du^\varepsilon\|_q + \|{\vartheta_\varepsilon}^{p-2}\partial_1 \nabla u^\varepsilon\|_q + V_\varepsilon^{p-2} + 1\big\} := H.$$

Nečas' theorem on negative norms gives us

$$\|\xi G\|_q \le C(\|\xi G\|_{-1,q} + \|\nabla(\xi G)\|_{-1,q}) \le H.$$

From definition of G and symmetry of Du we obtain

$$\partial_{12}\mathcal{S}_{12}^\varepsilon \partial_2 Du_{12}^\varepsilon = \frac{G}{2} - \frac{1}{2}\partial_{11}\mathcal{S}_{12}^\varepsilon \partial_2 Du_{11}^\varepsilon - \frac{1}{2}\partial_{22}\mathcal{S}_{12}^\varepsilon \partial_2 Du_{22}^\varepsilon.$$

Using $\partial_{12}\mathcal{S}_{12}^\varepsilon \ge C{\vartheta_\varepsilon}^{p-2}$ and the condition $\operatorname{div} u^\varepsilon = 0$ we get that $\|\xi {\vartheta_\varepsilon}^{p-2}\partial_2^2 u_1^\varepsilon\|_q \le H$. Thus,

$$\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_q \le C\|\xi G\|_q + \|\xi\vartheta_\varepsilon^{p-2}\nabla\partial_\tau u^\varepsilon\|_q + \tilde{C}\sup_{x_1\in(-3r,3r)}|\partial_1 a|\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_q, \tag{5.20}$$

where $\tilde{C}$ is absolute constant. Since we can choose r sufficiently small in order to $\tilde{C}\max_{P\in\partial\Omega}\sup_{x_1\in(-3r,3r)}|\partial_1 a| \le 1/2$, the last term (5.20) can be absorbed into the left hand side. We have

$$\|\xi\vartheta_\varepsilon^{p-2}\nabla^2 u^\varepsilon\|_{q_2} \le CV_\varepsilon^{p-2}V_\varepsilon^{p-2}. \tag{5.21}$$

From (5.12) the boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2\,\mathrm{d}x$ is obtained, in other words $\|{\vartheta_\varepsilon}^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_2 \le C$. Interpolation of this result with (5.21) gives us for $q \in (2, q_2)$

$$\|{\vartheta_\varepsilon}^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q \le CV_\varepsilon^{\beta 2(p-2)}, \tag{5.22}$$

where $1/q = \beta/q_2 + (1-\beta)/2$. Since it holds $\|\vartheta_\varepsilon^{p/2}\|_{1,q} \le \|\vartheta_\varepsilon^{p/2}\|_q + \|{\vartheta_\varepsilon}^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q$, we want to use the following lemma for $f = \vartheta_\varepsilon^{p/2}$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain and $f \in W^{1,q}(\Omega)$ for some $q > 2$. Then $f \in \mathcal{C}(\overline{\Omega})$ and there is $C > 0$ independent of q such that*

$$\sup_\Omega |f| \le C\Big(\frac{q-1}{q-2}\Big)^{1-1/q}\|f\|_{1,q}. \tag{5.23}$$

Proof. Follows from the proof of [22, Theorem 2.4.1]. The result holds also for $\Omega \subset \mathbb{R}^n$, with $q > n$ and $q - n$ instead of $q - 2$ in the denominator of (5.23). □

Because $\frac{q-1}{q-2} \le CV^{p-2}$, we obtain

$$V_\varepsilon^{\frac{p}{2}} \le CV^{(p-2)(1-\frac{1}{q})}V_\varepsilon^{\beta 2(p-2)}. \tag{5.24}$$

Note that $(p-2)(1-1/q) \to p/2 - 1$ as $q \to 2$ and the exponent containing the interpolation parameter β can be made arbitrarily small, therefore we can rewrite (5.24) as $V_\varepsilon \le \hat{C}$. This together with (5.22) gives us

$$\sup_{t\in I}\|\nabla^2 u^\varepsilon\|_q \le C.$$

Step 7. *passes from the regularized problem to the original one.*

In the previous step we showed $V_\varepsilon \le \hat{C}$, where $V_\varepsilon = \sup_Q |\vartheta_\varepsilon(|Du^\varepsilon|)|$. Since $\vartheta_\varepsilon(s) = \min\{(1+s^2)^{\frac{1}{2}}, \frac{1}{\varepsilon}\} \le \frac{1}{\varepsilon}$, it is sufficient to choose ε in order to have $\hat{C} \le \frac{1}{\varepsilon}$. Thus, $u^\varepsilon = u$ is the solution of the original problem (1.1) and it holds that $\sup_Q (1+|Du|^2)^{1/2} \le C$ which leads to $\sup_{t\in I} \|\nabla^2 u\|_q \le C$.

□

Acknowledgement: The work was supported by the project 281211/B-MAT/MFF financed by the Grant Agency of Charles University (GAUK) and by the project 201/09/0917 financed by Czech Science Foundation (GAČR). The author is a junior researcher in the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (Math MAC). The author would like to thank Petr Kaplický for productive discussions and for his support.

References

[1] Abels, H., Terasawa, Y.: *On Stokes operators with variable viscosity in bounded and unbounded domains*, Math. Ann. 344 (2009), no. 2, 381—429.

[2] Amann, H.:*Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory,* Birkhäuser, Basel, 1995.

[3] Amann, H.:*On the Strong Solvability of the Navier-Stokes Equations*, J. Math. Fluid Mech. 2 (2000), 16–98.

[4] Beirão da Veiga, H., Kaplický, P., Růžička, M.: *Boundary Regularity of Shear Thickening Flows*, Journal of Mathematical Fluid Mechanics **13** (3), 387–404, (2011).

[5] Bergh, J., Löfström, J.: *Interpolation spaces, an introduction.* Springer-Verlag, Berlin, Heidelberg, New York, 1976, x + 204 pp.

[6] Denk, R., Hieber, M., Prüss, J.: *R−Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Mem. Amer. Math. Soc. 166 (2003), no. 788, viii+114 pp.

[7] Diening, L.: *Theoretical and Numerical Results for Electrorheological Fluids*, Freiburg: Albert-Ludwigs-Universität 2002, Ph.D. Thesis.

[8] Engel, K.-J., Nagel, R.: *One-Parameter Semigroups of Linear Evolution Equations*, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000. xxii+586 pp.

[9] Frese, J., Seregin, G.: *Full regularity for a class of degenerated parabolic systems in two spatial variables* Manuscripta Math. 99 (1999), 517–539.

[10] Kaplický, P.: *Regularity of flows of a non-Newtonian fluid subject to Dirichlet boundary conditions*, Journal for Analysis and its Applications, **24**, no. 3 (2005), 467–486.

[11] Kaplický, P., Málek, J. and Stará, J.: *On Global existence of smooth two-dimensional steady flows for a class of non-Newtonian fluids under various boundary conditions,* Applied Nonlinear Analysis, New York, Kluwer/Plenum, 1999, 213–229.

[12] Kaplický, P., Málek, J. and Stará, J.: $C^{1,\alpha}$*-solutions to a class of nonlinear fluids in two dimensions - stationary Dirichlet problem,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **259** (1999), 89–121.

[13] Kaplický, P., Málek, J. and Stará, J.: *Global-in-time Hölder continuity of the velocity gradients for fluids with shear-dependent viscosities,* NoDEA Nonlinear Differential Equations Appl. 9 (2002), 175–195.

[14] Kaplický, P., Tichý, J.: *Boundary Regularity of Flows under Perfect Slip Boundary Conditions,* accepted to CEJM.

[15] Málek, J., Nečas, J., Rokyta, M. and Růžička, M.: *Weak and Measure-valued Solutions to Evolutionary PDEs,* Vol. 13 of Applied Mathematics and Mathematical Computation. Chapman & Hall, London, 1996.

[16] Málek, J., Nečas, J. and Růžička, M.: *On a weak solution to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case* $p \geq 2$, Adv. Differential Equations 6 (2001), 257–302.

[17] Nečas, J., Šverák, V.: On regularity of solutions of nonlinear parabolic systems Ann. Scuola Norm. Sup. Pisa Cl. Sci. 18 (1991)(4), 1–11.

[18] Shimada, R.: *On the* $L_p - L_q$ *maximal regularity for Stokes equations with Robin boundary condition in a bounded domain*, Math. Methods Appl. Sci. 30 (2007), no. 3, 257—289.

[19] Shibata, Y., Shimada, R.: *On a generalized resolvent estimate for the Stokes system with Robin boundary condition*, J. Math. Soc. Japan **59** (2007) (2), 469–519.

[20] Steiger, O.: *Navier-Stokes equations with first order boundary conditions*, J. Math. Fluid Mech. 8 (2006), no. 4, 456—481.

[21] Triebel, H.: *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, Amsterdam - New York - Oxford, 1978.

[22] Ziemer, W.P.: *Weakly differentiable functions,* Sobolev spaces and functions of bounded variation, vol. 120 of Graduate Texts in Mathematics. New York: Springer 1989.