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Abstract. In the calculus of variations, the first usual discussed property of a minimizer is the validity of the Euler-Lagrange equations which follows by using the variations with respect to the variable - unknown. On the other hand, doing the variations with respect to the independent variable - x one can deduce the so-called Noether equations. Such a property is usually derived under the additional hypothesis the the minimizer is a $\mathcal{C}^1$ function. Such a minimizer is then also called the fully stationary point and the importance of its existence naturally arises in many fields, in particular in the regularity theory. In this short note we show that the restriction on the smoothness of a minimizer is in fact not needed for the validity of the Noether equation and we prove its validity for all minimizers for general class of variational problems where only natural growth assumptions are required and/or for sufficiently smooth (but not $\mathcal{C}^1$) solutions to the Euler-Lagrange equations.

1. Introduction and statement of the result

We consider a variational integral

$$J(u) := \int_\Omega F(x, u(x), \nabla u(x)) - b(x) \cdot u(x)\, dx \tag{1.1}$$

for an unknown $u : \Omega \to \mathbb{R}^N$ with $N \in \mathbb{N}$ and for a given $b : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^d$ is an open bounded Lipschitz domain with dimension $d \geq 2$. Next, for a given set S we look for a minimizer u over such a set, i.e., we look for $u \in S$ such that

$$J(u) \leq J(v) \qquad \text{for all } v \in S. \tag{1.2}$$

In the paper, we are not interested whether such a minimizer exists but we are more interested in further qualitative properties of such a minimizer and we look for the assumptions on the potential F and the set S which will finally guarantee the validity of the so-called Noether equation. To simplify the setting of the paper, we consider that F has at most p-growth with respect to ∇u and that the set $S \subset W^{1,p}(\Omega; \mathbb{R}^N)$. More precisely, for the potential F, we assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \mathbb{R}$ is a Carathéodory mapping fulfilling for some $K > 0$, some nonnegative $f \in L^1(\Omega)$ and some $p \in (1, \infty)$ the following growth condition

$$|F(x, u, \eta)| \leq K(1 + |\eta|^p + |u|^q) + f(x), \tag{1.3}$$

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where

$$q \le \begin{cases} \dfrac{dp}{d-p} & \text{if } p < d, \\ q_0 < \infty & \text{if } p \ge d. \end{cases} \tag{1.4}$$

Concerning $b : \Omega \to \mathbb{R}^N$ the minimal natural requirement in such a setting seems to be

$$b \in (W^{1,p}(\Omega; \mathbb{R}^N))^*. \tag{1.5}$$

Note that under such assumptions all expressions in (1.2) are well-defined[1] provided we consider $S \subset W^{1,p}(\Omega; \mathbb{R}^N)$. On the other hand, the assumptions above do not guarantee the existence of a minimizer but as mentioned at the beginning we are not interested whether such a minimizer exists but we want to discuss its further properties. The first and well-known property of a minimizer is that it usually satisfies the so-called Euler-Lagrange equations that in the case when $S = W_0^{1,p}(\Omega; \mathbb{R}^N)$ take the form (in the sense of distribution)

$$-\operatorname{div} F_\eta(\cdot, u, \nabla u) + F_u(\cdot, u, \nabla u) = b \qquad \text{in } \Omega, \tag{1.6}$$

provided that all object in (1.6) are well defined. In (1.6) and also in what follows we use the following notation

$$\begin{aligned} F_\eta(x, u, \eta) &:= \frac{\partial F(x, u, \eta)}{\partial \eta} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \to \mathbb{R}^{N \times d}, \\ F_u(x, u, \eta) &:= \frac{\partial F(x, u, \eta)}{\partial u} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \to \mathbb{R}^N, \\ F_x(x, u, \eta) &:= \frac{\partial F(x, u, \eta)}{\partial x} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \to \mathbb{R}^d. \end{aligned}$$

In addition, to guarantee the meaning to (1.6), i.e., its validity in the sense of distribution, it is natural to prescribe additional growth assumption on the derivatives of F. Thus, we assume that F is for almost all x a $\mathcal{C}^1$ mapping satisfying

$$|F_x(x, u, \eta)| + |F_u(x, u, \eta)| + |F_\eta(x, u, \eta)|^{\frac{p}{p-1}} \le K(f(x) + |\eta|^p + |u|^q). \tag{1.7}$$

Under such growth assumptions, one can indeed “derive” the weak formulation of Euler-Lagrange equations (1.6) by computing

$$\left.\frac{d}{dh} J(u + h\varphi)\right|_{h=0} = 0, \tag{1.8}$$

where $\varphi \in \mathcal{D}(\Omega; \mathbb{R}^N)$ is arbitrary.

Interestingly, it was already observed by Noether [6] that minimizing $J(u)$ with respect to the internal variable x, i.e., computing[2]

$$\left.\frac{d}{dh} J(u(x + h\varphi(x)))\, dx\right|_{h=0} = 0, \tag{1.9}$$

[1] Note here that having (1.5) we need to replace $\int b(x)u(x)\, dx$ by the duality $\langle b, u \rangle$.

[2] Here it is assumed that at least formally $u(x + t\varphi(x)) \in S$.

with arbitrary $\varphi \in \mathcal{C}_0^1(\Omega;\mathbb{R}^d)$ leads formally to the following identity

$$\begin{aligned}
-\sum_{i=1}^{d}\sum_{\alpha=1}^{N} D_i\left(F_{\eta_i^\alpha}(\cdot,u,\nabla u)D_k u^\alpha\right) + D_k F(\cdot,u,\nabla u) - F_{x_k}(\cdot,u,\nabla u)& \\
= \sum_{\alpha=1}^{N} b^\alpha(\cdot,u,\nabla u)D_k u^\alpha \qquad \text{in } \Omega&
\end{aligned} \tag{1.10}$$

for all $k = 1,\dots,d$, where we denoted $D_i := \frac{\partial}{\partial x_i}$. Note that such an identity can be also formally achieved by multiplying the α-th equation in (1.6) by $D_k u^\alpha$ and then summing the result with respect to α. Relation (1.10) is sometimes called the Noether equation, or sometimes if u solves the Euler-Lagrange equations and also the Noether equation then it is called the "fully" stationary point, i.e., also stationary with respect to variations of the independent variables. The importance of Noether solution (or "fully stationary" point) was successfully demonstrated in [3] for proving the partial regularity for harmonic mappings and also in [1] for proving the Hölder continuity of solution to (1.6) with F independent of u and in [2] for F being dependent on u or in [4] for proving the regularity of certain variational integrals. On the other hand in [1, 3, 4] it was assumed that either u satisfies (1.10) or (1.10) was proved under some additional regularity, e.g. $u \in \mathcal{C}^1$, see also [5] for more details. The validity of the Noether equation without any further assumption on u was used in [2], where the author repeated a simpler version of the proof from this article.

Thus, the main purpose of the paper is to show that under some assumption on the structure of S any minimizer $u \in S$ of $J(u)$ in fact satisfies (1.11) provided that F fulfills (1.7) and $b \in L^{p'}(\Omega;\mathbb{R}^N)$. Moreover, in case of "smooth" boundary, we can show the validity of (1.10) up to the boundary. The strength of such a result might seem to be surprising in view of the fact that in (1.9) we compose in general two Sobolev functions whose result need not to be again a Sobolev function. However, this difficulty can be overcome, and it is also the main ingredient in the proof, by using the fact that the set S over which we minimize is rich enough. In particular, we use the fact that the smooth functions are dense in such a set. Note here that it does not have nothing to do with the convexity of the set but is more related to the fact that the "target" space where u lies is a closed set.

Thus, the first result we prove here is related to the simplest case when $S = W_0^{1,p}(\Omega;\mathbb{R}^N)$.

Theorem 1.1. *Let $\Omega \in \mathcal{C}^{0,1}$, F satisfy* (1.3) *and* (1.7) *and let $b \in L^{p'}(\Omega;\mathbb{R}^N)$. Then for any $u \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ being a weak solution to* (1.6) *the following identity holds*

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{d}\sum_{\alpha=1}^{N} F_{\eta_i^\alpha}(x,u(x),\nabla u(x))D_i\psi_j(x)D_j u^\alpha(x)\,dx \\
&-\int_\Omega F(x,u(x),\nabla u(x))\operatorname{div}\psi(x) + \sum_{j=1}^{d} F_{x_j}(x,u(x),\nabla u(x))\psi_j(x)\;\,dx \\
&= \int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{N} b^\alpha(x,u(x),\nabla u(x))\psi_i(x)D_i u^\alpha(x)\;\,dx
\end{aligned} \tag{1.11}$$

for all $\psi \in \mathcal{C}_0^{0,1}(\Omega;\mathbb{R}^d)$ *provided that one of the following holds*

(A1) *The function* u *has an additional regularity* $u \in W_0^{1,p+1} \cap W^{2,\frac{p+1}{2}}(\Omega;\mathbb{R}^N)$.

(A2) *The function* u *is a minimizer, i.e., for all* $v \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ *there holds*

$$\int_\Omega F(x,u(x),\nabla u(x)) - b(x)\cdot u(x)\,dx \le \int_\Omega F(x,v(x),\nabla v(x)) - b(x)\cdot v(x)\,dx.$$

Moreover, if $\Omega \in \mathcal{C}^{1,1}$, *the identity* (1.11) *holds for all* $\psi \in \mathcal{C}^{0,1}(\overline{\Omega};\mathbb{R}^d)$ *such that* $\psi\cdot\nu = 0$ *on* $\partial\Omega$, *where* ν *denotes the unit outer normal vector on* $\partial\Omega$.

We would like to mention here that Theorem 1.1 can be proved in a more general setting, i.e., for more general growth conditions and for more general boundary data. But since we want to overcome all technical details here, we present the result in the simplest form. We would also like to emphasize that the result of Theorem 1.1 might be clear to the experts in the field but up to our best knowledge one cannot find it in the existing literature. This is also a partial motivation of the paper, i.e., to have a well established reference for the validity of the Noether equation. The rest of the paper is devoted to the proof of Theorem 1.1.

The second result we have in mind is then related to the case which cover also the harmonic mapping.

Theorem 1.2. *Let* $\Omega \in \mathcal{C}^{0,1}$ *and* F *satisfy* (1.3) *and* (1.7) *and let* $b \in L^{p'}(\Omega;\mathbb{R}^N)$. *Assume that* $K \subset \mathbb{R}^N$ *is a closed (even unbounded) set and that* $u_d \in W^{1,p}(\Omega;K)$ *is given and define*

$$S := \{v \in W^{1,p}(\Omega;K);\ v = u_d \ \text{ on } \partial\Omega\}.$$

Then for any $u \in S$ *being a minimizer to* (1.2), *i.e., for all* $v \in S$ *satisfying* (1.2), *the identity* (1.11) *holds for all* $\psi \in \mathcal{C}_0^{0,1}(\Omega;\mathbb{R}^d)$.

According to our best knowledge, the property (1.11) is always required as an additional information about the minimizer. Here, we in fact shows, that such a property is automatically met by any minimizer. There is only one reasonable restriction, namely the target set K must be closed, which is however the most typical case when dealing with minimizers with some constraint.

We would also like to mention here, that the most important consequence of (1.11) is the so-called monotonicity formula

$$\frac{d}{dR}\int_{B_R}\frac{|\nabla u|^p}{R^{d-p}} \ge -C,$$

which holds for F satisfying some reasonable coercivity condition and also the splitting condition, see [1, 2, 3] for details. Note that from above formula we see that there is only ε step missing to prove Hölder continuity of the solution, which can be dome by the method developed in [1] for $K = \mathbb{R}^N$ and moreover the above information allows one to improve the estimates on the dimension of the singular set in case of harmonic mapping, see [3].

2. Proof of the results

The proof is split onto two parts. First, we prove Theorem 1.1 for the case (A1). Second, we focus on the proof of Theorem 1.1 - case (A2) and simultaneously onto the proof of Theorem 1.2.

2.1. **Theorem 1.1 - the case (A1).** We start with the case (A1). We recall the weak formulation of (1.6)

$$\int_\Omega F_\eta(x,u(x),\nabla u(x))\cdot\nabla\varphi(x)+F_u(x,u(x),\nabla u(x))\cdot\varphi(x)\,dx = \int_\Omega b(x)\cdot\varphi(x)\,dx \tag{2.1}$$

that is valid for all $\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. Since we assume more regularity on u, we can incorporate such a regularity with the assumptions on F and b, and we see that (2.1) is valid for all $\varphi\in W_0^{1,\frac{p+1}{2}}\cap L^{p+1}(\Omega;\mathbb{R}^N)$. Our goal is to set $\varphi:=\sum_{i=1}^d\psi_iD_iu$ in (2.1). Since ψ is Lipschitz, it is evident (due to our assumptions on u) that $\varphi\in W^{1,\frac{p+1}{2}}\cap L^{p+1}(\Omega;\mathbb{R}^N)$. Thus, we just need to check that ψ has zero trace (note that this is automatically fulfilled if ψ has compact support in Ω). First, because u is zero on the boundary, then necessarily ∇u has only normal component not equal to zero on the boundary. Thus, assuming that ψ has zero normal component identically zero on $\partial\Omega$ we deduce that also φ has zero trace. Therefore, using this choice of φ we get

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(x,u(x),\nabla u(x))D_i(\psi_j(x)D_ju^\alpha(x))\,dx\\&\quad+\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^d F_{u^\alpha}(x,u(x),\nabla u(x))\psi_i(x)D_iu^\alpha(x)\,dx\\&=\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^d b^\alpha(x)\psi_i(x)D_iu^\alpha(x)\,dx.\end{aligned}\tag{2.2}$$

Since the first term can be rewritten as

$$\begin{aligned}&\sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(x,u(x),\nabla u(x))D_i(\psi_j(x)D_ju^\alpha(x))\\&=\psi(x)\cdot\nabla F(x,u(x),\nabla u(x))-\sum_{\alpha=1}^N\sum_{i=1}^d F_{u^\alpha}(x,u(x),\nabla u(x))\psi_i(x)D_iu^\alpha(x)\\&\quad-\sum_{i=1}^d F_{x_i}(x,u(x),\nabla u(x))\psi_i(x)+\sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(x,u(x),\nabla u(x))D_i\psi_j(x)D_ju^\alpha(x),\end{aligned}$$

we immediately deduce (1.11), which finishes the first part of the proof.

2.2. **Theorem 1.1 - the case (A2) & Theorem 1.2.** In this case, we first assume that $\psi\in\mathcal{D}(\Omega;\mathbb{R}^d)$ is fixed and for arbitrary $t\in\mathbb{R}$ we define the mapping $g_t:\mathbb{R}^d\to\mathbb{R}^d$ by the following

$$g_t(x):=x+t\psi(x).\tag{2.3}$$

Then, it is clear that there exists $t_0>0$ such that for any $t\in(0,t_0)$ the mapping g_t is a bijection with Lipschitz inverse g_t^{-1}. Since ψ is compactly supported, it is evident that t_0 can be found in such a way that $g_t(\Omega)=\Omega$ for all $t\in(0,t_0)$.

Moreover, we can compute the Jacobian of g_t and g_t^{-1} and they are of the form

$$
\begin{aligned}
J_t(x) &:= \det \nabla g_t(x) = 1 + t \operatorname{div} \psi(x) + t^2 h(t,x),\\
J_t^{-1}(x) &:= \det \nabla g_t^{-1}(x) = \frac{1}{1 + t \operatorname{div} \psi(g_t^{-1}(x)) + t^2 h(t, g_t^{-1}(x))},
\end{aligned} \tag{2.4}
$$

where h is a bounded function. Note that since ψ is Lipschitz, it is evident that for some t_0 there holds $|J_t(x)| + |J_t^{-1}(x)| \le C$ for all $t \in (0, t_0)$ and almost all $x \in \Omega$. Since Ω is Lipschitz, we know that we can find a sequence $u^n \in \mathcal{C}^1(\Omega; \mathbb{R}^N)$ such that

$$
u^n \to u \qquad \text{strongly in } W^{1,p}(\Omega; \mathbb{R}^N) \tag{2.5}
$$

Then, for any $t \in (0, t_0)$ and any $n \in \mathbb{N}$ we define $v_t^n \in W^{1,p}(\Omega; \mathbb{R}^N)$ by the formula (note here, that now it is a meaningful definition due to the smoothness of u^n)

$$
v_t^n(x) := u^n(g_t(x)).
$$

Next, we would like to use (1.2) and to set there $v := v_t^n$. However, this would be possible only for Theorem 1.1, where S is the whole space. On the other hand, it is an incorrect setting in case of Theorem 1.2, since it is not true in principle that $v_t^n \in S$. Therefore, we proceed slightly differently. First, we show that v_t^n is a Cauchy sequence. Indeed, by using the substitution theorem and the definition of v_t^n, we have (I denotes the identity matrix)

$$
\begin{aligned}
&\int_\Omega |v_t^n(x) - v_t^m(x)|^p + |\nabla(v_t^n(x) - v_t^m(x))|^p \, dx\\
&\quad = \int_\Omega |u^n(x) - u^m(x)|^p J_t^{-1}(x) \, dx\\
&\qquad + \int_\Omega |(\nabla(u^n(x) - u^m(x)))(I + t\nabla\psi(g_t^{-1}(x)))J_t^{-1}(x) \, dx\\
&\quad \le C\|u^n - u^m\|_{1,p}^p,
\end{aligned} \tag{2.6}
$$

where for the last inequality, we used the facts that ψ is Lipschitz and J_t^{-1} is bounded. Consequently, since also u^n is Cauchy, we can find $v_t \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that

$$
v_t^n \to v_t \qquad \text{strongly in } W^{1,p}(\Omega; \mathbb{R}^N). \tag{2.7}
$$

In addition, since ψ has a compact support, it is evident that $v_t = u$ on $\partial\Omega$. Moreover, using the fact that K is closed we can deduce that due to the strong convergence of u^n and from the fact that g_t is a bijection and since J_t and J_t^{-1} are bounded that also necessarily $v_t \in W^{1,p}(\Omega; K)$. Therefore, it is a correct comparison function in (1.2), which directly implies

$$
\begin{aligned}
&\int_\Omega F(x, u(x), \nabla u(x)) - b(x) \cdot u(x) \, dx\\
&\quad \le \int_\Omega F(x, v_t(x), \nabla v_t(x)) - b(x) \cdot v_t(x) \, dx\\
&\quad = \lim_{n\to\infty} \int_\Omega F(x, v_t^n(x), \nabla v_t^n(x)) - b(x) \cdot v_t^n(x) \, dx,
\end{aligned} \tag{2.8}
$$

where the second equality follows from the properties of F (continuity and the growth assumption) and from the strong convergence result (2.7). Next, we identify the limit on the right hand side in a different way and we focus on the limit in the term with F. First, using the definition of v_t^n (and its smoothness), we observe that

$$\frac{\partial v_t^n(x)}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial u^n(g_t(x))}{\partial x_j} \frac{\partial (g_t(x))_j}{\partial x_i}$$

and by using the definition of g_t we deduce (now I denotes the identity matrix) that

$$\nabla v_t^n(x) = \nabla u^n(g_t(x))(I + t\nabla\psi(x)).$$

Thus, using this relation in the definition of F and arguing by the substitution theorem, we find that

$$\begin{aligned}\int_\Omega F(x, v_t^n(x), \nabla v_t^n(x))\, dx &= \int_\Omega F\left(x, u^n(g_t(x)), \nabla u^n(g_t(x))(I + t\psi(x))\right)\, dx\\ &= \int_\Omega F(g_t^{-1}(x), u^n(x), \nabla u^n(x)(I + t\nabla\psi(g_t^{-1}(x))))J_t^{-1}(x)\, dx.\end{aligned}$$

Due to the properties of g_t, the growth assumptions (1.3) and the convergence property (2.5), we can easily let $n \to \infty$ in (2.8) with the term with F to deduce

$$\begin{aligned}&\int_\Omega F(x, u(x), \nabla u(x)) - b(x)\cdot(u(x) - v_t(x))\, dx\\ &\le \int_\Omega F(g_t^{-1}(x), u(x), \nabla u(x)(I + t\nabla\psi(g_t^{-1}(x))))J_t^{-1}(x)\, dx.\end{aligned} \tag{2.9}$$

Finally, we divide (2.9) by t and let $t \to 0_+$. First, we focus on the term with b. We start with the observation that there exists t_0 such that for all $t \in (0, t_0)$ we have

$$\int_\Omega \frac{|v_t(x) - u(x)|^p}{t^p}\, dx \le C(u, \psi). \tag{2.10}$$

Indeed, using (2.5) and (2.7), we see that

$$\|v_t - u\|_p^p = \lim_{n\to\infty} \|v_t^n - u^n\|_p^p \tag{2.11}$$

and therefore we estimate only the term on the right hand side of (2.11). Since both v_t^n and u^n are smooth, we can observe by using the Jensen inequality that for all $x \in \Omega$ there holds

$$\begin{aligned}|v_t^n(x) - u^n(x)|^p = |u^n(g_t(x)) - u^n(x)|^p &= \left|\int_0^1 \frac{d}{d\tau} u^n(x + \tau(g_t(x) - x))\, d\tau\right|^p\\ &= \left|\int_0^1 \frac{d}{d\tau} u^n(x + \tau t\psi(x))\, d\tau\right|^p\\ &\le t^p \int_0^1 |\nabla u^n(x + \tau t\psi(x))|^p |\nabla\psi(x)|^p\, d\tau\\ &\le Ct^p \int_0^1 |\nabla u^n(x + \tau t\psi(x))|^p\, d\tau.\end{aligned}$$

Hence using the substitution and the Fubini theorem, we see that

$$\begin{aligned}\int_\Omega |v_t^n(x)-u^n(x)|^p\,dx &\le Ct^p\int_\Omega\int_0^1 |\nabla u^n(x+\tau t\psi(x))|^p\,d\tau\,dx\\ &= Ct^p\int_0^1\int_\Omega |\nabla u^n(x)|^p J_{\tau t}^{-1}(x)\,dx\,d\tau\\ &\le Ct^p\|\nabla u^n\|_p^p \le Ct^p\end{aligned}$$

and with the help of (2.11), the estimate (2.10) easily follows. Consequently, due to the reflexivity of L^p we can find a not relabeled subsequence such that

$$\frac{v_t-u}{t} \rightharpoonup \overline{U} \qquad \text{weakly in } L^p(\Omega;\mathbb{R}^N). \tag{2.12}$$

Due to the uniqueness of the weak limit, we can identify the weak limit $\overline{U}$ as follows. For an arbitrary smooth $z\in\mathcal{D}(\Omega;\mathbb{R}^N)$ we can use the substitution theorem, (2.5) and (2.7) to obtain

$$\begin{aligned}\int_\Omega z(x)\cdot\frac{v_t(x)-u(x)}{t}\,dx &= \lim_{n\to\infty}\int_\Omega z(x)\cdot\frac{u^n(g_t(x))-u(x)}{t}\,dx\\ &= \lim_{n\to\infty}\int_\Omega \frac{z(g_t^{-1}(x))\cdot u^n(x)J_t^{-1}(x)-z(x)\cdot u(x)}{t}\,dx.\\ &= \int_\Omega \frac{z(g_t^{-1}(x))J_t^{-1}(x)-z(x)}{t}\cdot u(x)\,dx.\end{aligned} \tag{2.13}$$

First, we can decompose the term on the right hand side as

$$\frac{z(g_t^{-1}(x))J_t^{-1}(x)-z(x)}{t} = z(g_t^{-1}(x))\frac{J_t^{-1}(x)-1}{t}+\frac{z(g_t^{-1}(x))-z(x)}{t}. \tag{2.14}$$

Then, using the definition of J_t^{-1} (2.4) we have

$$\lim_{t\to0}\frac{J_t^{-1}(x)-1}{t} = -\lim_{t\to0}\frac{\operatorname{div}\psi(g_t^{-1}(x))+th(t,g_t^{-1}(x))}{1+t\operatorname{div}\psi(g_t^{-1}(x))+t^2h(t,g_t^{-1}(x))} = -\operatorname{div}\psi(x) \tag{2.15}$$

Similarly, using the definition of g_t (2.3) we see that

$$\lim_{t\to0}\frac{g_t^{-1}(x)-x}{t} = \lim_{t\to0}\frac{g_t^{-1}(x)-g_t(g_t^{-1}(x))}{t} = -\lim_{t\to0}\psi(g_t^{-1}(x)) = -\psi(x), \tag{2.16}$$

and consequently, we have

$$\begin{aligned}\lim_{t\to0}\frac{z(g_t^{-1}(x))-z(x)}{t} &= \lim_{t\to0}\int_0^1 \frac{d}{d\tau}\frac{z(x-\tau(x-g_t^{-1}(x)))}{t}\,d\tau\\ &= \lim_{t\to0}\int_0^1 \frac{(g_t^{-1}(x))-x}{t}\cdot\nabla z(x-\tau(x-g_t^{-1}(x)))\,d\tau\\ &= -\psi(x)\cdot\nabla z(x).\end{aligned} \tag{2.17}$$

Finally, we substitute (2.14)–(2.17) into (2.13) and let $t \to 0$ to deduce (with the help of the Lebesgue dominated convergence theorem) that

$$\begin{aligned}\lim_{t\to 0_+} \int_\Omega z(x) \cdot \frac{v_t(x) - u(x)}{t}\, dx \\ &= -\int_\Omega z(x)\cdot u(x) \operatorname{div} \psi(x) + \nabla z(x) \cdot (u(x) \otimes \psi(x))\, dx \\ &= \int_\Omega \sum_{\alpha=1}^{N} \sum_{k=1}^{d} z^\alpha(x) D_k u^\alpha(x) \psi_k(x)\, dx,\end{aligned}$$

where the last identity follows from integration by parts. Consequently, due to the uniqueness of the weak limit we can identify $\overline{U}$ and we have

$$\lim_{t\to 0_+} \int_\Omega b(x) \cdot \frac{v_t(x) - u(x)}{t}\, dx = \int_\Omega \sum_{\alpha=1}^{N} \sum_{k=1}^{d} b^\alpha(x) D_k u^\alpha(x) \psi_k(x)\, dx. \tag{2.18}$$

Thus, we finished the limiting procedure in the term with b. Therefore, it remains to discuss also the limit $t \to 0_+$ in terms with F. First, we decompose the remaining terms as

$$\begin{aligned}&F(g_t^{-1}(x), u(x), \nabla u(x)(I + t\nabla\psi(g_t^{-1}(x))))J_t^{-1}(x) - F(x, u(x), \nabla u(x)) \\ &= F(g_t^{-1}(x), u(x), \nabla u(x)(I + t\nabla\psi(g_t^{-1}(x))))(J_t^{-1}(x) - 1) \\ &\quad + F(g_t^{-1}(x), u(x), \nabla u(x)(I + t\nabla\psi(g_t^{-1}(x)))) - F(g_t^{-1}(x), u(x), \nabla u(x)) \\ &\quad + F(g_t^{-1}(x), u(x), \nabla u(x)) - F(x, u(x), \nabla u(x)) \\ &=: tI_1(t,x) + tI_2(t,x) + tI_3(t,x).\end{aligned} \tag{2.19}$$

Then using (2.15) and the growth assumption (1.3), we can deduce with the help of the Lebesgue dominated convergence theorem (note that since ψ is Lipschitz it has bounded gradient almost everywhere in Ω) that

$$\lim_{t\to 0_+} \int_\Omega I_1(t,x)\, dx = -\int_\Omega F(x, u(x), \nabla u(x)) \operatorname{div} \psi(x)\, dx. \tag{2.20}$$

Second, since F is $\mathcal{C}^1$ we rewrite the last term in the following way

$$\begin{aligned}I_3(t,x) &= \frac{1}{t} \int_0^1 \frac{d}{d\tau} F(x - \tau(x - g_t^{-1}(x)), u(x), \nabla u(x))\, d\tau \\ &= \int_0^1 \sum_{i=1}^{d} F_{x_i}(x - \tau(x - g_t^{-1}(x)), u(x), \nabla u(x)) \frac{(g_t^{-1}(x))_i - x_i}{t}\, d\tau.\end{aligned} \tag{2.21}$$

Hence, we see that it follows from (1.7) and also the definition of g_t and (2.16) that

$$|I_3(t,x)| \le C(\psi)(|\nabla u(x)|^p + |u(x)|^q + 1 + f(x)), \tag{2.22}$$

which is an integrable function. Moreover, using (2.16), we directly obtain (using also the definition of g_t) that for almost all $x \in \Omega$

$$\lim_{t\to 0_+} I_3(t,x) = -\sum_{i=1}^{d} F_{x_i}(x, u(x), \nabla u(x)) \psi_i(x). \tag{2.23}$$

Consequently, using (2.22), (2.23) and the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{t\to 0_+}\int_\Omega I_3(t,x)\,dx = -\int_\Omega \sum_{i=1}^d F_{x_i}(x,u(x),\nabla u(x))\psi_i(x)\,dx. \tag{2.24}$$

Finally, for the last term we again use the fact that F is $\mathcal{C}^1$ function and we rewrite I_2 in the following way

$$\begin{aligned} I_2(t,x) &= \frac{1}{t}\int_0^1 \frac{d}{d\tau} F(g_t^{-1}(x),u(x),\nabla u(x)+t\tau(\nabla u(x)\nabla\psi(g_t^{-1}(x))))\,d\tau \\ &= \int_0^1 \sum_{\alpha=1}^N\sum_{i,j=1}^d F_{\eta_i^\alpha}(g_t^{-1}(x),u(x),\nabla u(x)+t\tau(\nabla u(x)\nabla\psi(g_t^{-1}(x))))\cdot \\ &\quad\cdot D_j u^\alpha(x) D_i\psi_j(g_t^{-1}(x))\,d\tau. \end{aligned} \tag{2.25}$$

Thus, using (1.7) and the Young inequality, we see that (note that ψ is Lipschitz)

$$|I_2(t,x)| \le C(\psi)(1+|\nabla u(x)|^p + |u(x)|^q + f(x)). \tag{2.26}$$

Moreover, it is easy to let $t\to 0_+$ in (2.25) to get that for almost all $x\in\Omega$ there holds

$$\lim_{t\to 0_+} I_2(t,x) = \sum_{\alpha=1}^N\sum_{i,j=1}^d F_{\eta_i^\alpha}(x,u(x),\nabla u(x)) D_j u^\alpha(x) D_i\psi_j(x). \tag{2.27}$$

Hence, it directly follows from (2.26), (2.27) and the Lebesgue dominated convergence theorem that

$$\lim_{t\to 0_+}\int_\Omega I_2(t,x)\,dx = \int_\Omega \sum_{\alpha=1}^N\sum_{i,j=1}^d F_{\eta_i^\alpha}(x,u(x),\nabla u(x)) D_j u^\alpha(x) D_i\psi_j(x)\,dx. \tag{2.28}$$

Thus, dividing (2.9) by t, letting $t\to 0_+$ and using (2.18), (2.19), (2.20), (2.24) and (2.28) we obtain the following inequality

$$\begin{aligned} &\int_\Omega \sum_{\alpha=1}^N\sum_{k=1}^d b^\alpha(x) D_k u^\alpha(x)\psi_k(x)\,dx + \int_\Omega F(x,u(x),\nabla u(x))\operatorname{div}\psi(x)\,dx \\ &\le -\int_\Omega \sum_{i=1}^d F_{x_i}(x,u(x),\nabla u(x))\psi_i(x)\,dx \\ &\quad + \int_\Omega \sum_{\alpha=1}^N\sum_{i,j=1}^d F_{\eta_i^\alpha}(x,u(x),\nabla u(x)) D_j u^\alpha(x) D_i\psi_j(x)\,dx. \end{aligned} \tag{2.29}$$

Since ψ was arbitrary, the same inequality must hold also for $-\psi$ and therefore (2.29) holds with the equality sign which is nothing else than (1.11).

Finally, we focus on the last part of Theorem 1.1, i.e., the case that ψ has zero normal component on the boundary. We proceed exactly in the same manner as above but we change the definition of g_t in order to preserve the zero trace of the test function. First, since $u\in W_0^{1,p}$ we extend it by 0 outside Ω. The same extension is then used for b. For simplicity (but without loss of generality) we assume in what follows that $F(x,0,0) = 0$. With this simplification we can extend the integration domain and to conclude that

$$\int_{\mathbb{R}^d} F(x, u(x), \nabla u(x)) - b(x) \cdot u(x)\, dx \le \int_{\mathbb{R}^d} F(x, v(x), \nabla v(x)) - b(x) \cdot v(x)\, dx$$

for all $v \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^N)$ being identically equal to 0 outside Ω. Our main goal is to choose v in a proper way. Note that the choice $v(x) := u(x + t\psi(x))$ is not allowed in general[3] since v does not have zero trace on $\partial\Omega$. Therefore we must correct the definition of g_t. Thus, let ψ be fixed Lipschitz function having zero normal component on Ω and for simplicity assume that it has compact support in $\mathbb{R}^d$. Since $\Omega \in \mathcal{C}^{1,1}$ there surely exists a Lipschtiz mapping $\tilde{\nu} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\tilde{\nu}(x) = \nu(x)$ on $\partial\Omega$ where $\nu(x)$ denotes the outer normal vector at point $x \in \partial\Omega$. Then we define

$$g_t(x) := x + t\psi(x) + Ct^2\tilde{\nu}(x).$$

Since ψ is tangential on the boundary and $\tilde{\nu}$ is normal, we see that (for fixed ψ) there exists $C > 0$ and t_0 such that for all $t \in (0, t_0)$ there holds

$$g_t(x) \notin \Omega \qquad \text{for all } x \in \partial\Omega.$$

Consequently, we see that $v := u \circ g_t \in W_0^{1,p}(\Omega; \mathbb{R}^N)$ can be used as a test function. Then the proof follows line by line the proof for ψ having compact support, since the pollution term with $\tilde{\nu}$ is quadratic with respect to t and therefore all dependence on $\tilde{\nu}$ vanishes as we let $t \to 0_+$. Thus, the proof is complete.

References

[1] M. Bulíček and J. Frehse. $\mathcal{C}^\alpha$-regularity for a class of non-diagonal elliptic systems with p-growth. *Calc. Var. Partial Differential Equations*, 43(3):441–462, 2012.

[2] M. Bulíček, J. Frehse, and M. Steinhauer. Everywhere $\mathcal{C}^\alpha$-estimates for a class of nonlinear elliptic systems with critical growth. *Advances in Calculus of Variations*, online first, 2013.

[3] Lawrence C. Evans. Partial regularity for stationary harmonic maps into spheres. *Arch. Rational Mech. Anal.*, 116(2):101–113, 1991.

[4] Mariano Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1983.

[5] Stefan Hildebrandt. Harmonic mappings of Riemannian manifolds. In *Harmonic mappings and minimal immersions (Montecatini, 1984)*, volume 1161 of *Lecture Notes in Math.*, pages 1–117. Springer, Berlin, 1985.

[6] Emmy Noether. Invariant variation problems. *Transport Theory Statist. Phys.*, 1(3):186–207, 1971. Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1918, 235–257).

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[3]However, it is a possible setting in case that Ω is convex. Indeed for convex domain it follows from the fact that ψ has zero normal component on boundary, that $x + t\psi(x) \notin \Omega$. Therefore since u is extended by zero outside Ω we get $u(x + t\varphi(x)) = 0$ for all $x \in \partial\Omega$.