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A mesoscopic thermomechanically-coupled model for thin-film shape-memory alloys by dimension reduction and scale transition

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Abstract

We design a new mesoscopic thin-film model for shape-memory materials which takes into account mutual thermomechanical effects. Starting from a microscopic thermodynamical bulk model we guide the reader through a suitable dimension-reduction procedure followed by a scale transition valid for specimen large in area up to a limiting model which describes microstructure by means of parametrized measures. All our models obey the second law of thermodynamics and possess suitable weak solutions. This is shown for the resulting thin-film models by making the procedure described above mathematically rigorous.The main emphasize is, thus, put on modeling and mathematical treatment of conjoint interactions of mechanical and thermal effects accompanying phase transitions and on overpassing specimen dimensions and material scales.

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1 Introduction

Shape-memory alloys (SMAs) belong to the group of so-called *smart* materials owing to their outstanding response to thermal and/or mechanical loads. In particular, they exhibit the *shape-memory effect* related to recovery from deformation by heat supply. The remarkable behavior of SMAs is due to a diffusionless solid-to-solid phase transition (*martensitic transformation*) characterized by a change in the crystal lattice; in particular, the specimen can transit from a phase of higher symmetry of the crystal lattice, called *austenite*, to a phase with a less symmetric lattice, referred to as *martensite*. Martensite exists in many symmetry-related variants. Hence, the aforementioned phase transition is often accompanied by fast spatial oscillations of the deformation gradient in martensite, the so-called *microstructure*. A SMA specimen can, then, by restructuring this microstructure (sometimes referred to as *reorientation*) compensate mechanical loads, which is a key ingredient for its thermo-mechanical response.

Due to their particular multiscale character, when changes of the crystal lattice lead to extra-ordinary response on macroscale, SMAs have been in the scope of research of physicists, mathematicians and engineers for the last decades, cf. the monographs [9, 18, 25, 38, 42] for example. In particular, developing reliable models on various time and length-scales as well as surpassing scales is still a big challenge to these communities [41].

Models of the behavior of SMAs then serve for experiment interpretation or when tailoring SMA samples to a specific application area like to surgical tools or stents (for which SMAs are already widely used nowadays [20]); cf. [48] also for other applications. Thus, a large number of models has been developed for specific scales and/or loading regimes, see, e.g., [46] for a survey.

Within this contribution, we consider only continuum-mechanics based models operating on the single-crystalline level. Following [46], such models can be divided into *microscopic* and *mesoscopic* ones; the crucial difference is that microscopic models operate on the scale of several μm's and record fully the oscillations of the deformation gradient while mesoscopic models record only asymptotics of fine oscillations, e.g. in terms of Young measures generated by gradients (cf. [28]) and are suited for laboratory-sized specimen. Even though, as mentioned, the modelling effort has been large in the past decades, a model for single-crystalline

SMAs on the mesoscopic scale that would reflect the thermo-mechanically coupled nature of SMAs has been proposed only very recently [7].

The main goal of this contribution is to *adapt* the aforementioned model [7] to the *special geometry of thin films*. Indeed, this adaptation is of importance since thin-film specimens are widely used for their microactuator behavior in micro-electro-mechanical (MEMS) devices as they are able to form, under certain circumstances, tents and tunnels [10, 19, 37]. They profit from the fact that the sizes of these components can be reduced significantly without affecting their functionality that, as explained above, stems merely from crystallographic changes; hence, actuators from SMAs possess a significant power–weight ratio [40].

Dimension reduction, i.e. the rigorous limit procedure when one dimension of the specimen becomes negligible, forms an important tool for obtaining models for the thin-film geometry. In the context of SMAs, this 3D-2D dimension reduction has been performed in the static case; see [10] for the static analysis on the micro- or [30] on the macro-scale (the transition from the first to the latter was shown by Shu [49]), or on a purely mesoscopic level [15, 24]; similar procedures are used also in the context of multimaterials [8]. Nevertheless, a dimension reduction in the evolutionary mesoscopic model capturing thermo-mechanical coupling is, to our best knowledge, still missing in the literature.

Thus, we fill this gap by rigorously deriving a thin-film model in the thermo-mechanically coupled setting. To reach this goal we propose (see Section 2) a two-step procedure: starting from the microscopic thermodynamically consistent hyperelastic bulk model [7], we perform the dimension reduction and then we upscale to a mesoscopic model.

This paper is structured as follows. First, in Section 2, we formally review bulk and thin-film microscopic models which are a starting point of our consideration and which furnish us with ingredients needed for the limiting mesoscopic one. Then in Section 3, we review the existence of a suitably defined weak solution to the microscopic model and, in Section 4, we pass to a thin-film limiting model as the material thickness goes to zero. Finally, Section 5 is devoted to the existence of a weak solution to a mesoscopic model stemming from the microscopic one by omitting surface energy terms.

2 Considered models and captured effects

In this section, let us shortly introduce the models considered in this contribution and highlight the main effects they capture. As mentioned, the goal of this contribution is to develop a mesoscopic, thermomechanically-coupled model in the thin-film geometry.

It is well known [17] that mesoscopic models form a good approximation of microscopic models when the size of the specimen becomes much larger than the size of the microstructure formed in the specimen; e.g., in [17] the volume of the specimen approached $+\infty$. Now, the assumption that the volume of the specimen is infinitely large does not seem to be compatible with the assumption that one dimension of the specimen becomes negligible—which characterizes the dimension reduction. Therefore, we perform the following two-step limiting procedure

$$\text{Microscopic bulk model} \to \text{Microscopic thin-film model} \to \text{Mesoscopic thin-film model},$$

i.e. we consider a thermomechanically coupled model for bulk SMAs that fully resolves the microstructure and let one dimension of the specimen vanish in the first step. We, thus, obtain a thin-film model that is again thermomechanically coupled and fully resolves the microstructure (*microscopic thin-film model*); in this model, we perform then the upscaling for thin-films large in area to obtain the mesoscopic thin-film model. This sequence of reasoning is kept throughout the article.

2.1 Microscopic bulk model

The starting point of our analysis shall be a microscopic bulk model, analogous to [7], defined in the framework of generalized standard materials, cf. [26]. Take $\Omega_\varepsilon \subset \mathbb{R}^3$ (the reference configuration of the body), $\varepsilon > 0$, such that

$$\Omega_\varepsilon := \omega \times (0, \varepsilon) \quad \text{for some} \quad \omega \subset \mathbb{R}^2, \tag{1}$$

as usual in dimension reduction problems; here ω, the plane of the film, is a bounded Lipschitz domain in the (x_1, x_2) plane with disjoint boundary segments $\gamma_D \cup \gamma_N \cup N = \partial\omega$, where γ_D is the part of the boundary where Dirichlet boundary condition is prescribed, on γ_N we demand a Neumann boundary conditions and N is a null set; moreover, ε is the thickness measure of the body. Furthermore, time $t \in [0, T]$ shall be considered on a finite time horizon $0 < T < +\infty$, and we denote $Q_\varepsilon := [0, T] \times \Omega_\varepsilon$ the space-time cylinder, its boundary $\Sigma^\varepsilon := [0, T] \times \partial\Omega_\varepsilon$, while $\Sigma^\varepsilon_N := [0, T] \times \Gamma^\varepsilon_N$ for $\Gamma^\varepsilon_N := \gamma_N \times (0, \varepsilon)$; Σ^ε_D and Γ^ε_D analogously.

In what follows, $y(t)\colon \Omega_\varepsilon \to \mathbb{R}^3$ will denote the deformation of Ω_ε at each time instance $t \in [0,T]$. The set of state variables further includes the *temperature* $\theta : Q_\varepsilon \to \mathbb{R}$ and an internal variable, namely a vectorial *phase field* $\lambda : Q_\varepsilon \to \mathbb{R}^{M+1}$ that, up to small mismatch, corresponds to the vector of volume fractions of the variants of martensite and/or the austenite phase. Indeed, when assuming that the considered material can exist in $M \in \mathbb{N}$ variants of martensite, together with the austenite we have possible $M+1$ states of the specimen. Hence, we may introduce $\mathcal{L} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{M+1}$ a continuous, frame-indifferent, (i.e. $\mathcal{L}(F) = \mathcal{L}(RF)$ for every $R \in \mathrm{SO}(3)$ and every $F \in \mathbb{R}^{3\times 3}$) bounded mapping such that

$$\mathcal{L}(\nabla y)_i = \begin{cases} \text{volume fraction of the } i\text{'s variant of martensite} & \text{if } i \leq M, \\ \text{volume fraction of austenite} & \text{if } i = M+1; \end{cases}$$

e.g. $\mathcal{L}(\cdot)_i$ can be chosen such that it equals one near the respective well and vanishes far from it [29]. We then assume that $\lambda \sim \mathcal{L}(\nabla y)$, the size of the mismatch is controlled by the penalty term in (2). Moreover, we follow the modelling assumption that the evolution of the internal variable leads to energy dissipation (so, indirectly, change of the ratio of the martensitic variants and/or austenite phase leads to dissipation).

Within the framework of generalized standard solids, we have to constitutively define two potentials: the Gibbs free energy $\mathcal{G}^\varepsilon_\eta$ and a dissipation potential $\mathcal{R}^\varepsilon_\eta$ (the two parameters denote the dependence on both the bulk thickness ε and the parameter η governing microscopic effects). Here we confine ourselves to the following forms of the two potentials:

$$\begin{aligned}\mathcal{G}^\varepsilon_\eta(t,y,\lambda,\theta) = \underbrace{\int_{\Omega_\varepsilon} H(\nabla y,\lambda,\theta)\,\mathrm{d}x}_{\text{Helmholtz free energy}} \underbrace{-\int_{\Omega_\varepsilon} f(t)\cdot y\,\mathrm{d}x - \int_{\Gamma^\varepsilon_N} g(t)\cdot y\,\mathrm{d}S}_{\text{external loading}} \\ + \eta\underbrace{\left(\left\|\nabla^2 y\right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times3\times3})} + \|\nabla\lambda\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{(M+1)\times 3})}\right)}_{\text{interfacial energy}} + \kappa\underbrace{\|\lambda - \mathcal{L}(\nabla y)\|^2_{W^{-1,2}(\Omega_\varepsilon;\mathbb{R}^{M+1})}}_{\text{penalty term}}\end{aligned} \tag{2}$$

following [21, 44], we propose the following partially linearized ansatz

$$H(F,\lambda,\theta) := W(F) + Z(\theta) + (\theta - \theta_{\mathrm{tr}})\mathfrak{a}\cdot\lambda, \quad \forall F \in \mathbb{R}^{3\times 3},\ \lambda \in \mathbb{R}^{M+1},\ \theta > 0, \tag{3}$$

where $\theta_{\mathrm{tr}} > 0$ is the temperature at which austenite and martensite are energetically equal, W is the purely mechanic part of the Helmholtz free energy, Z purely thermal part and $\mathfrak{a} := (0,0,\ldots,0,-s_{\mathrm{tr}})^\top$ with s_{tr} being a specific transformation entropy, which corresponds, roughly, to the Clausius–Clapeyron constant multiplied by the transformation strain, cf. [4, 29]. Also, the transformation entropy is proportional to the latent heat. Let us note that the thermomechanical coupling term is the leading order in the *chemical energy* [50].

When choosing W of a multi-well character with the individual wells manifesting the variants of martensite and the austenitic phase, this choice allows the model to predict the formation of microstructure, or in other words, oscillations of the deformation gradient. Now, as the interfacial energy in (2) (the form is chosen following, e.g., [9, 38]) has a compactifying effect, the size of the microstructure is controlled by η.

The dissipation potential is chosen in the form

$$\mathcal{R}^\varepsilon_\eta(\dot y,\dot\lambda) = \int_{\Omega_\varepsilon} \eta|\nabla\dot y| + \frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda)\,\mathrm{d}x, \tag{4}$$

with real constants $\alpha > 0$ and $q \geq 2$, the dot standing for $\dot h := \frac{\partial h}{\partial t}$. The last term $\delta^*_S(\dot\lambda)$, the Legendre–Fenchel conjugate of the indicator function of a bounded convex neighborhood S of the origin $0 \in \mathbb{R}^{M+1}$, is considered 1-homogeneous (to capture dissipation due to rate-independent processes—considered dominant) and non-smooth at $\delta^*_S(0)$ (to assure that the change of the phase variable—and, in particular, also the martensite/austenite transition—is an activated process). The term $\frac{\alpha}{q}|\dot\lambda|^q$ corresponds to dissipation due to rate-dependent processes which needs to be included at time-scales where heat conduction takes place, cf. [12]. Finally, the term $\eta|\nabla\dot y|$ models pinning effects, cf. [1], which will vanish on the mesoscopic scale.

The evolution of the state variables is then standardly [26], in quasistatic approximation, governed by the following inclusions accompanied with the balance of the entropy s:

$$\partial_{\dot y}\mathcal{R}^\varepsilon_\eta(\dot y,\dot\lambda) + \partial_y\mathcal{G}^\varepsilon_\eta(t,y,\lambda,\theta) \ni 0, \tag{5a}$$

$$\partial_{\dot\lambda}\mathcal{R}^\varepsilon_\eta(\dot y,\dot\lambda) + \partial_\lambda\mathcal{G}^\varepsilon_\eta(t,y,\lambda,\theta) \ni 0, \tag{5b}$$

$$\theta\dot s + \operatorname{div} j = \partial\left(\frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda)\right)\dot\lambda + \eta|\nabla\dot y|. \tag{5c}$$

In the last equation, j stands for the heat flux and shall be assumed to be governed by the Fourier law, i.e. $j = -\mathbb{K}(\lambda, \theta)\nabla\theta$ with $\mathbb{K}$ being the heat conductivity tensor. Moreover, ∂ is the convex sub-differential which we used in (5a) only formally (since $\mathcal{G}_\eta^\varepsilon(t, y, \lambda, \theta)$ is not convex). We shall give a rigorous weak formulation of the system (5) in Section 3 – here, for highlighting ideas, we believe the formal system is sufficient.

Remark 1 (Boundary conditions)**.** *The system* (5)*, of course, needs to be furnished with appropriate boundary conditions. As it turns out, this is rather non-trivial due to the fact that we included the second gradients in the Gibbs free energy through its interfacial part. Due to this fact, we have to work in the context of so-called* non-simple *continua where boundary conditions have to prescribed with special care (see e.g. [45]). We shall, thus, assume that the boundary conditions for* (5a) *in the strong formulation are such that they "vanish" in weak formulation. The entropy equation* (5c) *is, nonetheless, furnished by Robin-type boundary conditions, cf. Section 3.*

To summarize, the system (5) records formation of *microstructure of finite width* in martensite as well as its dissipative evolution that is linked to thermal effects, in particular, the shape-memory effect (i.e. recovery from deformation by heat supply) is captured; also, an "inverse" effect is included in the model, namely, the heating/cooling of the specimen during martensitic transformation – since the latent heat in SMAs is typically larger than dissipative effects the mentioned cooling can indeed be observed [50].

2.2 Microscopic thin-film model

Now when $\varepsilon \to 0_+$ in the potentials (2)–(4), we obtain (after suitable rescaling and a careful limit procedure exposed in Section 3) the following "thin-film Gibbs free energy and dissipation potential"

$$\begin{aligned}\mathcal{G}_\eta(t, y, b, \lambda, \theta) = \; & \underbrace{\int_\omega \mathscr{H}(\nabla_p y, b, \lambda, \theta)\,\mathrm{d}z_p}_{\text{in-plane Helmholtz free energy}} \; \underbrace{- \int_\omega f^0(t)\cdot y\,\mathrm{d}z_p - \int_{\gamma_N} g^0(t)\cdot y\,\mathrm{d}S_p}_{\text{external force acting in-plane}} + \\ & \underbrace{\eta\left(\left\|\nabla_p^2 y\right\|^2_{L^2(\omega;\mathbb{R}^{3\times 2\times 2})} + 2\left\|\nabla_p b\right\|^2_{L^2(\omega;\mathbb{R}^{3\times 2})} + \left\|\nabla_p \lambda\right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{(M+1)\times 2})}\right)}_{\text{interfacial energy}} + \\ & \underbrace{\kappa\left\|\lambda - \mathcal{L}(\nabla_p y|b)\right\|^2_{W^{-1,2}(\omega;\mathbb{R}^{M+1})}}_{\text{mismatch term}},\end{aligned} \tag{6a}$$

where $\mathscr{H}(\nabla_p y, b, \lambda, \theta) = W(\nabla_p y|b) + Z(\theta) + (\theta - \theta_{\mathrm{tr}})\mathfrak{a}\cdot\lambda$, and

$$\mathcal{R}_\eta(\dot{y}, \dot{b}, \dot{\lambda}) = \int_\omega \eta|(\nabla_p \dot{y}|\dot{b})| + \frac{\alpha}{q}|\dot{\lambda}|^q + \delta_S^*(\dot{\lambda})\,\mathrm{d}z_p. \tag{6b}$$

So, the potentials (6a) and (6b) are analogous to (2) and (4) but operate only on the two-dimensional domain ω and, following [10], we obtained a further state variable b that refers to the Cosserat vector and measures the deformation of the cross-section of the thin film. All state variables y, b, λ and θ in (6a) will be shown to be independent of the third variable x_3, likewise the external forces: $f^0(t, x_1, x_2) = f(t, x_1, x_2, 0)$, $g^0(t)$ analogously. Consistently, we introduced ∇_p, the in-plane gradient, more precisely,

$$(\nabla_p u)_{ij} = \partial u_i/\partial x_j \quad \text{for any } u\colon \omega \to \mathbb{R}^d \text{ and } i = 1, \dots, 3 \text{ and } j = 1, 2; \tag{7}$$

also a point $(x_1, x_2, x_3) \in \Omega_\varepsilon$ consists of an in-plane $x_p = (x_1, x_2)$ and a normal component x_3. Lastly, we introduce the notation $(F|z) \in \mathbb{R}^{3\times 3}$ if $F \in \mathbb{R}^{3\times 2}$ and $z \in \mathbb{R}^3$ is the last column of the matrix.

With the definition of the two needed potentials at hand, we have the evolution of the thin-film specimen governed by the following system analogous to (5)

$$\partial_{(\dot{y},\dot{b})}\mathcal{R}_\eta(\dot{y}, \dot{b}, \dot{\lambda}) + \partial_{(y,b)}\mathcal{G}_\eta(t, y, b, \lambda, \theta) \ni 0, \tag{8a}$$

$$\partial_{\dot{\lambda}}\mathcal{R}_\eta(\dot{y}, \dot{b}, \dot{\lambda}) + \partial_\lambda\mathcal{G}_\eta(t, y, b, \lambda, \theta) \ni 0, \tag{8b}$$

$$\theta\dot{s} + \operatorname{div} j = \partial\left(\frac{\alpha}{q}|\dot{\lambda}|^q + \delta_S^*(\dot{\lambda})\right)\dot{\lambda} + \eta\big|(\nabla_p \dot{y}|\dot{b})\big|. \tag{8c}$$

Since the structure of the model is pertained from the bulk model, its main features are analogous to the ones highlighted in the previous subsection.

2.3 Mesoscopic thin-film model

For thin films of large area passing to the limit $\eta \to 0_+$ is justified by scaling arguments similar to [7, 17]; this limit is sometimes referred to as *relaxation*.

In such a case the interfacial energy vanishes and so the microstructure—or, in other words, oscillations of the deformation gradient—become “infinitely fine”; therefore, we need a suitable mathematical tool to capture this phenomenon. To this end, we employ here the so-called gradient Young measure $\nu \in \mathcal{G}^p_{\Gamma_D}(\Omega;\mathbb{R}^{2\times 3})$ which we shortly, we introduce these measures in Section 5; at this point it is sufficient to think of them as representatives of the “infinitely fine” microstructure. We use the operator “$\bullet$” to indicate an application of the (gradient) Young measure on its dual, a continuous function with appropriate growth at infinity.

On the other hand, in the thin-film geometry, also the Cosserat vector can form fast spatial oscillations additionally to the deformation gradient. This is caused by the fact that a thin film can form an accordion-like structure; if the area of the thin film approaches infinity also the piling up of the film into the accordion-like structure may become infinitely fine causing again “infinitely fast” oscillations of the Cosserat vector. We capture these by introducing the Young measure $\mu \in \mathcal{Y}^p_{\Gamma_D}(\Omega;\mathbb{R}^3)$.

After passing $\eta \to 0_+$, the Gibbs free energy will read as

$$\mathcal{G}(t,y,\nu,\mu,\lambda,\theta) = \underbrace{\int_\omega W\bullet(\nu,\mu) + Z(\theta) + (\theta-\theta_{\rm tr})\mathfrak{a}\cdot\lambda(t)\,\mathrm{d}z_p}_{\text{(relaxed) Helmholtz free energy}} + \underbrace{\kappa\left\|\lambda - \mathcal{L}\bullet(\nu,\mu)\right\|^2_{W^{-1,2}(\omega;\mathbb{R}^{3\times 3})}}_{\text{mismatch term}}$$
$$-\underbrace{\int_\omega f^0(t)\cdot y\,\mathrm{d}z_p - \int_{\gamma_N} g^0(t)\cdot y\,\mathrm{d}S_p}_{\text{external forces}}, \tag{9}$$

here we denoted $\nabla y = \mathrm{id}\bullet\nu_{z_p}$ for a.a. $z_p \in \omega$ the “average deformation” induced by the microstructure. Notice that the interfacial energy is missing now. Similarly, we scale pinning effects in the dissipation potential to zero and obtain

$$\mathcal{R}(\dot\lambda) = \int_\omega \frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda)\,\mathrm{d}z.$$

Again, the evolution of the state variables is governed by the following set of equations/inclusions:

$$\partial_{(\nu,\mu)}\mathcal{G}(t,y,\nu,\mu,\lambda,\theta) \ni 0, \tag{10a}$$
$$\partial_{\dot\lambda}\mathcal{R}(\dot\lambda) + \partial_\lambda\mathcal{G}(t,y,\nu,\mu,\lambda,\theta) \ni 0, \tag{10b}$$
$$\theta\dot s + \mathrm{div}\,j = \partial\left(\frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda)\right)\dot\lambda. \tag{10c}$$

In this system, in particular, (10a) is merely a formal inclusion since the set of gradient Young measures is not convex, therefore the (convex) subdifferential loses sense here. However, we shall formulate (10a) later, in Section 5, via a minimization problem which will, additionally, capture the standard assumption in quasi-static processes that the Gibbs free energy is minimized in every $t \in [0,T]$.

Lastly, let us note that this mesoscopic model does predict several geometric properties of the microstructure, on the other side, the width of the microstructure is not captured anymore. In this approximation it is so fine that it becomes a characteristic of a single material point—in accord with our intentions with the upscaling. Still, all the important effects stemming from the interplay of formation of microstructure, dissipation and heat conduction in the specimen remain included.

3 Analysis of the microscopic bulk model

Let us now review the weak formulation of (5) and a proof of existence of weak solutions following [6, 7, 44]. We start with some preparatory paragraphs introducing the necessary notation and the so-called *enthalpy transformation* that will come in handy for the analysis performed later.

To perform the latter, we first transform the entropy equation (5c) into a *heat equation* by employing the standard Gibbs relation $s = -H'_\theta$; thus getting

$$c_{\rm v}(\theta)\dot\theta - \mathrm{div}\,(\mathbb{K}(\lambda,\theta)\nabla\theta) = \frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda) + \eta|\nabla\dot y| + \theta\mathfrak{a}\cdot\dot\lambda, \tag{11}$$

where $c_{\rm v}(\theta) = -\theta H''_{\theta\theta}$ is the specific heat capacity. Note that the adiabatic term $+\theta\mathfrak{a}\cdot\dot\lambda$ results from the proposed thermo-mechanical coupling and leads (as already announced) to heating/cooling during phase transition which is actually dominant over the dissipated energy transformed to heat, as observed in experiments [50].

Reformulating this heat equation (11) through the enthalpy transformation (cf. [44], for example) by introducing the enthalpy w through

$$w = \hat c_{\rm v}(\theta) = \int_0^\theta c_{\rm v}(r)\,{\rm d}r, \tag{12}$$

one arrives to the relation

$$\dot w - {\rm div}\,(\mathcal{K}(\lambda,\theta)\nabla w) = \alpha|\dot\lambda|^q + \delta^*_S(\dot\lambda) + \eta|\nabla\dot y| + \Theta(w)\mathfrak{a}\cdot\dot\lambda, \tag{13}$$

where

$$\Theta(w) := \begin{cases} \hat c_{\rm v}^{-1}(w) = \theta, & \text{if } w \ge 0,\\ 0, & \text{otherwise}\end{cases}, \qquad \text{and} \qquad \mathcal{K}(\lambda,\theta) := \frac{\mathbb{K}(\lambda,\Theta(w))}{c_{\rm v}(\Theta(w))}.$$

We refer to (13) as the *enthalpy equation*; notice that this will be more convenient for our analysis since the time derivative is not multiplied by the specific heat capacity anymore. Let us stress that in more complicated situations—when we do not have the partially linearized ansatz (3) for the Helmholtz free energy—it requires more care to perform the enthalpy transformation (12), cf. [47].

Let us consider the following Robin boundary condition for (13)

$$\big(\mathcal{K}(\lambda,\theta)\nabla w\big)\cdot n + \mathfrak{b}\Theta(w) = \mathfrak{b}\theta_{\rm ext} \qquad \text{on } \Sigma^\varepsilon,$$

for $\mathfrak{b},\theta_{\rm ext}\in\mathbb{R}$ a given heat-transfer coefficient, $\theta_{\rm ext}$ a given external temperature; cf. [7].

As far as additional notation is concerned, we will use $\mathfrak{G}^\varepsilon_\eta$ for the "deformation-related" part of the Gibbs free energy

$$\begin{aligned}\mathfrak{G}^\varepsilon_\eta(t,y(t),\lambda(t),\Theta(w(t))) := \int_{\Omega_\varepsilon} W(\nabla y(t)) + \eta\left|\nabla^2 y(t)\right|^2 + \frac{\kappa}{2}\left|\nabla\triangle^{-1}(\lambda(t)-\mathcal{L}(\nabla y(t)))\right|^2{\rm d}x\\ - \int_{\Omega_\varepsilon} f(t)\cdot y(t)\,{\rm d}x - \int_{\Gamma^\varepsilon_N} g(t)\cdot y(t)\,{\rm d}S,\end{aligned}$$

since this is the only part of the energy that contributes to the semi-stability (14).

Further, where it shall be obvious, we will denote the list of arguments of $\mathcal{G}^\varepsilon_\eta$ and $\mathfrak{G}^\varepsilon_\eta$ at time t simply by t, that is,

$$\mathcal{G}^\varepsilon_\eta(t) \equiv \mathcal{G}^\varepsilon_\eta(t,y(t),\lambda(t),\Theta(w(t))), \qquad \mathfrak{G}^\varepsilon_\eta(t) \equiv \mathfrak{G}^\varepsilon_\eta(t,y(t),\lambda(t),\Theta(w(t)))$$

Lastly,

$$(\!(u,v)\!)_\varepsilon = \int_{\Omega_\varepsilon} \nabla\triangle^{-1}u\cdot\nabla\triangle^{-1}v\,{\rm d}x$$

will stand for the inner product in $W^{-1,2}(\Omega_\varepsilon;\mathbb{R}^{M+1}) \simeq \big(W^{1,2}_0(\Omega_\varepsilon;\mathbb{R}^{M+1})\big)^*$, while ${\rm Var}_h(u;I\times M)$ shall be the time-variation of a map u with respect to $h\ge 0$, more precisely

$$\begin{aligned}{\rm Var}_h(u;I\times M) := \sup\Bigg\{\sum_{i=1}^n\int_M h(u(t_i,x)-u(t_{i-1},x))\,{\rm d}x :\\ \text{for all partitions } [t_0,t_n]=I,\ n\in\mathbb{N},\ \text{such that } t_0<t_1<\cdots<t_n\Bigg\};\end{aligned}$$

we shall omit the space argument $I\times M$ in case $I\times M = Q_\varepsilon$.

3.1 Weak Formulation

To define a suitable weak solution of the system (5), we shall call for the energetic-solution concept (see e.g. [35]) further adapted to combinations of rate-independent/rate-dependent processes in [44]. Let us note that, for further convenience, we will explicitly express the dependence of the solutions on the parameters ε and η in their notation.

Definition 1. *The triple* $(y^{\eta,\varepsilon},\lambda^{\eta,\varepsilon},w^{\eta,\varepsilon})$ *belonging to*

$$
\begin{aligned}
&y^{\eta,\varepsilon}\in BV(0,T;W^{1,1}(\Omega_\varepsilon;\mathbb{R}^3))\cap L^\infty(0,T;W^{2,2}(\Omega_\varepsilon;\mathbb{R}^3)),\\
&\lambda^{\eta,\varepsilon}\in W^{1,q}(0,T;L^q(\Omega_\varepsilon;\mathbb{R}^{M+1}))\cap L^\infty(0,T;W^{1,2}(\Omega_\varepsilon;\mathbb{R}^{(M+1)\times 3})),\\
&w^{\eta,\varepsilon}\in L^1(0,T;W^{1,1}(\Omega_\varepsilon)),
\end{aligned}
$$

satisfying the boundary condition $y^{\eta,\varepsilon}(t,x)=0$ *on* Σ_D^ε *with is called a weak solution of the system* (5) *if the following holds:*

1. semi-stability*:*
$$\mathcal{G}_\eta^\varepsilon(t)\le\mathcal{G}_\eta^\varepsilon(t,\bar y,\lambda^{\eta,\varepsilon}(t),\Theta(w^{\eta,\varepsilon}(t)))+\eta\int_{\Omega_\varepsilon}|\nabla\bar y-\nabla y^{\eta,\varepsilon}(t)|\,dx \tag{14}$$
for all $\bar y\in W^{2,2}(\Omega_\varepsilon;\mathbb{R}^3)$ *such that* $\bar y(x)=0$ *on* Γ_D^ε *and all* $t\in[0,T]$.

2. deformation-related energy equality*:*
$$\mathfrak{G}_\eta^\varepsilon(T)-\mathfrak{G}_\eta^\varepsilon(0)+\eta\mathrm{Var}_{|\cdot|}(\nabla y^{\eta,\varepsilon})=\int_0^T[\mathfrak{G}_\eta^\varepsilon]_t'(t)+2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,dt \tag{15}$$

3. flow rule*:*
$$
\begin{aligned}
&\int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla y^{\eta,\varepsilon}),v-\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,dt+\int_0^s\int_{\Omega_\varepsilon}(\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot(v-\dot\lambda^{\eta,\varepsilon})+2\eta\nabla\lambda^{\eta,\varepsilon}\cdot\nabla v+\frac{\alpha}{q}|v|^q+\delta_S^*(v)\,dx\,dt\\
&\qquad\ge\eta\|\nabla\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{M+1})}-\eta\|\nabla\lambda^{\eta,\varepsilon}(0)\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{M+1})}+\int_0^s\int_{\Omega_\varepsilon}\frac{\alpha}{q}|\dot\lambda^{\eta,\varepsilon}|^q+\delta_S^*(\dot\lambda^{\eta,\varepsilon})\,dx\,dt
\end{aligned}\tag{16}
$$
for all test functions $v\in L^q(0,T;L^q(\Omega_\varepsilon;\mathbb{R}^{M+1}))\cap L^\infty(0,T;W^{1,2}(\Omega_\varepsilon;\mathbb{R}^{M+1}))$ *and all* $s\in[0,T]$.

4. enthalpy equation*:*
$$
\begin{aligned}
&\int_{Q_\varepsilon}\mathcal{K}(\lambda^{\eta,\varepsilon},w^{\eta,\varepsilon})\nabla w^{\eta,\varepsilon}\cdot\nabla\zeta-w^{\eta,\varepsilon}\dot\zeta\,dx\,dt+\int_{\Sigma^\varepsilon}\mathfrak{b}\Theta(w^{\eta,\varepsilon})\zeta\,dS\,dt\\
&\quad=\int_{Q_\varepsilon}\big(\delta_S^*(\dot\lambda^{\eta,\varepsilon})+\alpha|\dot\lambda^{\eta,\varepsilon}|^q+\Theta(w^{\eta,\varepsilon})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\big)\zeta\,dx\,dt+\eta\int_{\overline{Q}_\varepsilon}\zeta\mathcal{H}_\varepsilon^\eta(\,dx\,dt)\\
&\qquad+\int_{\Omega_\varepsilon}w_0^{\eta,\varepsilon}\zeta(0)\,dx+\int_{\Sigma^\varepsilon}\mathfrak{b}\theta_{\rm ext}\zeta\,dS\,dt
\end{aligned}\tag{17}
$$
for all $\zeta\in C^1(\overline{Q}_\varepsilon)$ *such that* $\zeta(T)=0$*; the Radon measure* $\mathcal{H}_\varepsilon^\eta\in\mathcal{M}(\overline{Q}_\varepsilon)$*, representing the heat production stemming from the term* $|\nabla\dot y|$ *in* (4)*, is defined for every closed set* $A=[t,s]\times B$*, where* $[t,s]\subseteq[0,T]$ *and* $B\subset\Omega_\varepsilon$ *a Borel set, as*
$$\mathcal{H}_\varepsilon^\eta(A):=\mathrm{Var}_{|\cdot|}(\nabla y^{\eta,\varepsilon};[t,s]\times B).$$

5. initial conditions*:* $y^{\eta,\varepsilon}(0)=y_0$ *for some* $y_0\in W^{2,2}(\Omega_\varepsilon;\mathbb{R}^3)$ *and* $\lambda^{\eta,\varepsilon}(0)=\lambda_0$ *in* Ω_ε *with* $\lambda_0\in L^q(\Omega_\varepsilon;\mathbb{R}^{M+1})$.

Remark 2 (Weak formulation of the flow-rule (5b))**.** *The weak formulation* (16) *is a standard weak formulation of the differential inclusion* (5b) *together with a by-parts integration in the term*

$$
\begin{aligned}
&\int_0^s\int_{\Omega_\varepsilon}2\eta\nabla\lambda^{\eta,\varepsilon}\cdot(\nabla v-\nabla\dot\lambda^{\eta,\varepsilon})\,dx\,dt\\
&\quad\overset{\text{by parts}}{=}\int_0^s\int_{\Omega_\varepsilon}2\eta\nabla\lambda^{\eta,\varepsilon}\cdot\nabla v\,dx\,dt-\eta\|\nabla\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{M+1})}+\eta\|\nabla\lambda^{\eta,\varepsilon}(0)\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{M+1})}.
\end{aligned}
$$

Further, while standardly one would demand only that it holds for $s=T$*, we require that the flow-rule holds for all* $s\in[0,T]$*. Notice that if we did not perform the aforementioned by-parts integration, both requirements would be equivalent. Indeed, in such a case, taking a test function such that* $v\equiv\dot\lambda^{\eta,\varepsilon}$ *on* $(s,T]$ *would yield the flow-rule for any* $s\in[0,T]$ *if it were known for* $s=T$*.*

Here, since we used by-parts integration, the required weak formulation is a bit stronger *which shall be advantageous when performing the dimension reduction in Section 4.*

Remark 3. *(i) Note that the second law of thermodynamics holds, i.e. the entropy production will be non-negative, if we can show that* $\theta^{\eta,\varepsilon} \geq 0$ *(when the assumed positive semi-definiteness of* $\mathbb{K}$ *holds).*
(ii) Definition 1 is indeed selective, cf. [7].

3.2 Change of variables and rescaling

In order to prepare for the dimension reduction performed later, let us change variables in order to work on the fixed domain $\Omega := \Omega_1 = \omega \times (0,1)$ by introducing new coordinates $z\colon \Omega_\varepsilon \to \Omega$ as

$$z(x) := (z_1, z_2, z_3) = (x_1, x_2, x_3/\varepsilon) \quad \forall x = (x_1, x_2, x_3) \in \Omega_\varepsilon. \tag{18}$$

Subsequently, the scaled functionals (with unchanged notation)

$$\mathcal{G}^\varepsilon_\eta = \frac{1}{\varepsilon}\mathcal{G}^\varepsilon_\eta \circ z^{-1} \quad \text{and} \quad \mathcal{R}^\varepsilon_\eta = \frac{1}{\varepsilon}\mathcal{R}^\varepsilon_\eta \circ z^{-1}, \tag{19}$$

in terms of the new variables read as

$$\begin{aligned}\mathcal{G}^\varepsilon_\eta(t) = \int_\Omega & W\left(\nabla'_\varepsilon y^{\eta,\varepsilon}(t)\right) + \kappa \left|\nabla'_\varepsilon \triangle_\varepsilon^{-1}\big(\lambda^{\eta,\varepsilon}(t) - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}(t))\big)\right|^2 \\ &+ \eta\bigg(|\nabla_p^2 y^{\eta,\varepsilon}(t)|^2 + \frac{2}{\varepsilon^2}|\nabla_p y^{\eta,\varepsilon}_{,3}(t)|^2 + \frac{1}{\varepsilon^4}|y^{\eta,\varepsilon}_{,33}(t)|^2 + |\nabla_p \lambda^{\eta,\varepsilon}(t)|^2 + \frac{1}{\varepsilon^2}|\lambda^{\eta,\varepsilon}_{,3}(t)|^2\bigg) \\ &+ (\Theta(w^{\eta,\varepsilon}(t)) - \theta_{\mathrm{tr}})\mathfrak{a}\cdot\lambda^{\eta,\varepsilon}(t) - f(t)\cdot y^{\eta,\varepsilon}(t)\,\mathrm{d}z - \int_{\Gamma_N} g(t)\cdot y^{\eta,\varepsilon}(t)\,\mathrm{d}S\end{aligned} \tag{20a}$$

and

$$\mathcal{R}^\varepsilon_\eta(\dot{y}^{\eta,\varepsilon}(t), \dot{\lambda}^{\eta,\varepsilon}(t)) = \int_\Omega \eta\,|\nabla'_\varepsilon \dot{y}^{\eta,\varepsilon}(t)| + \frac{\alpha}{q}|\dot{\lambda}^{\eta,\varepsilon}(t)|^q + \delta^*_S(\dot{\lambda}^{\eta,\varepsilon})\,\mathrm{d}z. \tag{20b}$$

The scaling factor $1/\varepsilon$ corresponds to the stiffness of the material (in linearized elasticity to the Lamé coefficients of order $1/\varepsilon$).

Above, we denoted by $\nabla'_\varepsilon g$ the scaled gradient, namely,

$$\nabla'_\varepsilon g = \left(\nabla_p g \,\middle|\, \frac{1}{\varepsilon} g_{,3}\right)$$

with the 3×2 planar component $(\nabla_p g)_{ij}$ of the gradient, cf. (7), and $(g_{,3})_k := \partial g_k/\partial x_3$ for $k = 1,2,3$. The scaled inverse Laplace operator $\triangle_\varepsilon^{-1}\colon L^2(\Omega;\mathbb{R}^{M+1}) \to W^{1,2}(\Omega;\mathbb{R}^{M+1})$ stands for the relation $\triangle_\varepsilon^{-1} g = h$ whenever

$$\int_\Omega \nabla'_\varepsilon h(z)\cdot\nabla'_\varepsilon\varphi(z) - g(z)\varphi(z)\,\mathrm{d}z = 0 \tag{21}$$

for all $\varphi \in C^\infty(\Omega;\mathbb{R}^{M+1})$. Also, we will also keep the notation $(\!(\cdot,\cdot)\!)_\varepsilon$ for the scaled inner product in $W^{-1,2}(\Omega)$ defined as $(\!(f,g)\!)_\varepsilon = \int_\Omega \nabla'_\varepsilon \triangle_\varepsilon^{-1} f \cdot \nabla'_\varepsilon \triangle_\varepsilon^{-1} g\,\mathrm{d}z$.

In the same spirit, the transformed initial conditions shall be denoted as

$$\begin{aligned} y^{\eta,\varepsilon}(0,z) &= y_{0,\varepsilon}(z) := y_0(z_p, \varepsilon z_3), \\ \lambda^{\eta,\varepsilon}(0,z) &= \lambda_{0,\varepsilon}(z) := \lambda_0(z_p, \varepsilon z_3), \\ w^{\eta,\varepsilon}(0,z) &= w_{0,\varepsilon}(z) := w_0(z_p, \varepsilon z_3). \end{aligned} \tag{22}$$

In view of (18)–(20) the transformation of Definition 1 of the weak solution is straightforward.

3.3 Data qualification and existence of weak solutions

Throughout the article, we shall use the following data qualifications:

(D1) *Stored energy density:* $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is continuous and frame-indifferent, and there exist positive real constants c_1 and c_2 satisfying

$$c_1(-1 + |A|^p) \leq W(A) \leq c_2(1 + |A|^p)$$

for some $2 \leq p < 6$ and all $A \in \mathbb{R}^{3\times 3}$.

(D2) *External forces:*

$$f \in W^{1,\infty}(0,T;L^{p^{*'}}(\Omega_\varepsilon;\mathbb{R}^3)), \qquad g \in W^{1,\infty}(0,T;L^{p^{\sharp'}}(\Gamma_N^\varepsilon;\mathbb{R}^3)),$$

such that $f \circ z^{-1}$ and $g \circ z^{-1}$ (denoted again by f and g) are independent of the thickness ε.

(D3) *Phase-distribution function:* $\mathcal{L}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is continuous and bounded.

(D4) *Specific heat capacity:* $c_\mathrm{v}\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth

$$c_1(1+\theta)^{\varsigma_1 - 1} \le c_\mathrm{v}(\theta) \le c_2(1+\theta)^{\varsigma_2 - 1}$$

for some real positive constants c_1, c_2 and $q' \le \varsigma_1 \le \varsigma_2$.

(D5) *Heat-conductivity tensor:* $\mathcal{K}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{3\times 3}$ is continuous and there exist real positive constants ξ and Ξ such that

$$\mathcal{K}(\lambda, w) \le \Xi, \qquad \chi^\top \mathcal{K}(\lambda, w)\chi \ge \xi|\chi|^2$$

hold for all λ, $w \in \mathbb{R}$ and all $\chi \in \mathbb{R}^3$.

(D6) *Initial and boundary data:*

$$\mathfrak{b} \in L^\infty(\Sigma^\varepsilon),\ \mathfrak{b} \ge 0 \qquad \text{and} \qquad \theta_\mathrm{ext} \in L^1(\Sigma^\varepsilon),\ \theta_\mathrm{ext} \ge 0,$$
$$y_0 \in W^{2,2}(\Omega_\varepsilon;\mathbb{R}^3), \qquad \text{and} \qquad w_0 \in L^1(\Omega_\varepsilon) \text{ with } \theta_0 \ge 0,$$

and

$$\lambda_0 \in L^q(\Omega_\varepsilon;\mathbb{R}^{M+1}) \quad \text{is independent of } x_3.$$

Remark 4. *Note that (D1) excludes the constraint on the Helmholtz free energy that $W(F) \to \infty$ whenever* $\det(F) \to 0$*, or, in the thin-film setting, whenever the normal of the thin film approaches zero. The results of [2] would allow us to consider such a constraint in the static case when the Cosserat vector is minimized out. Here, however, the interplay between the Cosserat vector and the film normal makes the situation considerably more difficult and results of [2] are not applicable. Let us also point to [5] for further results on Young measure relaxation considering the non-interpenetration constraint.*

To ease notation, we shall from now on use C as a *generic constant* possibly depending on the given data but *never on ε, η.*

Proposition 1 (Existence of a bulk weak solution)**.** *Let (D1)–(D6) hold. Then, for every $\varepsilon > 0$, $\eta > 0$ fixed, there exists a weak solution of* (5) *in the spirit of Definition 1 such that the following a-priori estimates hold:*

$$\|y^{\eta,\varepsilon}(t)\|_{BV(0,T;W^{1,1}(\Omega;\mathbb{R}^3))} \le C\eta^{-1}, \tag{23a}$$
$$\sup_{t\in[0,T]} \|\nabla'_\varepsilon y^{\eta,\varepsilon}(t)\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} \le C, \tag{23b}$$
$$\sup_{t\in[0,T]} \left\|\frac{1}{\varepsilon^2} y^{\eta,\varepsilon}_{,33}(t)\right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\eta^{-1/2}, \tag{23c}$$
$$\sup_{t\in[0,T]} \|\nabla'_\varepsilon y^{\eta,\varepsilon}(t)\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})} \le C\eta^{-1/2} \tag{23d}$$

for the deformation,

$$\|\dot\lambda^{\eta,\varepsilon}\|_{L^q(0,T;L^q(\Omega;\mathbb{R}^{M+1}))} \le C, \tag{24a}$$
$$\sup_{t\in[0,T]} \|\nabla'_\varepsilon \lambda^{\eta,\varepsilon}(t)\|_{L^2(\Omega;\mathbb{R}^{(M+1)})} \le C\eta^{-1/2} \tag{24b}$$

for the phase field, and

$$\|w^{\eta,\varepsilon}\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{25a}$$
$$\|\nabla'_\varepsilon w^{\eta,\varepsilon}\|_{L^r(0,T;L^r(\Omega;\mathbb{R}^3)} \le C(r) \text{ for any } r < \frac{5}{4}, \tag{25b}$$
$$\|\dot w^{\eta,\varepsilon}\|_{\mathcal{M}(0,T;(W^{1,\infty}(\Omega))^*)} \le C \tag{25c}$$

for the enthalpy.

Note that in (25c) $\mathcal{M}$ denotes the set of Radon measures.

Proof. The proof follows a rather standard procedure, cf. [6, 7] or [44], of showing that the interpolants of a particular discrete approximation converge to the sought bulk solution, therefore a detailed proof is omitted. Let us, however, sketch its main ingredients.

Step 1: Time discretization of the weak formulation. Define the discrete weak solution of (5) at time level k, $k = 1, \dots, T/\tau$, as a triple $(y_k^\tau, \lambda_k^\tau, w_k^\tau) \in W^{2,2}(\Omega;\mathbb{R}^3) \times L^{2q}(\Omega;\mathbb{R}^{M+1}) \times W^{1,2}(\Omega)$ satisfying

1. time-incremental minimization problem:

$$\begin{aligned}\text{Minimize}\quad & \mathcal{G}_\eta^\varepsilon(t_k, y, \lambda, \Theta(w_k^\tau)) + \int_\Omega \tau|\lambda|^{2q} + \eta|\nabla'_\varepsilon y - \nabla'_\varepsilon y_{k-1}^\tau| \\ & \qquad + \delta_S^*\left(\frac{\lambda - \lambda_{k-1}^\tau}{\tau}\right) + \frac{\tau\alpha}{q}\left|\frac{\lambda - \lambda_{k-1}^\tau}{\tau}\right|^q \mathrm{d}z \\ \text{subject to}\quad & (y, \lambda) \in W^{2,2}(\Omega;\mathbb{R}^3) \times L^{2q}(\Omega;\mathbb{R}^{M+1}), \\ & y(z) = 0 \text{ for } z \in \Gamma_D. \end{aligned} \tag{26}$$

2. enthalpy equation:

$$\begin{aligned}&\int_\Omega \frac{w_k^\tau - w_{k-1}^\tau}{\tau} + \mathcal{K}(\lambda_k^\tau, w_k^\tau)\nabla'_\varepsilon w_k^\tau \cdot \nabla'_\varepsilon \zeta \,\mathrm{d}z + \int_{\partial\Omega} \mathfrak{b}_k^\tau \Theta(w_k^\tau)\zeta - \mathfrak{b}_k^\tau \theta_{\mathrm{ext}}\zeta \,\mathrm{d}S \\ &\quad = \int_\Omega \delta_S^*\left(\frac{\lambda_k^\tau - \lambda_{k-1}^\tau}{\tau}\right)\zeta + \alpha\left|\frac{\lambda_k^\tau - \lambda_{k-1}^\tau}{\tau}\right|^q \zeta + \left|\frac{\nabla'_\varepsilon y_k^\tau - \nabla'_\varepsilon y_{k-1}^\tau}{\tau}\right|\zeta + \Theta(w_k^\tau)\mathfrak{a}\cdot\left(\frac{\lambda_k^\tau - \lambda_{k-1}^\tau}{\tau}\right)\zeta \,\mathrm{d}z\end{aligned}$$

for all $\zeta \in W^{1,2}(\Omega)$.

3. initial conditions:

$$y_0^\tau = y_{0,\varepsilon}, \qquad \lambda_0^\tau = \lambda_{0,\varepsilon}^\tau, \qquad w_0^\tau = w_{0,\varepsilon}^\tau \quad \text{a.e. in } \Omega,$$

where $\mathfrak{b}_k^\tau, \lambda_{0,\varepsilon}^\tau, w_{0,\varepsilon}^\tau$ are suitable approximations of the original data (D6).

Notice the added regularization term $\int_\Omega \tau|\lambda|^{2q}\,\mathrm{d}z$ which allows for a rather standard proof of existence of a discrete weak solution but vanishes as $\tau \to 0$. Details are to be found, e.g., in [6].

Step 2: A-priori estimates. Let us outline the proof of the a-priori estimates (23)–(25) merely heuristically, on the continuum level instead of the discrete setting, where a rigorous proof would follow the same ideas but be technically more demanding, cf. [6] again.

First, from the energy equality (15) integrated only to some $s \in [0, T]$ (note that we actually need only the lower inequality—this can be, on the discrete level, got from (26) integrated to any arbitrary $s \in [0, T]$), we get by exploiting the coercivity assumptions (D1) on the left-hand side and the bounds (D2)-(D3) as well as (D6) on the right-hand side

$$\begin{aligned}&\int_\Omega C|\nabla'_\varepsilon y^{\eta,\varepsilon}(s)|^p + \eta\left(\left|\nabla_p^2 y^{\eta,\varepsilon}(s)\right|^2 + 2\left|\frac{1}{\varepsilon}\nabla_p y_{,3}^{\eta,\varepsilon}(s)\right|^2 + \left|\frac{1}{\varepsilon^2}y_{,33}^{\eta,\varepsilon}(s)\right|^2\right)\mathrm{d}z \\ &\qquad + \eta\mathrm{Var}_{|\cdot|}(\nabla'_\varepsilon y^{\eta,\varepsilon}; \Omega\times[0,s]) \le \int_0^s\int_\Omega\left(\frac{\alpha}{4q}|\dot\lambda^{\eta,\varepsilon}|^q + C|\nabla'_\varepsilon y^{\eta,\varepsilon}|^p\right)\mathrm{d}z\mathrm{d}t + C.\end{aligned} \tag{27}$$

Further, by testing the flow rule (16) (after the change of scale) with by $v = 0$ on $[0, s]$ (note that this test essentially executes the standard test of the strong flow-rule by $\dot\lambda^{\eta,\varepsilon}$) we get

$$\begin{aligned}&\int_0^s\int_\Omega \delta_S^*(\dot\lambda^{\eta,\varepsilon}) + \frac{\alpha}{q}|\dot\lambda^{\eta,\varepsilon}|^q\,\mathrm{d}z\,\mathrm{d}t + \eta\|\nabla'_\varepsilon\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{(M+1)\times3})} + \eta\|\nabla_p\lambda_0\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{(M+1)\times2})} \\ &\qquad \le -2\kappa\int_0^s (\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), \dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t + \int_0^s\int_\Omega |\Theta(w^{\eta,\varepsilon}) - \theta_{\mathrm{ext}}|\cdot|\dot\lambda^{\eta,\varepsilon}|\,\mathrm{d}z\mathrm{d}t,\end{aligned} \tag{28}$$

where we used that $[\lambda_0]_{,3} = 0$ due to (D6). This, after plugging in the by-parts integration formula

$$2\int_0^s (\!(\lambda^{\eta,\varepsilon}, \dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t = \int_\Omega |\nabla'_\varepsilon\triangle_\varepsilon^{-1}\lambda^{\eta,\varepsilon}(s)|^2 - |\nabla'_\varepsilon\triangle_\varepsilon^{-1}\lambda^{\eta,\varepsilon}(0)|^2\,\mathrm{d}z, \tag{29}$$

yields (with the help of Young's inequality and (D6) again) the estimate

$$\int_0^s \int_\Omega \left(\delta_S^*(\dot\lambda^{\eta,\varepsilon}) + \frac{\alpha}{q}|\dot\lambda^{\eta,\varepsilon}|^q\right) \mathrm{d}z\mathrm{d}t + \int_\Omega \kappa|\nabla'_\varepsilon\triangle_\varepsilon^{-1}\lambda^{\eta,\varepsilon}(s)|^2\,\mathrm{d}z + \eta|\nabla'_\varepsilon\lambda^{\eta,\varepsilon}(s)|^2\,\mathrm{d}z \\ \leq \int_0^s \int_\Omega \frac{\alpha}{4q}|\dot\lambda^{\eta,\varepsilon}|^q + C|w|\,\mathrm{d}z\mathrm{d}t + C. \tag{30}$$

Lastly, testing enthalpy equation (13) by α/lq, with some $l \geq 8$ such that $\alpha \leq lq$, and integrating again over Ω and $[0,s]$ gives (notice that this test can be straightforwardly executed on the discrete level)

$$\frac{\alpha}{lq}\int_0^s \int_\Omega \dot w^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t \leq \int_0^s \int_\Omega \frac{2\alpha}{lq}|\dot\lambda^{\eta,\varepsilon}|^q + C|w^{\eta,\varepsilon}|\,\mathrm{d}z\mathrm{d}t + \frac{\alpha\varepsilon}{lq}\mathrm{Var}_{|\cdot|}(\nabla'_\varepsilon y^{\eta,\varepsilon};\Omega\times[0,s]) + C. \tag{31}$$

Adding (27), (30) and (31) gives then the bounds (23), (24) and (25a). The estimate (25b) on the scaled gradient of $w^{\eta,\varepsilon}$ follows by fine technique due to [13, 14] from the test of the enthalpy equation in (13) by $1 - 1/(1+w^{\eta,\varepsilon})^\alpha$, while (25c) is a standard dual estimate stemming from the enthalpy equation (17) itself.

Step 3: Convergence $\tau \to 0$. The proof of convergence for $\tau \to 0$ can be performed similarly as in [6, 44], or the methods exposed in the proof of Theorem 1 are easily applicable to this case, too. □

□

4 Dimension reduction to the microscopic thin-film model

Let us now concentrate on the microscopic thin-film model given through the system of inclusion / equations (8). As mentioned above, particularly the inclusion (8a) is rather formal, therefore we propose its weak formulation in the spirit of semi-energetic solutions, due to [44], similarly to the previous section. Also, again, we transformed the heat equation into a enthalpy equation.

4.1 Weak Formulation

To shorten the notation, we shall denote hereinafter $\mathcal{Q} := [0,T]\times\omega$, while the in-plane inner product in $W^{-1,2}(\omega;\mathbb{R}^{M+1})$ will be denoted as $(\!(u,v)\!)_p := \int_\omega \nabla_p\triangle_p^{-1}u\cdot\nabla_p\triangle_p^{-1}v\,\mathrm{d}z_p$, for all $u,v\in W^{-1,2}(\omega;\mathbb{R}^{M+1})$, whereas $\triangle_p^{-1}\colon L^2(\omega;\mathbb{R}^{M+1}) \to W^{1,2}(\omega;\mathbb{R}^{M+1})$ is the in-plane inverse Laplace operator, more precisely, $\triangle_p^{-1}g = h$ whenever

$$\int_\omega \nabla_p h(z_p)\cdot\nabla_p\phi(z_p) - g(z_p)\phi(z_p)\,\mathrm{d}z_p = 0$$

for every $\phi\in C^\infty(\omega;\mathbb{R}^{M+1})$.

Definition 2. *Let us call the quadruple* $(y^\eta, b^\eta, \lambda^\eta, w^\eta)$ *belonging to*

$$y^\eta \in BV(0,T;W^{1,1}(\omega;\mathbb{R}^3))\cap L^\infty(0,T;W^{2,2}(\omega;\mathbb{R}^3)), \tag{32a}$$
$$b^\eta \in BV(0,T;L^1(\omega;\mathbb{R}^3))\cap L^\infty(0,T;W^{1,2}(\omega;\mathbb{R}^3)), \tag{32b}$$
$$\lambda^\eta \in W^{1,q}(0,T;L^q(\omega;\mathbb{R}^{M+1}))\cap L^\infty(0,T;W^{1,2}(\omega;\mathbb{R}^{M+1})), \tag{32c}$$
$$w^\eta \in L^1(0,T;W^{1,1}(\omega)), \tag{32d}$$

such that $(y^\eta,b^\eta)(t,z_1,z_2) = 0$ *for all* $t\in[0,T]$ *and a.e. on* γ_D*, a weak solution of the evolutionary thin-film problem* (8) *if it satisfies*

1. semi-stability:

$$\mathcal{G}_\eta(t) \leq \mathcal{G}_\eta(t,\bar y,\bar b,\lambda^\eta(t),\Theta(w^\eta(t))) + \int_\omega \eta\left|(\nabla_p y^\eta(t)|b^\eta(t)) - \left(\nabla_p\bar y|\bar b\right)\right|\,dz_p \tag{33}$$

for every $(\bar y,\bar b)\in W^{2,2}(\omega;\mathbb{R}^3)\times W^{1,2}(\omega;\mathbb{R}^3)$ *such that* $(\bar y,\bar b) = 0$ *a.e. on* γ_D *(recall the definition* (6a) *of the Gibbs free energy* $\mathcal{G}_\eta(t)$*);*

2. deformation-related energy equality:

$$\mathfrak{G}_\eta(T) - \mathfrak{G}_\eta(0) + \eta \mathrm{Var}_{|\cdot|}((\nabla_p y^\eta|b^\eta); \mathcal{Q}) = \int_0^T [\mathfrak{G}_\eta]'_t(t) + 2\kappa(\!(\lambda^\eta - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), \dot{\lambda}^\eta)\!)_p \, dt \tag{34}$$

where $\mathfrak{G}_\eta(t)$ is defined as

$$\mathfrak{G}_\eta(t) := \int_\omega W(\nabla_p y^\eta|b^\eta) + \eta\left(|\nabla_p^2 y^\eta|^2 + 2|\nabla_p b^\eta|^2\right) \, dz_p + \kappa \, \|\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta)\|^2_{W^{-1,2}(\omega;\mathbb{R}^{M+1})} - \int_\omega f^0 \cdot y^\eta \, dz_p - \int_{\gamma_N} g^0 \cdot y^\eta \, dS_p; \tag{35}$$

3. flow rule:

$$\int_0^s 2\kappa(\!(\lambda^\eta - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), v - \dot{\lambda}^\eta)\!)_p \, dt + \int_0^s \int_\omega (\Theta(w^\eta) - \theta_{\mathrm{tr}})\mathfrak{a} \cdot (v - \dot{\lambda}^\eta) + 2\eta \nabla_p \lambda^\eta \cdot \nabla_p v + \frac{\alpha}{q}|v|^q + \delta_S^*(v) \, dz_p \, dt$$
$$\geq \eta \|\nabla_p \lambda^\eta(T)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \eta \|\nabla_p \lambda^\eta(0)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} + \int_0^s \int_\omega \frac{\alpha}{q}|\dot{\lambda}^\eta|^q + \delta_S^*(\dot{\lambda}^\eta) \, dz_p \, dt \tag{36}$$

for all test functions $v \in L^q(0,T;L^q(\omega;\mathbb{R}^{M+1})) \cap L^\infty(0,T;W^{1,2}(\omega;\mathbb{R}^{M+1}))$ and every $s \in [0,T]$.

4. enthalpy equation:

$$\int_\mathcal{Q} \mathcal{K}(\lambda^\eta, w^\eta)\nabla_p w^\eta \cdot \nabla_p \zeta - w^\eta \dot{\zeta} \, dz_p \, dt + \int_0^T \int_{\partial\omega} \mathfrak{b}\Theta(w^\eta)\zeta \, dS_p \, dt = \int_\omega w_0 \zeta(0) \, dz_p +$$
$$\int_\mathcal{Q} \left(\delta_S^*(\dot{\lambda}^\eta) + \alpha|\dot{\lambda}^\eta|^q + (\Theta(w^\eta) - \theta_{\mathrm{tr}})\mathfrak{a} \cdot \dot{\lambda}^\eta\right)\zeta \, dz_p \, dt + \eta \int_{\overline{\mathcal{Q}}} \zeta \mathcal{H}^\eta(\, dz_p \, dt) + \int_0^T \int_{\partial\omega} \mathfrak{b}\theta_{\mathrm{ext}}\zeta \, dS_p \, dt \tag{37}$$

for all $\zeta \in C^1(\overline{\mathcal{Q}})$ such that $\zeta(T) = 0$. Analogously to (37), here again the Radon measure $\mathcal{H}^\eta \in \mathcal{M}(\overline{\mathcal{Q}})$, $\eta > 0$ represents the heat production due to $\eta|(\nabla_p \dot{y}|\dot{b})|$ and is defined for any closed set $A = [t,s] \times B$, where $[t,s] \subseteq [0,T]$ and $B \subset \omega$ a Borel set, as

$$\mathcal{H}^\eta(A) := \mathrm{Var}_{|\cdot|}((\nabla_p y^\eta|b^\eta); [t,s] \times B).$$

5. initial conditions:

$$\begin{aligned} y^\eta(0,z_p) &= y_{0,0}(z_p) := y_0(z_p,0), \\ b^\eta(0,z_p) &= b_0(z_p) := (y_0)_{,3}(z_p,0), \\ \lambda^\eta(0,z_p) &= \lambda_{0,0}(z_p) := \lambda_0(z_p,0), \end{aligned} \tag{38}$$

4.2 Existence of weak solutions

Theorem 1. *Let (D1)–(D6) hold. Then there exists a quadruple $(y^\eta, b^\eta, \lambda^\eta, w^\eta)$ satisfying (32) such that $(y^\eta, b^\eta)(t, z_1, z_2) = 0$ for all $t \in [0,T]$ and a.e. on γ_D and a (not relabeled) a subsequence $\varepsilon \to 0_+$ such that the following holds*

$$y^{\eta,\varepsilon}(t) \to y^\eta(t) \quad \text{in } W^{2,2}(\Omega;\mathbb{R}^3) \text{ for all } t \in [0,T], \tag{39a}$$

$$\frac{1}{\varepsilon} y^{\eta,\varepsilon}_{,3}(t) \to b^\eta(t) \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^3) \text{ for all } t \in [0,T], \tag{39b}$$

$$\lambda^{\eta,\varepsilon} \to \lambda^\eta \quad \text{in } W^{1,q}(0,T;L^q(\Omega;\mathbb{R}^{M+1})), \tag{39c}$$

$$\nabla'_\varepsilon \lambda^{\eta,\varepsilon} \to (\nabla_p \lambda^\eta|0) \quad \text{in } L^2(\Omega;\mathbb{R}^{(M+1)\times 3}) \text{ for all } t \in [0,T] \tag{39d}$$

$$\nabla_p w^{\eta,\varepsilon} \rightharpoonup \nabla_p w^\eta \quad \text{in } L^r(0,T;L^r(\Omega)) \text{ for any } 1 \leq r < \frac{5}{4} \tag{39e}$$

$$w^{\eta,\varepsilon} \to w^\eta \quad \text{in } L^s(Q) \text{ for any } 1 \leq s < \frac{5}{3}, \tag{39f}$$

with $\{(y^{\eta,\varepsilon}, \lambda^{\eta,\varepsilon}, w^{\eta,\varepsilon})\}_{\varepsilon>0}$ a family of weak solutions of (5) obtained in Proposition 1; $(y^\eta, b^\eta, \lambda^\eta, w^\eta)$ is then a weak solution to (8) in the spirit of Definition 2.

Proof. For the sake of transparency, let us divide the proof into separate distinct steps.

STEP 1: SELECTION OF SUBSEQUENCES. The a-priori estimates (23) ensure—by Helly's selection principle—the existence of two vector fields $y^\eta \in BV(0,T;W^{1,1}(\Omega;\mathbb{R}^3))$, $b^\eta \in BV(0,T;L^1(\Omega;\mathbb{R}^3))$ such that

$$y^{\eta,\varepsilon}(t) \rightharpoonup y^\eta(t) \qquad \text{in } W^{2,2}(\Omega;\mathbb{R}^3) \text{ for all } t \in [0,T], \tag{40a}$$

$$\frac{1}{\varepsilon} y^{\eta,\varepsilon}_{,3}(t) \rightharpoonup b^\eta(t) \qquad \text{in } W^{1,2}(\Omega;\mathbb{R}^3) \text{ for all } t \in [0,T]. \tag{40b}$$

Similarly, using standard selection and embedding theorems, estimate (24a) ensures the existence of a limit phase field λ^η such that

$$\lambda^{\eta,\varepsilon} \rightharpoonup \lambda^\eta \quad \text{ in } W^{1,q}(0,T;L^q(\Omega;\mathbb{R}^{M+1})). \tag{40c}$$

By exploiting further the estimate (24b) and the continuous embedding of $W^{1,q}(0,T;L^q(\Omega;\mathbb{R}^{M+1}))$ into $C(0,T;L^q(\Omega;\mathbb{R}^{M+1})$ we get that

$$\nabla_p\lambda^{\eta,\varepsilon}(t) \rightharpoonup \nabla_p\lambda^\eta(t) \qquad \text{in } L^2(\Omega;\mathbb{R}^{(M+1)\times 2}) \text{ for all } t \text{ in } [0,T]. \tag{40d}$$

The situation is more complicated for the third component of $\nabla'_\varepsilon\lambda^{\eta,\varepsilon}$, we shall return to it later in Step 3, where also the strong convergence (39d) will be shown. The strong convergences (39a)–(39b) will be obtained in Step 5.

Lastly, we may extract a (not relabeled) subsequence of $\{w^{\eta,\varepsilon}\}_{\varepsilon>0}$ such that (39e) and (39f) are satisfied; notice that the latter convergence stems from the dual estimate (25c) and the generalized Aubin–Lions lemma, cf. [43, Corollary 7.8 and 7.9] and [44, equation (4.55)]. Moreover, (39f) yields, together with the assumption (D4), the strong convergence

$$\Theta(w^{\eta,\varepsilon}) \to \Theta(w^\eta) \text{ in } L^{q'}(Q)\ . \tag{41}$$

In order to see this, we exploit the first inequality in assumption (D4)

$$w^{\eta,\varepsilon} = \int_0^{\theta^{\eta,\varepsilon}} c_{\rm v}(r)\,\mathrm{d}r \geq c_1 \int_0^{\Theta(w^{\eta,\varepsilon})} (1+r)^{\varsigma_1-1}\,\mathrm{d}r \geq c_1\left((1+\Theta(w^{\eta,\varepsilon}))^{\varsigma_1} - 1\right),$$

where we used that $\theta^{\eta,\varepsilon} \geq 0$, together with the assumption $\varsigma_1 \geq q'$ to get the bound

$$|\Theta(w^{\eta,\varepsilon})| \leq C\left(1 + |w^{\eta,\varepsilon}|^{1/q'}\right).$$

Hence, by the continuity of the Nemytskii mapping induced by Θ, one arrives to (41).

STEP 2: INDEPENDENCE OF z_3. It follows from the estimates (23d) and the weak lower semicontinuity of the norm that

$$0 = \liminf_{\varepsilon\to 0_+} c\varepsilon \geq \liminf_{\varepsilon\to 0_+} \|y^{\eta,\varepsilon}_{,3}(t)\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \geq \|y^\eta_{,3}(t)\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \geq 0.$$

This means that y^η is independent of z_3 for all $t \in [0,T]$. Analogously, the independence of λ^η and b^η of z_3 follows from the estimate (24b), resp. (23c). For w^η we get that it is independent of z_3 only for a.a. $t \in [0,T]$ from (25b).

STEP 3: THIN-FILM FLOW RULE. Recall the bulk flow (16) which we rescale and and in which we expand the matrix ∇'_ε into its planar and normal components, namely

$$\begin{aligned}\int_0^s\int_\Omega (\Theta(w^{\eta,\varepsilon}) - \theta_{\rm tr})\mathfrak{a}\cdot(v - \dot\lambda^{\eta,\varepsilon}) + \frac{\alpha}{q}|v|^q + \delta^*_S(v)\,\mathrm{d}z\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), v - \dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t \\ + \int_0^s\int_\Omega 2\eta\nabla_p\lambda^{\eta,\varepsilon}\cdot\nabla_p v + \frac{2\eta}{\varepsilon^2}\lambda^{\eta,\varepsilon}_{,3}\cdot v_{,3}\,\mathrm{d}z\mathrm{d}t + \eta\|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} \\ \geq \eta\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{\eta}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})} + \int_0^s\int_\Omega \frac{\alpha}{q}|\dot\lambda^{\eta,\varepsilon}|^q + \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t\end{aligned}$$

where we used that, due to (D6), λ_0 does not depend on the third component. Let us admit *only test functions independent of* z_3 which simplifies the flow rule to

$$\int_0^s \int_\Omega (\Theta(w^{\eta,\varepsilon}) - \theta_{\rm tr})\mathfrak{a}\cdot(v - \dot\lambda^{\eta,\varepsilon}) + \frac{\alpha}{q}|v|^q + \delta_S^*(v)\,{\rm d}z{\rm d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), v - \dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,{\rm d}t$$
$$+ \int_0^s \int_\Omega 2\eta\nabla_p\lambda^{\eta,\varepsilon}\cdot\nabla_p v\,{\rm d}z{\rm d}t + \eta\|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})}$$
$$\ge \eta\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{\eta}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})} + \int_0^s\int_\Omega \frac{\alpha}{q}|\dot\lambda^{\eta,\varepsilon}|^q + \delta_S^*(\dot\lambda^{\eta,\varepsilon})\,{\rm d}z{\rm d}t. \quad (42)$$

Let us take an $s \in [0,T]$ arbitrary but fixed. Then, from (24b), we can choose a further subsequence of ε's *dependent on* s, labeled $\varepsilon_{k(s)}$, such that

$$\frac{1}{{\varepsilon_{k(s)}}^2}\|\lambda^{\eta,\varepsilon_{k(s)}}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)})} \to d_s \in \mathbb{R}^{M+1}.$$

Let us work, for the moment, only with this special subsequence and pass to the limit $\varepsilon_{k(s)} \to 0_+$ in (42) to obtain

$$\int_0^s \int_\omega (\Theta(w^{\eta}) - \theta_{\rm tr})\mathfrak{a}\cdot(v - \dot\lambda^{\eta}) + \frac{\alpha}{q}|v|^q + \delta_S^*(v)\,{\rm d}z_p{\rm d}t + \int_0^s 2\kappa(\!(\lambda^{\eta} - \mathcal{L}(\nabla_p y^{\eta}|b^\eta), v - \dot\lambda^{\eta})\!)_p\,{\rm d}t$$
$$+ \int_0^s \int_\omega 2\eta\nabla_p\lambda^{\eta}\cdot\nabla_p v\,{\rm d}z_p{\rm d}t + \eta\|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}$$
$$\ge \eta\|\nabla_p\lambda^{\eta}(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} + \eta d_s + \int_0^s\int_\omega \frac{\alpha}{q}|\dot\lambda^{\eta}|^q + \delta_S^*(\dot\lambda^{\eta})\,{\rm d}z_p{\rm d}t, \quad (43)$$

for all $v \in L^q(0,T;L^q(\omega;\mathbb{R}^{M+1})) \cap L^\infty(0,T;W^{1,2}(\omega;\mathbb{R}^{(M+1)\times 2}))$.

To see this, we employ (40c) and (41) on the left-hand side to pass to the limit (even for the whole sequence $\varepsilon \to 0_+$) in $\int_0^s\int_\Omega(\Theta(w^{\eta,\varepsilon}) - \theta_{\rm tr})\mathfrak{a}\cdot(v - \dot\lambda^{\eta,\varepsilon})\,{\rm d}z{\rm d}t$.

Further, let us choose $t \in [0,T]$ arbitrarily but fixed, and denote, for the sake of simplicity, $\Lambda^{\eta,\varepsilon}_t := \lambda^{\eta,\varepsilon}(t) - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}(t))$. Then the weak convergences (40a)–(40b), shown in Step 1, yield that $\nabla'_\varepsilon y^{\eta,\varepsilon}(t) \to (\nabla_p y^\eta|b^\eta)(t)$ strongly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Thus, by (D3), Nemytskii continuity and the estimate (24b) we also get that $\Lambda^{\eta,\varepsilon}_t \to \lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)) =: \Lambda^\eta_t$ strongly in $L^2(\Omega;\mathbb{R}^{M+1})$.

Let us show that in such a case, for $\varepsilon \to 0_+$,

$$\nabla'_\varepsilon \triangle_\varepsilon^{-1}\Lambda^{\eta,\varepsilon}_t \to \nabla_p\triangle_p^{-1}\Lambda^\eta_t \qquad \text{in } L^2(\Omega;\mathbb{R}^{M+1}).$$

Indeed, denote $h^\varepsilon_t = \triangle_\varepsilon^{-1}\Lambda^{\eta,\varepsilon}_t$; then h^ε_t solves

$$\int_\Omega \nabla_p h^\varepsilon_t\cdot\nabla_p\phi + \frac{1}{\varepsilon^2}h^\varepsilon_{,3}\phi_{,3} - \Lambda^{\eta,\varepsilon}_t\phi\,{\rm d}z = 0 \qquad \forall\phi \in W^{1,2}_0(\Omega;\mathbb{R}^{M+1}). \quad (44)$$

Taking ϕ independent of z_3 this simplifies to

$$\int_\Omega \nabla_p h^\varepsilon\cdot\nabla_p\phi - \Lambda^{\eta,\varepsilon}_t\phi\,{\rm d}z = 0 \qquad \forall\phi \in W^{1,2}_0(\omega;\mathbb{R}^{M+1}). \quad (45)$$

Since $\|\nabla_p h^\varepsilon_t\|_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})}$ is uniformly bounded (owing to the bounds on $\Lambda^{\eta,\varepsilon}_t$) we pass to the limit $\varepsilon \to 0_+$ in (45) and get that $\nabla_p h^\varepsilon_t \rightharpoonup \nabla_p h_t$ in $L^2(\Omega;\mathbb{R}^{(M+1)\times 2})$ where h_t solves

$$\int_\omega \nabla_p h_t\cdot\nabla_p\phi - \Lambda^{\eta}_t\phi\,{\rm d}z_p = 0 \qquad \forall\phi \in W^{1,2}_0(\omega;\mathbb{R}^{M+1}). \quad (46)$$

Here we relied on the fact that the limit difference Λ^η_t does not depend on z_3, i.e. $h = \triangle_p^{-1}\Lambda^\eta_t$.

Next, test (44) $\varepsilon\phi$ and notice that $\frac{1}{\varepsilon}\|h^\varepsilon_{t,3}\|_{L^2(\Omega;\mathbb{R}^{M+1})}$ is uniformly bounded (owing to the bounds on $\Lambda^{\eta,\varepsilon}_t$) to get $\frac{1}{\varepsilon}h^\varepsilon_{t,3} \rightharpoonup 0$ in $L^2(\Omega;\mathbb{R}^{M+1})$. Finally, by testing the difference of (44) and (46) with $h^\varepsilon_t - h_t$, we obtain even that $\nabla'_\varepsilon h^\varepsilon_t \to (\nabla_p h_t|0)$ *strongly* in $L^2(\Omega;\mathbb{R}^{(M+1)\times 3})$. Note that all the above would stay valid even if we had only $\Lambda^{\eta,\varepsilon}_t \rightharpoonup \Lambda^\eta_t$ in $L^2(\Omega;\mathbb{R}^{M+1})$ at hand.

Thus, relying on Lebegue's dominated convergence theorem, $\int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), v - \dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,{\rm d}t \to \int_0^s 2\kappa(\!(\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta), v - \dot\lambda^\eta)\!)_p\,{\rm d}t$. Finally, on the left-hand side of (42) in term $\int_0^s\int_\Omega 2\eta\nabla_p\lambda^{\eta,\varepsilon}\cdot\nabla_p v\,{\rm d}z{\rm d}t$

we use (40d) again combined with Lebegue's dominated convergence; on the right-hand side of (42) we rely on the weak lower semicontinuity of the involved convex terms to obtain (43).

Next, we aim to show that $d_s \equiv 0$. Clearly, $d \geq 0$ and the opposite inequality could be immediately seen if we allowed to put $v = \dot{\lambda}^\eta$ in (43). Yet, $\dot{\lambda}^\eta$ does not need to have the required regularity. So we introduce a sequence of smooth functions $\{\lambda^\eta_\ell\}_{\ell>0}$ such that $\lambda^\eta_\ell \to \lambda^\eta$ strongly in $W^{1,q}(0,T;L^q(\omega;\mathbb{R}^{M+1}))$ and $\nabla_p \lambda^\eta_\ell(t) \to \nabla_p\lambda^\eta(t)$ strongly in $L^2(\omega;\mathbb{R}^{M+1})$ for $\ell \to 0_+$ for all $t \in [0,T]$. Putting then $v = \dot{\lambda}^\eta_\ell$ in (43) yields

$$\int_0^s \int_\omega (\Theta(w^\eta) - \theta_{\rm tr})\mathfrak{a}\cdot(\dot{\lambda}^\eta_\ell - \dot{\lambda}^\eta) + \frac{\alpha}{q}|\dot{\lambda}^\eta_\ell|^q + \delta^*_S(\dot{\lambda}^\eta_\ell)\,\mathrm{d}z_p\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta), \dot{\lambda}^\eta_\ell - \dot{\lambda}^\eta)\!)_p\,\mathrm{d}t$$
$$+ \int_0^s \int_\omega 2\eta\nabla_p\lambda^\eta{\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t + \eta\|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}$$
$$\geq \eta\|\nabla_p\lambda^\eta(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} + \eta d_s + \int_0^s\int_\omega \frac{\alpha}{q}|\dot{\lambda}^\eta|^q + \delta^*_S(\dot{\lambda}^\eta)\,\mathrm{d}z_p\mathrm{d}t. \quad (47)$$

Reformulating, by means of by parts integration, $\int_0^s\int_\omega 2\eta\nabla_p\lambda^\eta{\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t$ as

$$\int_0^s \int_\omega 2\eta\nabla_p\lambda^\eta{\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t = \int_0^s\int_\omega 2\eta(\nabla_p\lambda^\eta - \nabla_p\lambda^\eta_\ell){\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t + \int_0^s\int_\Omega 2\eta\nabla_p\lambda^\eta_\ell{\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t$$
$$= \int_0^s\int_\omega 2\eta(\nabla_p\lambda^\eta - \nabla_p\lambda^\eta_\ell){\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t$$
$$+ \eta\big(\|\nabla_p\lambda^\eta_\ell(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda^\eta_\ell(0)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}\big) \quad (48)$$

and passing to the limit $\ell \to 0_+$ yiels that

$$\int_0^s\int_\omega 2\eta\nabla_p\lambda^\eta{\cdot}\nabla_p\dot{\lambda}^\eta_\ell\,\mathrm{d}z_p\mathrm{d}t \to \eta\big(\|\nabla_p\lambda^\eta(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}\big).$$

Therefore, passing $\ell \to 0_+$ in (47) gives that $d_s \leq 0$.

Last but not least, note that the s-dependent subsequence $\varepsilon_{k(s)}$ was used to pass to the limit merely in $\frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)})}$, all other limit passages hold in the whole sequence of ε's. Hence, we get that $\frac{1}{\varepsilon_{k(s)}{}^2}\|\lambda^{\eta,\varepsilon_{k(s)}}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)})} \to 0$ for all subsequences $\varepsilon_{k(s)}$ in which the left-hand side converges, and, by uniqueness of the limit, we conclude that

$$\frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)})} \to 0 \quad (49)$$

in the original sequence of ε's, independently of the chosen $s \in [0,T]$. Thus, we conclude that normal part of (39d) and (36) hold.

Step 4: Phase-field related energy equality and strong convergence of $\dot{\lambda}^{\eta,\varepsilon}$. In this step, let us deduce an energy equality that is related to the phase field. To this end, we reformulate the flow rule (16) (exploiting the convexity of $|\cdot|^q$) into the following equivalent form

$$\int_0^s\int_\Omega \alpha|\dot{\lambda}^{\eta,\varepsilon}|^{q-2}\dot{\lambda}^{\eta,\varepsilon}{\cdot}(v - \dot{\lambda}^{\eta,\varepsilon}) + (\Theta(w^{\eta,\varepsilon}) - \theta_{\rm tr})\mathfrak{a}{\cdot}(v - \dot{\lambda}^{\eta,\varepsilon}) + \delta^*_S(v)\,\mathrm{d}z\mathrm{d}t$$
$$+ \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), v - \dot{\lambda}^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t + \int_0^s\int_\Omega 2\eta\nabla_p\lambda^{\eta,\varepsilon}{\cdot}\nabla_p v + \frac{2\eta}{\varepsilon^2}\lambda^{\eta,\varepsilon}_{,3}{\cdot}v_{,3}\,\mathrm{d}z\mathrm{d}t$$
$$\geq \eta\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \eta\|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{\eta}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})} + \int_0^s\int_\Omega \delta^*_S(\dot{\lambda}^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t \quad (50)$$

and test (50) by $v = 0$ to get

$$-\left(\int_0^s\int_\Omega \alpha|\dot{\lambda}^{\eta,\varepsilon}|^{q-2}\dot{\lambda}^{\eta,\varepsilon}{\cdot}\dot{\lambda}^{\eta,\varepsilon} + (\Theta(w^{\eta,\varepsilon}) - \theta_{\rm tr})\mathfrak{a}{\cdot}\dot{\lambda}^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon} - \mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}), \dot{\lambda}^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t\right)$$
$$\geq \eta\left(\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2}} + \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\right) + \int_0^s\int_\Omega \delta^*_S(\dot{\lambda}^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t \quad (51)$$

and also by $v = 2\dot\lambda^{\eta,\varepsilon}$ to get (if $\dot\lambda^{\eta,\varepsilon}$ does not have the required regularity we can proceed as in Step 3 above, namely, we can smoothen $\dot\lambda^{\eta,\varepsilon}$, perform by parts integration analogous to (48) and pass to limit with the mollifying parameter which gives the desired result)

$$\begin{aligned}&\bigg(\int_0^s\int_\Omega \alpha|\dot\lambda^{\eta,\varepsilon}|^{q-2}\dot\lambda^{\eta,\varepsilon}\cdot\dot\lambda^{\eta,\varepsilon} + (\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t\bigg)\\ &\qquad + 2\eta\bigg(\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg)\\ &\geq \eta\bigg(\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg) - \int_0^s\int_\Omega \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t,\end{aligned}$$

where we relied on the one-homogeneity of $\delta^*_S(\cdot)$. In other words,

$$\begin{aligned}&-\bigg(\int_0^s\int_\Omega \alpha|\dot\lambda^{\eta,\varepsilon}|^{q-2}\dot\lambda^{\eta,\varepsilon}\cdot\dot\lambda^{\eta,\varepsilon} + (\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t\bigg)\\ &\leq \eta\bigg(\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg) + \int_0^s\int_\Omega \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t;\end{aligned} \tag{52}$$

combining this with (51) we obtain the *phase-field related energy equality* in the bulk, more precisely

$$\begin{aligned}\int_0^s\int_\Omega \alpha|\dot\lambda^{\eta,\varepsilon}|^q\,\mathrm{d}z\mathrm{d}t &= -\int_0^s\int_\Omega(\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t - \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t\\ -\eta\bigg(\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} &- \|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} + \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg) - \int_0^s\int_\Omega \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t.\end{aligned} \tag{53}$$

By an analogous procedure, we get from (36) the *phase-field related energy equality* in the thin film

$$\begin{aligned}\int_0^s\int_\omega \alpha|\dot\lambda^{\eta}|^q\,\mathrm{d}z_p\mathrm{d}t &= -\int_0^s\int_\omega(\Theta(w^{\eta})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta}\,\mathrm{d}z_p\mathrm{d}t - \int_0^s 2\kappa(\!(\lambda^{\eta}-\mathcal{L}(\nabla_p y^{\eta}|b^\eta),\dot\lambda^{\eta})\!)_p\,\mathrm{d}t\\ &\quad -\eta\big(\|\nabla_p\lambda^{\eta}(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}\big) - \int_0^s\int_\omega \delta^*_S(\dot\lambda^{\eta})\,\mathrm{d}z_p\mathrm{d}t.\end{aligned} \tag{54}$$

Having (53) and (54) at hand, we prove the *strong* convergences (39c) and the in-plane part of (39d). Indeed, we have

$$\begin{aligned}&\int_0^s\int_\omega \alpha|\dot\lambda^{\eta}|^q\,\mathrm{d}z_p\mathrm{d}t \leq \liminf_{\varepsilon\to 0_+}\int_0^s\int_\Omega \alpha|\dot\lambda^{\eta,\varepsilon}|^q\,\mathrm{d}z\mathrm{d}t \leq \limsup_{\varepsilon\to 0_+}\int_0^s\int_\Omega \alpha|\dot\lambda^{\eta,\varepsilon}|^q\,\mathrm{d}z\mathrm{d}t\\ &\overset{\rm (I)}{=}\limsup_{\varepsilon\to 0_+}\bigg(-\int_0^s\int_\Omega(\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon} + \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t - \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t\\ &\qquad + \eta\bigg(\|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \frac{1}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg)\bigg)\\ &\overset{\rm (II)}{=}-\lim_{\varepsilon\to 0_+}\bigg(\int_0^s\int_\Omega(\Theta(w^{\eta,\varepsilon})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\,\mathrm{d}z\mathrm{d}t + \int_0^s 2\kappa(\!(\lambda^{\eta,\varepsilon}-\mathcal{L}(\nabla'_\varepsilon y^{\eta,\varepsilon}),\dot\lambda^{\eta,\varepsilon})\!)_\varepsilon\,\mathrm{d}t + \frac{\eta}{\varepsilon^2}\|\lambda^{\eta,\varepsilon}_{,3}(s)\|^2_{L^2(\Omega;\mathbb{R}^{M+1})}\bigg)\\ &\qquad + \eta\big(\|\nabla_p\lambda_0\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} - \liminf_{\varepsilon\to 0_+}\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})}\big) - \liminf_{\varepsilon\to 0_+}\int_0^s\int_\Omega \delta^*_S(\dot\lambda^{\eta,\varepsilon})\,\mathrm{d}z\mathrm{d}t\\ &\overset{\rm (III)}{\leq}-\int_0^s\int_\omega(\Theta(w^{\eta})-\theta_{\rm tr})\mathfrak{a}\cdot\dot\lambda^{\eta}\,\mathrm{d}z_p\mathrm{d}t - \int_0^s 2\kappa(\!(\lambda^{\eta}-\mathcal{L}(\nabla_p y^{\eta}|b^\eta),\dot\lambda^{\eta})\!)_p\,\mathrm{d}t\\ &\qquad \|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \|\nabla_p\lambda^{\eta}(s)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} - \int_0^s\int_\omega \delta^*_S(\dot\lambda^{\eta})\,\mathrm{d}z_p\mathrm{d}t\\ &\overset{\rm (IV)}{=}\int_0^s\int_\omega \alpha|\dot\lambda^{\eta}|^q\,\mathrm{d}z_p\mathrm{d}t,\end{aligned}$$

where the inequalities on the first line follow from the weak lower semicontinuity of the norm and a general $\liminf \leq \limsup$ relation, the equality (I) is due to (53), the equality (II) follows from general $\limsup$, $\liminf$

inequalities, the inequality (III) was obtained by lower semicontinuity of the convex terms and (40c) and (40d), the limit $\varepsilon \to 0_+$ uses (41), (49) and similar techniques as when passing to the limit in the flow rule in *Step 3*. Finally, (IV) is due to (54).

So, we conclude that $\|\dot\lambda^{\eta,\varepsilon}\|_{L^q(Q;\mathbb{R}^{M+1})} \to \|\dot\lambda^{\eta}\|_{L^q(Q;\mathbb{R}^{M+1})}$ and, as the space $L^q(Q;\mathbb{R}^{M+1})$ is uniformly convex, also

$$\dot\lambda^{\eta,\varepsilon} \to \dot\lambda^{\eta} \qquad \text{in } L^q(Q;\mathbb{R}^{M+1}). \tag{55}$$

Moreover, using (55) and passing to the limit $\varepsilon \to 0_+$ in (53) and comparing to (54) yields that

$$\|\nabla_p\lambda^{\eta,\varepsilon}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} \to \|\nabla_p\lambda^{\eta}(s)\|^2_{L^2(\Omega;\mathbb{R}^{(M+1)\times 2})} \qquad \forall s\in[0,T]. \tag{56}$$

Step 5: Thin-film semi-stability. Fix again some $t\in[0,T]$ arbitrarily. Then, we test (14) (formulated only in the deformation-related energy) by $\bar y^\varepsilon_\delta(z) := \tilde y(z_p) + \varepsilon z_3 b_\delta(z_p)$ with some arbitrary $\tilde y \in W^{2,2}(\omega;\mathbb{R}^3)$ and a smooth approximation $\{b_\delta\}_{\delta>0}$ of an arbitrary $\tilde b \in W^{1,2}(\omega;\mathbb{R}^3)$ (the smoothing is required in order to obtain the test function in $W^{2,2}(\Omega;\mathbb{R}^3)$) such that $\tilde y(z_p)+\varepsilon z_3 b_\delta(z_p)=0$ a.e. on Γ_D. Then, by taking first $\liminf_{\varepsilon\to 0}$ then $\liminf_{\delta\to 0_+}$ one arrives to

$$\begin{aligned}
\mathfrak{G}_\eta(t) &\le \liminf_{\varepsilon\to 0_+} \mathfrak{G}^\varepsilon_\eta(t)\\
&\le \lim_{\delta\to 0_+}\left(\liminf_{\varepsilon\to 0_+}\mathfrak{G}^\varepsilon_\eta(t,\bar y^\varepsilon_\delta,\lambda^{\eta,\varepsilon}(t)) + \int_\Omega \eta\,|\nabla'_\varepsilon \bar y^\varepsilon_\delta - \nabla'_\varepsilon y^{\eta,\varepsilon}(t)|\,\mathrm{d}z\right)\\
&\le \lim_{\delta\to 0_+}\left(\limsup_{\varepsilon\to 0_+}\mathfrak{G}^\varepsilon_\eta(t,\bar y^\varepsilon_\delta,\lambda^{\eta,\varepsilon}(t)) + \int_\Omega \eta\,|\nabla'_\varepsilon \bar y^\varepsilon_\delta - \nabla'_\varepsilon y^{\eta,\varepsilon}(t)|\,\mathrm{d}z\right)\\
&= \mathfrak{G}_\eta(t,\tilde y,\tilde b,\lambda^\eta(t)) + \int_\omega \eta\left|(\nabla_p\tilde y|\tilde b) - (\nabla_p y^\eta(t)|b^\eta(t))\right|\mathrm{d}z_p,
\end{aligned}$$

where we used (39c),(39d) and the compact embedding $L^q(\Omega;\mathbb{R}^{M+1}) \Subset W^{-1,2}(\Omega;\mathbb{R}^{M+1})$ (recall that $q\ge 2$) to pass to the limit in $\mathfrak{G}^\varepsilon_\eta(t,\bar y^\varepsilon_\delta,\lambda^{\eta,\varepsilon}(t))$ while (40a), (40b) was used to pass to the limit in the term $\int_\Omega \eta\,|\nabla'_\varepsilon \bar y^\varepsilon_\delta - \nabla'_\varepsilon y^{\eta,\varepsilon}(t)|\,\mathrm{d}z$. Observe that this is equivalent to (33).

Moreover, letting $\tilde y := y^\eta(t)$ and $\tilde b := b^\eta(t)$ yields

$$\lim_{\varepsilon\to 0_+}\mathfrak{G}^\varepsilon_\eta(t) = \mathfrak{G}_\eta(t) \qquad \text{for all } t\in[0,T]\,. \tag{57}$$

From this we may, similarly as in [10], deduce that

$$\begin{aligned}
\nabla_p^2 y^{\eta,\varepsilon}(t) &\to \nabla_p^2 y^\eta(t) \text{ in } L^2(\Omega;\mathbb{R}^{3\times2\times2}),\\
\nabla_p\frac{1}{\varepsilon}y^{\eta,\varepsilon}_{,3}(t) &\to \nabla_p b^\eta(t) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 2}),\\
\frac{1}{\varepsilon^2}y^{\eta,\varepsilon}_{,33}(t) &\to 0 \text{ in } L^2(\Omega;\mathbb{R}^3),
\end{aligned}$$

thus showing (39a)–(39b). As we will not need this improved convergence in the following, we omit a detailed proof.

Step 6: Thin-film deformation-related energy equality. We show the deformation-related energy equality (34) as two inequalities. One follows from the bulk inequality by taking $\liminf_{\varepsilon\to 0_+}$ with the aid of the convergences (39a)–(39d), the data qualification (D2) as

$$\begin{aligned}
&\mathfrak{G}_\eta(T) - \mathfrak{G}_\eta(0) + \eta\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta) \le\\
&\qquad \liminf_{\varepsilon\to 0_+}\left(\mathfrak{G}^\varepsilon_\eta(T) - \mathfrak{G}^\varepsilon_\eta(0) + \eta\mathrm{Var}_{|\cdot|}(\nabla'_\varepsilon y^{\eta,\varepsilon})\right) \le\\
&\qquad \limsup_{\varepsilon\to 0_+}\int_0^T [\mathfrak{G}^\varepsilon_\eta]'_t(t) + \left\langle [\mathfrak{G}^\varepsilon_\eta]'_\lambda(t), \dot\lambda^{\eta,\varepsilon}(t)\right\rangle \mathrm{d}t \le\\
&\qquad\qquad \int_0^T [\mathfrak{G}_\eta]'_t(t) + \left\langle [\mathfrak{G}_\eta]'_\lambda(t), \dot\lambda^{\eta}(t)\right\rangle \mathrm{d}t
\end{aligned} \tag{58}$$

as far as the first inequality is concerned, recall that $\mathrm{Var}_{|\cdot|}$ is lower-semicontinuous under the convergences (39a)-(39b).

The opposite inequality is a consequence of the thin-film semistability (33) (cf. [23, 29, 44] and [7] for an analogous (and more detailed) proof as the one given below). To see this, we introduce a partition of $[0,T]$, $0 = t_0 < t_1 < \cdots < t_{N(\beta)} = T$, such that $\max\{|t^\beta_{i-1} - t^\beta_i| : \ i = 1 \ldots N(\beta)\} \le \beta$ and test (33) at the time t^β_{i-1} by $(y^\eta(t^\beta_i), b^\eta(t^\beta_i))$, $i = 1, \ldots, N(\beta)$. Summing from 0 to $N(\beta)$ reveals that

$$\mathfrak{G}_\eta(T) - \mathfrak{G}_\eta(0) + \eta \mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta) \ge \sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} [\mathfrak{G}_\eta]'_t(t, y^\eta(t^\beta_i))\,\mathrm{d}t \\ + \sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} \Big\langle [\mathfrak{G}_\eta]'_\lambda(y^\eta(t^\beta_i), b^\eta(t^\beta_i), \lambda^\eta(t)), \dot{\lambda}^\eta(t) \Big\rangle \,\mathrm{d}t\,, \tag{59}$$

where

$$\sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} \Big\langle [\mathfrak{G}_\eta]'_\lambda(y^\eta(t^\beta_i), b^\eta(t^\beta_i), \lambda^\eta(t)), \dot{\lambda}^\eta(t) \Big\rangle = 2\kappa \sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} ((\lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t^\beta_i)|b^\eta(t^\beta_i)), \dot{\lambda}^\eta(t)))_p \\ + \underbrace{\sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} ((\lambda^\eta(t) - \lambda^\eta(t^\beta_i), \dot{\lambda}^\eta))_p}_{\text{(i)}} + \underbrace{\sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} ((\lambda^\eta(t^\beta_i) - \mathcal{L}(\nabla_p y^\eta(t^\beta_i)|b^\eta(t^\beta_i)), \dot{\lambda}^\eta(t^\beta_i)))_p}_{\text{(ii)}} \\ + \underbrace{\sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} ((\dot{\lambda}^\eta(t^\beta_i) - \mathcal{L}(\nabla_p y^\eta(t^\beta_i)|b^\eta(t^\beta_i)), \dot{\lambda}^\eta(t) - \dot{\lambda}^\eta(t^\beta_i)))_p}_{\text{(iii)}} \tag{60}$$

To make the limit passage for $\beta \to 0_+$, one makes use of the fact (cf. [16]) that every Bochner integrable $h\colon [0,T] \to X$, with X a Banach space, can be approached by its piecewise constant interpolant h_β defined on $[0,T]$ as $h_\beta|_{[t^\beta_{i-1}, t^\beta_i)} := h(t^\beta_i)$, $i = 1, \ldots, N(\beta)$ strongly to h in $L^1(0,T;X)$; more precisely

$$\lim_{\beta \to 0_+} \sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} \|h_\beta(t) - h(t)\|_X \,\mathrm{d}t = 0$$

in the partition of $[0,T]$ chosen above (in fact this holds for a.a. partitions of [0,T]). Hence, one may assume that

$$\begin{aligned}
\lambda^\eta_\beta &\rightharpoonup \lambda^\eta && \text{in } L^q(0,T;L^q(\omega;\mathbb{R}^{M+1})), && (61a)\\
y^\eta_\beta &\rightharpoonup y^\eta && \text{in } L^p(0,T;W^{1,p}(\omega;\mathbb{R}^3)), && (61b)\\
b^\eta_\beta &\rightharpoonup b^\eta && \text{in } L^2(0,T;L^2(\omega;\mathbb{R}^3)), && (61c)\\
\dot{\lambda}^\eta_\beta &\to \dot{\lambda}^\eta && \text{in } L^1(0,T;L^q(\omega;\mathbb{R}^{M+1})), && (61d)\\
\Big[((\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta), \dot{\lambda}^\eta))_p\Big]_\beta &\to ((\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta), \dot{\lambda}^\eta))_p && \text{in } L^1(0,T). && (61e)
\end{aligned}$$

Using (61b) we establish that $\sum_{i=1}^{N(\beta)} \int_{t^\beta_{i-1}}^{t^\beta_i} [\mathfrak{G}_\eta]'_t(t, y^\eta(t^\beta_i))\,\mathrm{d}t \to \int_0^T [\mathfrak{G}_\eta]'_t(t, y^\eta(t))\,\mathrm{d}t$; moreover, (61a) assures that (i) in (60) converges to 0, by (61e) we immediately see that (ii) in (60) converges to $\int_0^T ((\lambda^\eta - \mathcal{L}(\nabla_p y^\eta|b^\eta), \dot{\lambda}^\eta))_p\,\mathrm{d}t$ and, finally, by the uniform boundedness of the term $\dot{\lambda}^\eta(t^\beta_i) - \mathcal{L}(\nabla_p y^\eta(t^\beta_i)|b^\eta(t^\beta_i))$ in $L^\infty(0,T;W^{-1,2}(\omega;\mathbb{R}^{M+1}))$ and (61d) (iii) in (60) converges to 0.

Thus, we got that

$$\mathfrak{G}^\varepsilon_\eta(T) - \mathfrak{G}^\varepsilon_\eta(0) + \eta \mathrm{Var}_{|\cdot|}(\nabla'_\varepsilon y^{\eta,\varepsilon}) \ge \int_0^T [\mathfrak{G}_\eta]'_t(t) + \Big\langle [\mathfrak{G}_\eta]'_\lambda(t), \dot{\lambda}^\eta(t) \Big\rangle \,\mathrm{d}t$$

and combining this with (58) as well as (57) we obtain that

$$\mathrm{Var}_{|\cdot|}(\nabla'_\varepsilon y^{\eta,\varepsilon}) \to \mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta) \tag{62}$$

Step 7: Thin-film enthalpy equation. Recall that the bulk enthalpy equation reads as

$$\int_Q \mathcal{K}(\lambda^{\eta,\varepsilon},w^{\eta,\varepsilon})\nabla'_\varepsilon w^{\eta,\varepsilon}\cdot\nabla'_\varepsilon\zeta - w^{\eta,\varepsilon}\dot\zeta\,\mathrm{d}z\mathrm{d}t + \int_\Sigma \mathfrak{b}\Theta(w^{\eta,\varepsilon})\zeta\,\mathrm{d}S\mathrm{d}t = \int_Q \big(\delta_S^*(\dot\lambda^{\eta,\varepsilon}) + \alpha|\dot\lambda^{\eta,\varepsilon}|^q + \Theta(w^{\eta,\varepsilon})\mathfrak{a}\cdot\dot\lambda^{\eta,\varepsilon}\big)\zeta\,\mathrm{d}z\mathrm{d}t + \eta\int_{\overline{Q}}\zeta\mathcal{H}_\varepsilon(\,\mathrm{d}x\mathrm{d}t) + \int_\Omega w_0^{\eta,\varepsilon}\zeta(0)\,\mathrm{d}z + \int_\Sigma \mathfrak{b}\theta_{\mathrm{ext}}\zeta\,\mathrm{d}S\mathrm{d}t \quad (63)$$

with $\bar\zeta\in C^1(\overline{\mathcal{Q}})$ and $\bar\zeta(T)=0$. Let us restrict ourselves to test functions independent of z_3. When taking $\varepsilon\to 0_+$ in (63), we aim to get (37).

First, let us show that

$$\lim_{\varepsilon\to 0_+}\int_{\overline{Q}}\zeta\mathcal{H}^\eta_\varepsilon(\mathrm{d}z\mathrm{d}t) = \int_{\overline{Q}}\zeta\mathcal{H}^\eta(\mathrm{d}z\mathrm{d}t)\ . \quad (64)$$

To this end, recall that from the a-priori estimates (23) follows the existence of a limit measure $\overline{\mathcal{H}}$ such that

$$\mathcal{H}^\eta_\varepsilon \overset{*}{\rightharpoonup} \overline{\mathcal{H}} \quad \text{in } \mathcal{M}(\overline{Q})\ , \quad (65)$$

while, on the other hand, (62) ensures that

$$\lim_{\varepsilon\to 0}\mathcal{H}^\eta_\varepsilon(\overline{Q}) = \mathcal{H}^\eta(\overline{Q})\ . \quad (66)$$

Now, the contradiction argument in [44, Proposition 4.3] supports that (65)–(66) indeed yield (64). More precisely, if, by contradiction, it held that $\mathcal{H}^\eta\neq\overline{\mathcal{H}}$, we could define the Borel set $\mathfrak{B}:=\mathrm{supp}\,(\mathcal{H}^\eta-\overline{\mathcal{H}})\subset\overline{Q}$ and (66) would imply that

$$\int_{\mathfrak{B}}(\mathcal{H}^\eta-\overline{\mathcal{H}})\,\mathrm{d}z\mathrm{d}t>0$$

(otherwise (66) would be violated), which immediately contradicts the weak* lower semicontinuity of the map $\varepsilon\mapsto\int_{\mathfrak{B}}\mathcal{H}^\eta_\varepsilon(\,\mathrm{d}z\mathrm{d}t)$.

For the other terms in (63), we use $\lambda^{\eta,\varepsilon}\to\lambda^\eta$ in $L^q(0,T;L^q(\Omega;\mathbb{R}^{M+1}))$, $w^{\eta,\varepsilon}\to w^\eta$ in $L^s(0,T;L^s(\Omega))$, for any $1\le s<5/3$; the latter convergence ensures also that $w^{\eta,\varepsilon}\to w^\eta$ in $L^1(0,T;L^1(\Sigma))$ which allows us to pass to the limit in the boundary terms on the left-hand side of (63) as well as that $\mathcal{K}(\lambda^{\eta,\varepsilon},w^{\eta,\varepsilon})\to\mathcal{K}(\lambda^\eta,w^\eta)$ in $L^\beta(0,T;L^\beta(\Omega;\mathbb{R}^{3\times3}))$ for any $1\le\beta<+\infty$. Hence, we obtain (37). □

□

5 Relaxation in the microscopic thin-film model

In this section, we surpass scales to rigorously obtain the mesoscopic model formally given by (10a)–(10c).

As mentioned in Section 2, this upscaling lets the interfacial energy vanish; this may lead to fast spatial oscillations of the deformation gradient, on one hand, as well as of the Cosserat vector, on the other hand. A standard tool to capture these oscillations is the theory of (gradient) *Young measures* [28, 51, 38].

Let $\mathcal{O}\subset\mathbb{R}^l$ be a Lebesgue measurable subset with finite measure. Young measures are weakly measurable and essentially bounded mappings $\nu\in L^1(\mathcal{O};C_0(\mathbb{R}^d))^*\cong L^\infty_{\mathrm{w}}(\mathcal{O};\mathcal{M}(\mathbb{R}^d))$; here $C_0(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity, so that $\mathcal{M}(\mathbb{R}^d)$ denotes the space of Radon measures on $\mathbb{R}^d$. Having a bounded sequence $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\mathcal{O};\mathbb{R}^d)$ for $1\le p<+\infty$ then there is a subsequence (not relabeled) and a Young measure ν such that $\lim_{k\to\infty}\int_{\mathcal{O}}h(x,u_k(x))\,\mathrm{d}x=\int_{\mathcal{O}}\int_{\mathbb{R}^d}h(x,F)\,\nu_x(\mathrm{d}F)\mathrm{d}x$ whenever $\{h(\cdot,u_k)\}_{k\in\mathbb{N}}\subset L^1(\mathcal{O})$ is uniformly integrable, where $h:\mathcal{O}\times\mathbb{R}^d\to\mathbb{R}$ is a Carathéodory integrand. We then say that ν is generated by $\{u_k\}_{k\in\mathbb{N}}$. The set of mappings from $L^\infty_{\mathrm{w}}(\mathcal{O};\mathcal{M}(\mathbb{R}^d))$ generated by bounded sequences in $L^p(\mathcal{O};\mathbb{R}^d)$ is denoted by $\mathscr{Y}^p(\mathcal{O};\mathbb{R}^d)$.

An important subset of $\mathscr{Y}^p(\mathcal{O};\mathbb{R}^d)$ is the set of so-called p-gradient Young measures ($1<p<+\infty$) which consists of measures generated by $\{\nabla y_k\}_{k\in\mathbb{N}}$ of a bounded sequence of mappings $\{y_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\mathcal{O};\mathbb{R}^d)$. The set of p-gradient Young measures (shortly gradient Young measures) is denoted by $\mathscr{G}^p(\mathcal{O};\mathbb{R}^{d\times l})$. Occasionally, we may write $\mathscr{G}^p_{\gamma_D}(\mathcal{O};\mathbb{R}^{d\times l})$ to indicate that $y_k=0$ on $\gamma_D\subset\partial\mathcal{O}$.

Further, we use the the shorthand notation (momentum operator) " $\bullet$ " defined through

$$[f\bullet\nu](x):=\int_{\mathbb{R}^{d\times l}}f(s)\nu_x(\mathrm{d}s).$$

Denoting id : $\mathbb{R}^{d\times l} \to \mathbb{R}^{d\times l}$ the identity mapping, we speak of $\mathrm{id}\bullet\nu$ as the *mean value* of the gradient Young measure $\nu \in \mathscr{G}^p(\mathcal{O};\mathbb{R}^{d\times l})$. It can be proved, cf. [28], that whenever $\nu \in \mathscr{G}^p(\mathcal{O};\mathbb{R}^{d\times l})$ there exists $y \in W^{1,p}(\mathcal{O};\mathbb{R})$ such that $\nabla y = \mathrm{id}\bullet\nu$ a.e. on $\mathcal{O}$. Additionally, ν is an element of $\mathscr{G}^p_{\gamma_D}(\mathcal{O};\mathbb{R}^{d\times l})$ if and only if $y = 0$ on γ_D.

5.1 Weak formulation

Let us now state the weak formulation of (10a)–(10c).

Definition 3. *We call the quintuple* (y,ν,μ,λ,w), *where*

$$y \in B(0,T;W^{1,p}(\omega;\mathbb{R}^3)), \tag{67a}$$
$$\nu \in (\mathscr{G}^p_{\gamma_D}(\omega;\mathbb{R}^{3\times 2}))^{[0,T]}, \tag{67b}$$
$$\mu \in (\mathscr{Y}^p(\omega;\mathbb{R}^3))^{[0,T]}, \tag{67c}$$
$$\lambda \in W^{1,q}(0,T;L^q(\mathbb{R}^{M+1})), \tag{67d}$$
$$w \in L^\infty(0,T;L^1(\omega)), \tag{67e}$$

such that $y(t) = \mathrm{id}\bullet\nu_{z_p}(t)$ *for a.a.* $z_p \in \omega$ *and all* $t \in [0,T]$ *a weak solution of* (10a)–(10c) *if it satisfies*

1. minimization property :

$$\mathcal{G}(t,y(t),\nu(t),\mu(t),\lambda(t),\Theta(w(t))) \le \mathcal{G}(t,\bar{y},\bar{\nu},\bar{\mu},\lambda(t),\Theta(w(t))) \tag{68}$$

for every $(\bar{y},\bar{\nu},\bar{\mu}) \in W^{1,p}(\omega;\mathbb{R}^3)\times\mathscr{G}^p_{\gamma_D}(\omega;\mathbb{R}^{3\times 2})\times\mathscr{Y}^p(\omega;\mathbb{R}^3)$ *such that* $\bar{y} = \mathrm{id}\bullet\bar{\nu}_{z_p}$ *for almost all* $z_p \in \omega$ *and* $\mathcal{G}$ *defined in* (9).

2. flow rule*:*

$$\int_0^T 2\kappa(\!(\lambda - \mathcal{L}\bullet(\nu,\mu), v - \dot{\lambda})\!)_p\,dt + \int_0^T\int_\omega (\Theta(w^{\eta,\varepsilon}) - \theta_{\mathrm{tr}})\mathfrak{a}\cdot(v-\dot{\lambda}) + \frac{\alpha}{q}|v|^q + \delta^*_S(v)\,dz_p\,dt \ge \int_0^T\int_\omega \frac{\alpha}{q}|\dot{\lambda}|^q + \delta^*_S(\dot{\lambda})\,dz_p\,dt \tag{69}$$

for all test functions $v \in L^q(0,T;L^q(\omega;\mathbb{R}^{M+1}))$.

3. enthalpy equation*:*

$$\int_\mathcal{Q} \mathcal{K}(\lambda,w)\nabla_p w\cdot\nabla_p\zeta - w\dot{\zeta}\,dz_p\,dt + \int_0^T\int_{\partial\omega}\mathfrak{b}\Theta(w)\zeta\,dS_p\,dt = \int_\mathcal{Q}\big(\delta^*_S(\dot{\lambda}) + \alpha|\dot{\lambda}|^q + (\Theta(w))\mathfrak{a}\cdot\dot{\lambda}\big)\zeta\,dz_p\,dt + \int_\omega w_0\zeta(0)\,dz_p + \int_0^T\int_{\partial\omega}\mathfrak{b}\theta_{\mathrm{ext}}\zeta\,dS_p\,dt \tag{70}$$

for every $\zeta \in C^1(\overline{\mathcal{Q}})$ *such that* $\zeta(T) = 0$.

4. *the remaining* initial conditions:

$$\nu_{z_p}(0) = \delta_{y_{0,0}(z_p)}, \qquad \mu_{z_p}(0) = \delta_{b_0(z_p)}, \qquad \lambda(0) = \lambda_{0,0}, \tag{71}$$

with $y_{0,0}(z_p)$, $b_0(z_p)$ *and* $\lambda_{0,0}$ *referring to* (38).

Notice that in this formulation we used the (not completely standard) notation $B(0,T;X)$ for the space of function $[0,T] \mapsto X$, X a Banach space, that are bounded but not necessarily Lebesgue measurable. Also, we used the notation

$$\Psi\bullet(\nu,\mu)(z_p) := \int_{\mathbb{R}^{3\times 2}}\int_{\mathbb{R}^3}\Psi(A|b)\,\mathrm{d}\nu_{z_p}(A)\,\mathrm{d}\mu_{z_p}(b),$$

with Ψ a continuous function with at most p-growth.

Remark 5 (Deformation-related energy equality). *Note that we omit a deformation-related energy equality analogous to* (34)*. Since we scale down the rate-independent dissipation due to* $\eta|(\nabla_p \dot{y}^\eta|b^\eta)|$ *to zero, such an equality is a direct consequence of* (68) *and, hence, becomes redundant. To see this, we may proceed as Step 6 of the proof of Theorem 1 and introduce a partition of the interval* $[0,T]$*,* $0 = t_0^\beta \leq t_1^\beta \dots t_{K(\beta)}^\beta = T$ *and test* (68) *at* $t = t_{i-1}^\beta$ *by* $(y(t_i^\beta), \nu(t_i^\beta), \mu(t_i^\beta))$*; summing and passing to the limit* $\beta \to 0$ *leads, as in Step 6 of the proof of Theorem 1, to the inequality*

$$\mathfrak{G}(T) - \mathfrak{G}(0) \geq \int_0^T \mathfrak{G}'_t(t) + \left\langle \mathfrak{G}'_\lambda(t), \dot{\lambda}(t) \right\rangle \, dt, \tag{72}$$

where

$$\mathfrak{G}(t) = \mathfrak{G}(t, y(t), \nu(t), \mu(t), \lambda(t)) := \int_\omega W \bullet (\nu, \mu) + \kappa \left\| \lambda - \mathcal{L} \bullet (\nu, \mu) \right\|^2_{W^{-1,2}(\omega;\mathbb{R}^{3\times 3})} - \int_\omega f^0 \cdot y \, dz_p - \int_{\gamma_N} g^0 \cdot y \, dS_p \ , \tag{73}$$

is the deformation-related part of the mesoscopic Gibbs free energy.

The other inequality is then obtained by the same procedure: We test, however, (68) *at* $t = t_i^\beta$ *with by* $(y(t_{i-1}^\beta), \nu(t_{i-1}^\beta), \mu(t_{i-1}^\beta))$*. We obtain an "energy-related" inequality because the dissipation component related to* $\eta|(\nabla_p \dot{y}^\eta|b^\eta)|$ *is not present in* (68) *anymore.*

5.2 Existence of weak solutions

Theorem 2. *Let* $\{(y^\eta, b^\eta, \lambda^\eta, w^\eta)\}_{\eta>0}$ *be a family of weak solutions of the thin-film problem* (8a)–(8c) *as found in Theorem 1. Then there exist a quintuple* $(y, \nu, \mu, \lambda, w)$*, satisfying* (67)*, and a sequence* $\eta \to 0_+$ *such that*

$$\lambda^\eta \to \lambda \quad \text{in } W^{1,q}(0,T; L^q(\omega; \mathbb{R}^{M+1})), \tag{74}$$

and

$$w^\eta \rightharpoonup w \quad \text{in } L^r(0,T; W^{1,r}(\omega)), \text{ for every } r < \frac{5}{4}, \tag{75a}$$

$$w^\eta \to w \quad \text{in } L^s([0,T] \times \omega; \mathbb{R}^{M+1}), \text{ for every } s < \frac{5}{3}. \tag{75b}$$

Moreover, for each $t \in [0,T]$ *there exists a subsequence* $\eta_{k(t)}$ *such that* $\nabla y_{\eta_{k(t)}}(t)$ *generates a gradient Young measure* $\nu(t)$*,* $y_{\eta_{k(t)}}(t) \rightharpoonup y(t)$ *in* $W^{1,p}(\omega; \mathbb{R}^3)$ *and* $b_{\eta_{k(t)}}(t)$ *generates a Young measure* $\mu(t)$*.*

At least one cluster point found in this way is then a weak solution to (10a)–(10c) *in the sense of Definition 3.*

Proof. For lucidity, let us divide the proof into several steps. Let us note that the idea of the proof, in particular the technique of selecting a suitable cluster point, roughly follows [7].

Step 1: Selection of subsequences and reformulation of the flow rule. Similarly as in Step 1 of the proof of Theorem 1, we choose, owing to the a-priori estimates (24)–(25) (and the Aubin-Lions theorem), a (not relabeled) subsequence of $\eta \to 0_+$ and find (λ, w) such that

$$\lambda^\eta \rightharpoonup \lambda \quad \text{in } W^{1,q}(0,T; L^q(\omega; \mathbb{R}^{M+1})) \tag{76}$$

and (75) hold as well as the limit $\lim_{\eta\to 0_+} \mathfrak{G}_\eta(T)$ is well defined. Recall that, again as in Step 1 in the proof of Theorem 1, we have the additional convergences $\lambda^\eta(t) \rightharpoonup \lambda(t)$ in $L^q(\omega; \mathbb{R}^{M+1})$ for all $t \in [0,T]$ and $\Theta(w^\eta) \to \Theta(w)$ in $L^{q'}(\mathcal{Q})$.

Now, let us turn our attention to the flow rule (36), more specifically to the penalty term

$$\int_0^T 2\kappa(\!(\lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), v - \dot{\lambda}^\eta)\!)_p \, \mathrm{d}t \tag{77}$$

involved in $\int_0^T \langle [\mathcal{G}_\eta]'_t, v - \dot{\lambda}^\eta \rangle \, \mathrm{d}t$, which turns out to be the most troublesome. Indeed, note that since the limit for $(\nabla y^\eta, b^\eta)$ is evaluated point-wise in $t \in [0,T]$ the limit of $\mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t))$ (taken again point-wise) is not guaranteed to be measurable in time. Moreover, $\dot{\lambda}^\eta$ converges only weakly in $L^q(\mathcal{Q}; \mathbb{R}^{M+1})$ and, thus, convergence for a.a. $t \in [0,T]$ of this term cannot be expected. To handle the latter obstacle, we plug the energy equality (34) into (36) with $s = T$ to obtain a weaker *reformulated flow rule*:

$$
\begin{aligned}
&\mathfrak{G}_\eta(T) + \eta \mathrm{Var}_{|\cdot|}(\nabla_p y^\eta | b^\eta) + \eta \|\nabla_p \lambda^\eta(T)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})} + \int_{\mathcal{Q}} \frac{\alpha}{q}|\dot\lambda^\eta|^q + \delta^*_S(\dot\lambda^\eta)\,\mathrm{d}z_p\mathrm{d}t \\
&\quad \le \mathfrak{G}_\eta(0) + \int_0^T [\mathfrak{G}_\eta]'_t(t, y^\eta(t))\,\mathrm{d}t + \int_{\mathcal{Q}} (\Theta(w^\eta) - \theta_{\mathrm{tr}})\mathfrak{a}\cdot(\tilde v - \dot\lambda^\eta) + 2\eta \nabla_p \lambda^\eta \cdot \nabla_p \tilde v + \frac{\alpha}{q}|\tilde v|^q + \delta^*_S(\tilde v)\,\mathrm{d}z_p\mathrm{d}t \\
&\qquad + \int_0^T 2\kappa(\!(\lambda^\eta - \mathcal{L}(\nabla_p y^\eta | b^\eta), \tilde v)\!)_p\,\mathrm{d}t + \eta\|\nabla_p \lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times 2})}. \qquad (78)
\end{aligned}
$$

Indeed, the term $\int_0^T 2\kappa(\!(\lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), \dot\lambda)\!)_p\,\mathrm{d}t$ is no longer present in (78).

Further, inspired by [7, 23, 16], we define

$$
\mathfrak{P}^v(t) = \limsup_{\eta\to 0} 2\kappa(\!(\lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), v(t))\!)_p\,\mathrm{d}t \quad \text{and} \quad \mathcal{F}(t) = \limsup_{\eta\to 0}[\mathfrak{G}_\eta]'_t(t, y^\eta(t))
$$

for any $v \in L^q(\mathcal{Q};\mathbb{R}^{M+1})$ and every $t \in [0,T]$; notice that both $\mathfrak{P}^v$ and $\mathcal{F}$ are measurable. Moreover, by Fatou's lemma, we have

$$
\begin{aligned}
\int_0^T \mathfrak{P}^v(t)\,\mathrm{d}t &\ge \limsup_{\eta\to 0_+} \int_0^T 2\kappa(\!(\lambda^\eta(t) - \mathcal{L}(\nabla_p y^\eta(t)|b^\eta(t)), v(t))\!)_p\,\mathrm{d}t, \\
\int_0^T \mathcal{F}(t)\,\mathrm{d}t &\ge \limsup_{\eta\to 0_+} \int_0^T [\mathfrak{G}_\eta]'_t(t, y^\eta(t))\,\mathrm{d}t.
\end{aligned}
$$

Since $L^q(\mathcal{Q};\mathbb{R}^{M+1})$ is separable, we consider, for now, the test functions $v = v^\ell$ only from a countable dense subset of $L^q(\mathcal{Q};\mathbb{R}^{M+1})$, denoted by $\mathcal{V}$. Next, we fix $t \in [0,T]$ and choose a subsequence of η's labeled η_{t,v^ℓ} such that

$$
\mathfrak{P}^{v^\ell}(t) = \lim_{\eta_{t,v^\ell}\to 0_+} 2\kappa(\!(\lambda^{\eta_{t,v^\ell}}(t) - \mathcal{L}(\nabla_p y^{\eta_{t,v^\ell}}(t)|b^{\eta_{t,v^\ell}}(t)), v^\ell(t))\!)_p, \qquad (79a)
$$

$$
\mathcal{F}(t) = \lim_{\eta_{t,v^\ell}\to 0_+} . \qquad (79b)
$$

By a diagonal selection, we can find a further subsequence labeled η_t such that (79) holds for all v^ℓ. Note that the chosen subsequence remains to be time-dependent.

Now, owing to the a-priori estimates (23b) and (23c), we choose yet another subsequence of $\eta_{k(t)}$ (not relabeled) such that $\{\nabla_p y_{\eta_{k(t)}}(t)\}_{k\in\mathbb{N}}$ generates the gradient Young measure $\nu_{z_p}(t)$ and $\{b_{\eta_{k(t)}}(t)\}_{k\in\mathbb{N}}$ generates the Young measure $\mu_{z_p}(t)$; so,

$$
\begin{aligned}
\mathfrak{P}^v(t) &= \lim_{\eta_{k(t)}\to 0_+} 2\kappa(\!(\lambda^{\eta_{k(t)}}(t) - \mathcal{L}(\nabla_p y^{\eta_{k(t)}}(t)|b^{\eta_{k(t)}}(t)), v(t))\!)_p\,\mathrm{d}t = 2\kappa(\!(\lambda(t) - \mathcal{L}\bullet(\nu,\mu), v(t))\!)_p, \\
\mathcal{F}(t) &= \lim_{\eta_{k(t)}\to 0_+} [\mathfrak{G}_\eta]'_t(t, y_{\eta_{k(t)}}(t)) = \mathfrak{G}'_t(t, y(t)).
\end{aligned}
$$

Thus, when passing to the limit $\eta \to 0_+$ in (78), using weak-lower semicontinuity of the convex terms and non-negativity of $\eta\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta) + \eta\|\nabla_p\lambda^\eta(T)\|^2_{W^{-1,2}(\omega;\mathbb{R}^{(M+1)\times 2})}$ we get, similarly as in Step 3 of the proof of Theorem 1, the reformulated mesoscopic flow rule

$$
\begin{aligned}
\mathfrak{G}(T) + \int_{\mathcal{Q}} \frac{\alpha}{q}|\dot\lambda|^q + \delta^*_S(\dot\lambda)\,\mathrm{d}z_p\mathrm{d}t &\le \mathfrak{G}(0) + \int_0^T \mathfrak{G}'_t(t, y(t))\,\mathrm{d}t \\
&+ \int_{\mathcal{Q}} (\Theta(w) - \theta_{\mathrm{tr}})\mathfrak{a}\cdot(v - \dot\lambda) + \frac{\alpha}{q}|v|^q + \delta^*_S(v)\,\mathrm{d}z_p\mathrm{d}t + \int_0^T 2\kappa(\!(\lambda - \mathcal{L}\bullet(\nu,\mu), v)\!)_p\,\mathrm{d}t, \qquad (80)
\end{aligned}
$$

where, by density, the test functions can be taken from the whole of $L^q(\mathcal{Q};\mathbb{R}^{M+1})$.

Step 2: Minimization principle, back to the original flow-rule. First, we notice that (68) is equivalent to

$$
\mathfrak{G}(t, y, \nu, \mu, \lambda(t)) \le \mathfrak{G}(t, \bar y, \bar\nu, \bar\mu, \lambda(t))
$$

for every $(\bar y,\bar\nu,\bar\mu)\in W^{1,p}(\omega;\mathbb{R}^3)\times\mathscr{G}^p_{\Gamma_D}(\omega;\mathbb{R}^{3\times2})\times\mathscr{Y}^p(\omega;\mathbb{R}^3)$ such that $\bar y=\mathrm{id}\bullet\bar\nu_{z_p}$ for a.a. $z_p\in\omega$

Thus, thanks to (33), we have

$$\begin{aligned}\mathfrak{G}(t,y,\nu,\mu,\lambda(t))&\le\liminf_{\eta_{k(t)}\to0_+}\mathfrak{G}_{\eta_{k(t)}}(t,y^{\eta_{k(t)}}(t),b^{\eta_{k(t)}}(t),\lambda^{\eta_{k(t)}}(t))\\&\le\liminf_{\eta_{k(t)}\to0_+}\mathfrak{G}_{\eta_{k(t)}}(t,\tilde y,\tilde b,\lambda^{\eta_{k(t)}}(t))+\int_\omega\eta_{k(t)}|(\nabla_py^{\eta_{k(t)}}(t)|b^{\eta_{k(t)}}(t))-(\nabla_p\tilde y|\tilde b)|\,\mathrm{d}z_p\\&=\int_\omega W(\nabla_p\tilde y|\tilde b)\,\mathrm{d}z_p+\kappa\|\lambda(t)-\mathcal{L}(\nabla_p\tilde y|\tilde b)\|^2_{W^{-1,2}(\omega;\mathbb{R}^{M+1})}-\int_\omega f^0\cdot\tilde y\,\mathrm{d}z_p-\int_{\gamma_N}g^0\cdot\tilde y\,\mathrm{d}S_p\end{aligned}$$

for every $\tilde y\in W^{2,2}(\omega;\mathbb{R}^3)$ and $\tilde b\in W^{1,2}(\omega;\mathbb{R}^3)$, such that $y=0$ on γ_D. By density, we have that

$$\mathfrak{G}(t,y,\nu,\mu,\lambda(t))\le\int_\omega W(\nabla_p\tilde y|\tilde b)\,\mathrm{d}z_p+\kappa\|\lambda(t)-\mathcal{L}(\nabla_p\tilde y|\tilde b)\|^2_{W^{-1,2}(\omega;\mathbb{R}^{M+1})}-\int_\omega f^0\cdot\tilde y\,\mathrm{d}z_p-\int_{\gamma_N}g^0\cdot\tilde y\,\mathrm{d}S_p$$

even for all $\tilde y\in W^{1,2}(\omega;\mathbb{R}^3)$ satisfying $y=0$ on γ_D and all $\tilde b\in L^2(\omega;\mathbb{R}^3)$. Take an arbitrary pair of admissible Young measure $(\tilde\nu,\tilde\mu)\in\mathscr{G}^p_{\gamma_D}(\omega;\mathbb{R}^{3\times2})\times\mathscr{Y}^p(\Omega;\mathbb{R}^3)$, then we can always find its bounded generating sequence $\{(\nabla_p\tilde y_k,\tilde b_k)\}_{k\in\mathbb{N}}\subset L^p(\omega;\mathbb{R}^{3\times2})\times L^p(\omega;\mathbb{R}^3)$ such that $\{|\nabla_p\tilde y_k|^p+|\tilde b_k|^p\}_{k\in\mathbb{N}}$ is equi-integrable [22], $\{y_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\omega;\mathbb{R}^3)$ is bounded and $y_k(z_1,z_2)=0$ for $z\in\gamma_D$ for all $k\in\mathbb{N}$. Passing to the limit for $k\to\infty$ in the previous inequality with $\tilde y_k$ and $\tilde b_k$ in place of $\tilde y$ and $\tilde b$ we get that $\mathfrak{G}(t,y,\nu,\mu,\lambda(t))\le\mathfrak{G}(t,\tilde y,\tilde\nu,\tilde\mu,\lambda(t))$ where $\tilde y$ is the weak limit of $\tilde y_k$. Hence, (68) is shown.

Note that as a side product of the above procedure we obtained also that

$$\mathfrak{G}(0):=\mathfrak{G}(0,y(0),\nu(0),\mu(0),\lambda(0))=\lim_{\eta\to0_+}\mathfrak{G}_\eta(0),\tag{81a}$$

$$\mathfrak{G}(T):=\mathfrak{G}(T,y(T),\nu(T),\mu(T),\lambda(T))=\lim_{\eta\to0_+}\mathfrak{G}_\eta(T).\tag{81b}$$

Hence, the reformulated flow rule reads as

$$\begin{aligned}\mathfrak{G}(T)+\int_{\mathcal{Q}}\frac{\alpha}{q}|\dot\lambda|^q+\delta^*_S(\dot\lambda)\,\mathrm{d}z_p\mathrm{d}t\le\mathfrak{G}(0)+\int_0^T\mathfrak{G}'_t(t,y(t))\,\mathrm{d}t\\+\int_{\mathcal{Q}}(\Theta(w)-\theta_{\rm tr})\mathfrak{a}\cdot(v-\dot\lambda)+\frac{\alpha}{q}|v|^q+\delta^*_S(v)\,\mathrm{d}z_p\mathrm{d}t+\int_0^T2\kappa(\!(\lambda-\mathcal{L}\bullet(\nu,\mu),v)\!)_p\,\mathrm{d}t,\end{aligned}\tag{82}$$

and exploiting the balance of the mesoscopic deformation-related energy equality—cf. Remark 5 and (73)—we also get the *mesoscopic flow rule* (69).

Step 3: Strong convergence of $\dot\lambda^\eta$. This convergence is obtained from the monotonicity properties of the dissipation term $|\cdot|^q$ in the reformulated flow rule. Indeed, let us rewrite (78) (relying on the convexity of $|\cdot|^q$) as

$$\begin{aligned}\mathfrak{G}_\eta(T)+\eta\mathrm{Var}_{|\cdot|}(\nabla_py^\eta|b^\eta)+\eta\|\nabla_p\lambda^\eta(T)\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times2})}+\int_{\mathcal{Q}}\delta^*_S(\dot\lambda^\eta)\,\mathrm{d}z_p\mathrm{d}t\le\int_0^T[\mathfrak{G}_\eta]'_t(t,y^\eta(t))\,\mathrm{d}t\\+\mathfrak{G}_\eta(0)+\int_{\mathcal{Q}}\alpha|\dot\lambda^\eta|^{q-2}\dot\lambda^\eta\cdot(\tilde v-\dot\lambda^\eta)+(\Theta(w^\eta){-}\theta_{\rm tr})\mathfrak{a}\cdot(\tilde v{-}\dot\lambda^\eta)+\delta^*_S(\tilde v)+2\eta\nabla_p\lambda^\eta\cdot\nabla_p\tilde v\,\mathrm{d}z_p\mathrm{d}t\\+\int_0^T2\kappa(\!(\lambda^\eta-\mathcal{L}(\nabla_py^\eta|b^\eta),\tilde v)\!)_p\,\mathrm{d}t+\eta\|\nabla_p\lambda_0\|^2_{L^2(\omega;\mathbb{R}^{(M+1)\times2})};\end{aligned}\tag{83}$$

similarly, (82) is rewritten as

$$\begin{aligned}\mathfrak{G}(T)+\int_{\mathcal{Q}}\frac{\alpha}{q}|\dot\lambda|^q+\delta^*_S(\dot\lambda)\,\mathrm{d}z_p\mathrm{d}t\le\mathfrak{G}(0)+\int_0^T\mathfrak{G}'_t(t,y(t))\,\mathrm{d}t+\int_0^T2\kappa(\!(\lambda-\mathcal{L}\bullet(\nu,\mu),v)\!)_p\,\mathrm{d}t\\+\int_{\mathcal{Q}}\alpha|\dot\lambda|^{q-2}\dot\lambda\cdot(v-\dot\lambda)+(\Theta(w)-\theta_{\rm tr})\mathfrak{a}\cdot(v-\dot\lambda)+\delta^*_S(v)\,\mathrm{d}z_p\mathrm{d}t.\end{aligned}\tag{84}$$

Then, let us test (84) by $\dot\lambda^\eta$ and, symmetrically, (83) by λ'_j, a member of the sequence $\{\lambda'_j\}_{j\in\mathbb{N}}\subset\mathcal{V}\cap C(0,T;W^{1,2}(\omega;\mathbb{R}^{M+1}))$ such that $\lambda'_j\to\dot\lambda$ in $L^q(\mathcal{Q};\mathbb{R}^{M+1})$ for $j\to\infty$ (recall that $\mathcal{V}$ is the dense countable subset of $L^q(\mathcal{Q};\mathbb{R}^{M+1})$ used in Step 1), as $\dot\lambda$ does not have the required smoothness to be used as a test function in (83) and, moreover, we wish to use (79) (as well as the resulting convergences in Step 1) which is only available for test functions from $\mathcal{V}$.

Let us add (83) and (84) and apply $\lim_{j\to\infty}\limsup_{\eta\to 0}$ to get

$$
\begin{aligned}
&\alpha \lim_{\eta\to 0}\Big(\|\dot\lambda^\eta\|^{q-1}_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}-\|\dot\lambda\|^{q-1}_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}\Big)\Big(\|\dot\lambda^\eta\|_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}-\|\dot\lambda\|_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}\Big)\\
&\quad\le \limsup_{\eta\to 0}\ \alpha\int_0^T\int_\omega \Big(|\dot\lambda^\eta|^{q-2}\dot\lambda^\eta-|\dot\lambda|^{q-2}\dot\lambda\Big)\cdot(\dot\lambda^\eta-\dot\lambda)\,\mathrm{d}z_p\mathrm{d}t\\
&\quad\le \lim_{j\to\infty}\limsup_{\eta\to 0}\Big(\mathfrak{G}(0)-\mathfrak{G}(T)+\underbrace{\mathfrak{G}_\eta(0)-\mathfrak{G}_\eta(T)}_{\text{(I)}}-\eta\underbrace{\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta)}_{\text{(II)}_1}+\eta\int_\omega|\nabla_p\lambda_0|^2-\underbrace{|\nabla_p\lambda^\eta(T)|^2}_{\text{(II)}_2}\,\mathrm{d}z_p\\
&\qquad+\int_0^T\mathfrak{G}'_t(t,y)+\underbrace{[\mathfrak{G}_\eta]'_t(t,y^\eta)}_{\text{(III)}}\,\mathrm{d}t+\int_{\mathcal{Q}}\alpha\underbrace{|\dot\lambda^\eta|^{q-2}\dot\lambda^\eta(\lambda'_j-\dot\lambda)+\delta^*_S(\lambda'_j)-\delta^*_S(\dot\lambda)}_{\text{(IV)}}\,\mathrm{d}z_p\mathrm{d}t\\
&\qquad+\int_0^T\underbrace{2\kappa(\!(\lambda^\eta-\mathcal{L}(\nabla_p y^\eta|b^\eta),\lambda'_j)\!)_p}_{\text{(V)}}+\underbrace{2\kappa(\!(\lambda-\mathcal{L}\bullet(\nu,\mu),\dot\lambda^\eta)\!)_p}_{\text{(VI)}}\,\mathrm{d}t\\
&\qquad+\int_{\mathcal{Q}}\underbrace{(\Theta(w^\eta)-\theta_{\mathrm{tr}})(\lambda'_j-\dot\lambda^\eta)+(\Theta(w)-\theta_{\mathrm{tr}})(\dot\lambda^\eta-\dot\lambda)\,\mathrm{d}z\mathrm{d}t}_{\text{(VII)}}+\underbrace{2\eta\nabla_p\lambda^\eta\cdot\nabla_p\lambda'_j}_{\text{(VIII)}}\,\mathrm{d}z_p\mathrm{d}t\Big)\\
&\quad\le 2\mathfrak{G}(0)-2\mathfrak{G}(T)+\int_0^T 2\mathfrak{G}'_t(t,y)+4\kappa(\!(\lambda-\mathcal{L}\bullet(\nu,\mu),\dot\lambda)\!)_p\,\mathrm{d}t=0.
\end{aligned}
$$

Here, the first inequality in (85) is due to Hölder's inequality. Further, we used that term (I) is not smaller than $\mathfrak{G}(0)-\mathfrak{G}(T)$ by (81) and the non-negativity of (II)_1 and (II)_2. The convergence of the term between them to 0 is obvious. Term (III) is, owing to Step 1, bounded from above by $\mathfrak{G}'_t(t,y)$. Now, as $j\to\infty$ term (IV) converges to 0 as $\dot\lambda^\eta$ is bounded uniformly in $L^q(\mathcal{Q};\mathbb{R}^{M+1})$. The limsup of the term (V), again by Step 1, is bounded from above by $(\!(\lambda-\mathcal{L}\bullet(\nu,\mu),\dot\lambda)\!)_p$; for the terms (VI) and (VII) we proceed analogously as in Step 1, while the term (VIII) converges to 0 as the limit $\eta\to 0_+$ is executed first.

Finally, note that the last equality is due to the balance of the deformation related energy; cf. Remark 5. Hence, we obtained $\|\dot\lambda^\eta\|_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}\to\|\dot\lambda\|_{L^q(\mathcal{Q};\mathbb{R}^{M+1})}$ and from (76) by the uniform convexity of $L^q(\mathcal{Q};\mathbb{R}^{M+1})$ also (74).

Step 4: Enthalpy equation. It only remains to prove the enthalpy equation (70); to obtain it, we pass to the limit $\eta\to 0_+$ in (37) following ideas of Step 7 in the proof of Theorem 1. In order to pass to the limit in the terms expressing the heating due to dissipation, however, we need to show that $\eta\int_{\bar{Q}}\zeta\,\mathcal{H}^\eta(\,\mathrm{d}z_p\mathrm{d}t)\to 0$. To see this, we actually need only to show that $\lim_{\eta\to 0}\eta\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta)=0$ which we obtain by passing to the limit in (34). Indeed,

$$
\begin{aligned}
\limsup_{\eta\to 0}\eta\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta)\le\limsup_{\eta\to 0}\Big(&-\mathfrak{G}_\eta(T)+\mathfrak{G}_\eta(0)\\
&+\int_0^T\Big\langle[\mathfrak{G}_\eta]'_\lambda(y^\eta(t),b^\eta(t),\lambda^\eta(t)),\dot\lambda^\eta\Big\rangle+[\mathfrak{G}_\eta]'_t(t,y^\eta(t))\,\mathrm{d}t\Big).
\end{aligned}\tag{85}
$$

To pass to the limit on the right-hand side, we rewrite

$$\Big\langle[\mathfrak{G}_\eta]'_\lambda(y^\eta(t),b^\eta(t),\lambda^\eta(t)),\dot\lambda^\eta\Big\rangle=\Big\langle[\mathfrak{G}_\eta]'_\lambda(y^\eta(t),b^\eta(t),\lambda^\eta(t)),\dot\lambda\Big\rangle+\Big\langle[\mathfrak{G}_\eta]'_\lambda(y^\eta(t),\lambda^\eta(t)),\dot\lambda^\eta-\dot\lambda\Big\rangle.\tag{86}$$

Note that for the first term we get by Step 1 (if necessary, we can approximate $\dot\lambda$ by $\{\dot\lambda_\ell\}_{\ell\in\mathbb{N}}$ belonging to the dense countable subset of $L^q(\mathcal{Q};\mathbb{R}^{M+1})$ used in Step 1)

$$\Big\langle[\mathfrak{G}_\eta]'_\lambda(y^\eta(t),b^\eta(t),\lambda^\eta(t)),\dot\lambda\Big\rangle\le\int_0^T\Big\langle\mathfrak{G}'_\lambda(\nu(t),\mu(t),\lambda(t)),\dot\lambda\Big\rangle\,\mathrm{d}t,\tag{87}$$

while the second term converges to 0 in $L^1([0,T])$ owing to Step 3. Thus, we get

$$0\le\limsup_{\eta\to 0_+}\eta\mathrm{Var}_{|\cdot|}(\nabla_p y^\eta|b^\eta)\le\mathfrak{G}(0)-\mathfrak{G}(T)+\int_0^T\Big\langle\mathfrak{G}'_\lambda(\nu(t),\mu(t),\lambda(t)),\dot\lambda\Big\rangle+\mathfrak{G}'_t(t,y(t))\,\mathrm{d}t\le 0,\tag{88}$$

where the last inequality follows from Remark 5. □

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References

[1] R. Abeyaratne, C. Chu, R.D. James, *Kinetics of materials with wiggle energies: theory and application to the evolution of twinning microstructures in a Cu-Al-Ni shape memory alloy.* Philosophical Magazine A **73** (1996), 457–497.

[2] O. Anza Hafsa, J.-P. Mandallena, *Relaxation theorems in nonlinear elasticity.* Ann. I.H. Poincaré-AN **25** (2008), 135–148.

[3] O. Anza Hafsa, J.-P. Mandallena, *Relaxation and 3D-2D passage theorems in hyperelasticity.* J. Convex Anal. **19** (2012), 759–794.

[4] M. Arndt, M. Griebel, V. Novák, T. Roubíček, P. Šittner, *Martensitic transformation in NiMnGa single crystals: numerical simulation and experiments*, Int. J. Plasticity **22** (2006), 1943–1961.

[5] B. Benešová, M. Kružík, G. Pathó, *Young measures supported on invertible matrices.* Appl. Anal. (2013), DOI 10.1080/00036811.2012.760039.

[6] B. Benešová, M. Kružík, T. Roubíček, *Thermodynamically-consistent mesoscopic model for the ferro/para-magnetic transition.* Zeit. angew. Math. Phys. **64** (2013), 1-28.

[7] B. Benešová, T. Roubíček, *Micro-to-meso scale limit for shape-memory-alloy models with thermal coupling.* Multiscale Modeling and Simulation **10** (2012), 1059–1089.

[8] A.L. Bessoud, F. Krasucki, G. Michaille, *A relaxation process for bifunctionals of displacement-Young measure state variables: a model of multi-material with micro-structured strong interface.* Ann I. H. Poincaré–AN **27** (2010), 447–469.

[9] K. Bhattacharya, *Microstructure of martensite. Why it forms and how it gives rise to the shape-memory effect.* Oxford University Press, Oxford (2003).

[10] K. Bhattacharya, R.D. James, *A theory of thin films of martensitic materials with applications to microactuators.* J. Mech. Phys. Solids **47** (1999), 531–576.

[11] K. Bhattacharya, P. Purohit, B. Craciun, *The mobility of twin and phase boundaries.* J. Phys. IV France **112** (2003), 163–166.

[12] K. Bhattacharya, A. Sadjadpour, *A micromechanics inspired constitutive model for shape-memory alloys: the one dimensional case.* Smart Mater. Struct. **16** (2007), S51–62.

[13] L. Boccardo, A. Dall'aglio, T. Gallouët, L. Orsina, *Nonlinear parabolic equations with measure data.* J. Funct. Anal. **147** (1997), 237–258.

[14] L. Boccardo, T. Gallouët, *Non-linear elliptic and parabolic equations involving measure data.* J. Funct. Anal. **87** (1989), 149–169.

[15] M. Bocea, *Young measure minimizers in the asymptotic analysis of thin films.* Sixth Mississippi State Conference on Differential Equations and Computational Simulations; Electron. J. Diff. Eqns., Conf., **15** (2007), 41–50.

[16] G. Dal Maso, G.A. Francfort, R. Toader, *Quasistatic crack growth in nonlinear elasticity.* Arch. Ration. Mech. Anal. **176** (2005), 165–225.

[17] A. De Simone, *Energy minimizers for large ferromagnetic bodies.* Arch. Rational Mech. Anal. **125** (1993), 99–143.

[18] G. Dolzmann, *Variational methods for crystalline microstructure. Analysis and computation.* Springer, Berlin (2003).

[19] P.W. Dondl, C.P. Shen, K. Bhattacharya, *Computational analysis of martensitic thin films using subdivision surfaces.* Int. J. Num. Meth. Eng. **72** (2007), 72–94.

[20] T. Duerig, A. Pelton, D. Stöckel, *An overview of nitinol medical applications.* Materials Science and Engineering **A273–275** (1999), 149–160.

[21] F. Falk, P. Konopka, *Three-dimensional Landau theory describing the martensitic phase transformation of shape-memory alloys.* J. Condens. Matter **2** (1990), 61–77.

[22] I. Fonseca, S. Müller, P. Pedregal, *Analysis of concentration and oscillation effects generated by gradients.* SIAM. J. Math. Anal. **29** (1998), 736–756.

[23] G. Francfort, A. Mielke, *Existence results for a class of rate-independent material models with nonconvex elastic energies.* J. Reine Angew. Math. **595** (2006), 55–91.

[24] L. Freddi, R. Paroni, *The energy density of martensitic thin films via dimension reduction.* Interface. Free Bound. **6** (2004), 439–459.

[25] M. Frémond, S. Miyazaki, *Shape memory alloys.* Springer, Wien (1996).

[26] B. Halphen, Q.S. Nguyen, *Sur les materiaux standards généralisés.* J. Mécanique **14** (1975), 39–63.

[27] R.D. James, *Hysteresis in phase transformations.* In: ICIAM 95, Proc. 3rd Int. Congress Industr. Appl. Math. (eds. K. Kirchässner et al.) Akademie Verlag, Berlin, Math. Res. **87** (1996), 135–154.

[28] D. Kinderlehrer, P. Pedregal, *Gradient Young measures generated by sequences in Sobolev spaces.* J. Geom. Anal. **4**, 59–90.

[29] M. Kružík, A. Mielke, T. Roubíček, *Modelling of microstructure and its evolution in shape-memory-alloy single-crystals, in particular in CuAlNi.* Meccanica **40** (2005), 389–418.

[30] H. Le Dret, A. Raoult, *The nonlinear membrane model as variational limit in nonlinear three-dimensional elasticity.* J. Math. Pures Appl. **74** (1995), 549–578.

[31] M.L. Leghmizi, C. Licht, G. Michaille, *The nonlinear membrane model: a Young measure and varifold formulation.* ESAIM: Control Optim. Calc. Var. **11** (2005), 449–472.

[32] D.R. Mahapatra, R.V.N. Melnik, *A dynamic model for phase transformations in 3D samples of shape memory alloys.* LNCS Springer-Verlag **3516** (2005), 25–32.

[33] A. Mielke, T. Roubíček, *A rate-independent model for inelastic behavior of shape-memory alloys.* Multiscale Model. Simul. **1** (2003), 571–597.

[34] A. Mielke, T. Roubíček, U. Stefanelli, *Γ-limits and relaxations for rate-independent evolutionary systems.* Calc. Var. Partial Differ.Equat. **31** (2008), 387–416.

[35] A. Mielke, F. Theil, V.I. Levitas, *A variational formulation of rate-independent phase transformations using an extremum principle.* Arch. Rational Mech. Anal. **162** (2002), 137–177.

[36] A. Mielke, L. Truskinovsky, *From discrete visco-elasticity to continuum rate-independent plasticity: Rigorous results.* Arch. Ration. Mech. Anal. **203** (2012), 577-619.

[37] S. Miyazaki, Y.Q. Fu, W.M. Huang, *Thin Film Shape Memory Alloys: Fundamentals and Device Applications.* Cambridge University Press, New York (2009).

[38] S. Müller, *Variational models for microstructure and phase transitions.* In: Calc. of Var. and Geometric Evol. Problems (Eds.: S.Hildebrandt et al.) L. N. in Math. **1713**, Springer, Berlin, (1999), 85–210.

[39] J. Musielak, W. Orlicz, *On generalized variations (I).* Studia Math. **18** (1959), 13–41.

[40] G. Pathó, *Modelling of thin films of shape-memory alloys.* Technische Mechanik **32** (2012), 507–517.

[41] E. Patoor, *An overview of dierent approaches used to model SMAs and SMA structures*, keynote lecture at ESOMAT Prague, 2009.

[42] M. Pitteri, G. Zanzotto, *Continuum models for phase transitions and twinning in crystals.* CRC Press, Boca Raton, (2003).

[43] T. Roubíček, *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, Basel (2005).

[44] T. Roubíček, *Thermodynamics of rate independent processes in viscous solids at small strains.* SIAM J. Math. Anal. **42** (2010), 256–297.

[45] T. Roubíček, *Thermodynamics of perfect plasticity.* Discrete and Cont. Dynam. Syst. - S **6** (2013), 193–214.

[46] T. Roubíček, *Models of microstructure evolution in shape memory materials*, Chapter in Nonlinear Homogenization and its Application to Composites, Polycrystals and Smart Materials,(Eds. P.Ponte Castaneda, J.J.Telega, B.Gambin), NATO Sci.Ser.II/170, Kluwer, Dordrecht, (2004), pp. 269-304.

[47] T. Roubíček, G. Tomassetti: *Phase transformations in electrically conductive ferromagnetic shape-memory alloys, their thermodynamics and analysis.* Arch, Ration. Mech. Anal., accepted.

[48] S. Saadat, J. Salichs, M. Noori, Z. Hou, H. Davooli, I. Baron, Y. Suzuki, A. Masuda, *An overview of vibration and seismic applications of NiTi shape memory alloy.* Smart Mater. Struct. **11** (2002), 218–229.

[49] Y.C. Shu, *Heterogeneous thin films of martensitic materials.* Arch. Rational Mech. Anal. **153** (2000), 39–90.

[50] P. Sedlák, M. Frost, B.Benešová, T. Ben Zineb, P. Šittner: *Thermomechanical model for NiTi-based shape memory alloys including R-phase and material anisotropy under multi-axial loadings.* Int. J. Plasticity, **39** (2012), 132–151.

[51] L.C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations.* C. R. Soc. Sci. Varsovie, **30** (1937), 212–234.