Nečas Center for Mathematical Modeling

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Preprint no. 2013-013

http://ncmm.karlin.mff.cuni.cz/

A model of rupturing lithospheric faults with re-occurring earthquakes

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Abstract: An isothermal small-strain model based on the concept of generalized standard materials is devised, combining Maxwell-type rheology, damage, and perfect plasticity in the bulk. An interface analogue of the model is prescribed at the lithospheric faults, exploiting concepts of adhesive contacts with interfacial plasticity. The model covers simultaneously features such as rupturing of the fault zone accompanied with weakening/healing effects and also seismic waves emission and propagation connected with the sudden ruptures of the fault or a fluidic-like aseismic response between the ruptures. Stable numerical strategy based on semi-implicit discretisation in time is devised and its convergence is shown. Numerical simulations documenting the capacity of the model to simulate earthquakes with repeating occurrence are performed, too.

Keywords: seismic fault rupture, tectonic earthquakes, activated processes, aseismic slip, stable time discretisation, weak solution, convergence.

AMS Subj. Class. 35K85, 35Q86, 49S05, 74L05, 86A15.

1. Introduction.

A physically and mathematically sound description of the evolution and properties of a seismic fault zone and of the processes in the surrounding bulk material represents a very challenging task due to the great complexity of the processes involved. Tectonic earthquakes occur at the dynamic contacts of lithospheric plates, in regions where their mutual motion driven by the mantle convection has been restrained or totally disabled by a localized locking of the lithospheric blocks. In these so-called seismic gaps, the elastic strain energy gets gradually accumulated until the critical point when the corresponding stresses exceed the rigidity of the material (or typically the much lower rigidity of the material contact) leading to a sudden energy release by a rupture - earthquake. Here we are concerned only with the so-called tectonic earthquakes resulting from the lithospheric processes just outlined, in contrast to the so-called volcanic or explosive earthquakes which originate from different energy storage and release mechanisms.

During an earthquake, a rupture typically spreads from a particular spot, called the hypocenter, along a geologically predefined fault or possibly also extends the rupture zone into a previously intact medium. The strain energy, released typically on the fault and in its vicinity, is mostly absorbed by dissipative processes of frictional and plastic heat, damage and crack propagation, and a smaller part of it (typically $\leq$10%) is radiated away in the form of seismic waves.

In the case of a geologically complex fault zone, since the earthquake substantially changes the stress regime in the vicinity of the fault, this may lead to either an earthquake triggering at the neighboring faults in the cases when the stress is increased there, or, vice-versa, to a decrease of the seismic hazard when the stress gets reduced.

The rheological properties of both the contact zone and the surrounding bulk undergo quite a complex evolution during the earthquake, as documented by the observed (and in laboratory measured) phenomena such as slip and/or rate-of slip weakening or strengthening of the contact, fluid pressurization of the fault zone, partial melting of the frictional contact, material damage and damage-induced weakening of the elastic moduli in the bulk, etc.; see e.g. [25].

The rupture is followed by a process of successive healing of the fault and the bulk and of a gradual recreation of the surface bonds, leading possibly to a repeated locking of the fault zone and future earthquake reoccurrence. It is also possible that the fault resumes in a totally aseismic regime exhibiting a relatively smooth relative movement of the plates without further substantial elastic energy storage.

The processes described above cover the time span of units of seconds (rupture itself) up to tens of years (healing, earthquake reoccurrence). On much longer geological time scales of thousands up to millions of years the bulk material is also subject to a visco-elastic or even fluidic-like deformation and flow.

One of the key features of the studied problem is thus its obvious multi-scale nature both in space and time. Any suitable mathematical and physical model should be able to cover very long periods of slow healing and fluid-like flow and at the same time also the very fast processes during the earthquakes. The same concerns the space dimension, since the most dynamic part of the process is typically confined to narrow fault regions with the overall volume very small compared to the volume of the bulk of the lithosphere – it is thus worth modelling these zones as contact surfaces rather than layers.

In spite of the huge computational activity in geophysical modelling of seismic rupture processes during many past decades, it seems that there does not exist a model which would address the phenomena mentioned above and simultaneously bear a rigorous mathematical and numerical analysis as far as mere existence of its solution and stability and mere convergence of its numerical approximations is concerned.

Our goal is just to devise such models which facilitate rigorous mathematical treatment and devise an efficient computational scheme that allows for rigorous numerical analysis as far as stability and convergence concerns.

The philosophy of the models relies on the concept of suitably chosen internal parameters based on Halphen-Nguyen's generalized standard materials [21] inspired in particular by models of damage, plasticity, and adhesive contacts, combined with the modern concepts from the mathematical theory of rate-independent processes. We apply the simplified approach to handle the multi-scale character of the processes in time, namely that some fast processes are considered as rate-independent, i.e. they can even be infinitely fast (=jumping) in comparison with the other (relatively) slowly evolving processes.

In contrast to the conventional models used in seismic simulations, which mostly rely on combining the elasto-dynamic equations (and possibly plasticity) in the bulk with some - often empirically derived - yield/sliding criteria on the fault plane (to name a few, see e.g, [2, 4, 5, 8, 9, 11, 26] or further references in Remarks 2.2 and 3.1 below.), the advantage of the models proposed below is that they simultaneously:

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- ▷ originate from lucid general constructions and allow for verification of the ultimate physical principles as the energy conservation and the non-negative dissipation rate (or, in other words, entropy production), cf. also Remark 3.2 below,
- ▷ are able to capture much of the high complexity of the problem,
- ▷ use formally the same concepts with the analogous set of internal parameters for the bulk model as well as for the interface (fault) model,
- ▷ bear a rigorous mathematical and numerical analysis giving some solid background to the computational simulations.

While in this article we will confine ourselves merely to an isothermal case, a thermodynamically consistent completion by including the heat transport and the temperature dependence of the material parameters would be relatively straightforward, like e.g. [41, 44].

The plan of this paper is as follows: In Section 2, a generic model of the bulk material is introduced, its description being based on the specification of appropriate storage energy and dissipation potentials. This, accompanied by a certain variational principle, fully describes both the reversible and irreversible components of the energy budget and material evolution. In Section 3, the model from the bulk is "projected" to the fault surface using a similar approach as in the bulk. In Section 4, we devise a semi-implicit time discretisation that allows for an efficient computer implementation, and show basic a-priori estimates. Then, in Section 5, we devise a weak formulation of the model, and prove the existence of the corresponding solutions as well as the convergence of the discretization assuming a damage-independent viscous attenuation. We conclude in Section 6 by a simple demonstration of the computational capabilities of the model for a simplified single-degree-of-freedom slider experiment.

For readers' convenience, we first state the list of the main notation together with the corresponding physical dimensions:

Table 1.1
List of notation.

Symbol	Quantity	Valued in	Dimension (for $d=3$)
u	displacement	$\mathbb{R}^d$	[m]
$e(u)$	small strain tensor, $e(u)=\frac{1}{2}\nabla u^\top+\frac{1}{2}\nabla u$	$\mathbb{R}^{d\times d}_{\rm sym}$	[1]
$e_{\rm i}=[\![u]\!]$	displacement jump across the fault $\Gamma_{\rm C}$	$\mathbb{R}^d$	[m]
π	plastic strain	$\mathbb{R}^{d\times d}_{\rm dev}$	[1]
$\pi_{\rm i}$	plastic interfacial slip	$\mathbb{R}^{d-1}$	[m]
ζ	damage parameter	$\mathbb{R}$	[1]
$\zeta_{\rm i}$	interfacial damage (=delamination) parameter	$\mathbb{R}$	[1]
ε	Maxwellian strain	$\mathbb{R}^{d\times d}_{\rm sym}$	[1]
$\varepsilon_{\rm i}$	interfacial Maxwellian slip	$\mathbb{R}^{d-1}$	[m]
ϱ	mass density	$\mathbb{R}$	[kg m^{-3}]
$\mathbb{C}(\zeta)$	tensor of elastic moduli	$\mathbb{R}^{d\times d\times d\times d}$	[Pa=J m^{-3}]
$\mathbb{C}_{\rm i}(\zeta_{\rm i})$	tensor of interfacial elastic moduli	$\mathbb{R}^{(d-1)\times(d-1)}$	[Pa m^{-1}]
$\mathbb{D}$, $\mathbb{D}_0$	tensors of viscosity moduli	$\mathbb{R}^{d\times d\times d\times d}$	[Pa s]
$\mathbb{D}_{\rm i}$	interfacial viscous moduli	$\mathbb{R}^{(d-1)\times(d-1)}$	[Pa s m^{-1}]
$\mathfrak{c}(\zeta)$	stored energy of damage	$\mathbb{R}$	[Pa]
$\mathfrak{c}_{\rm i}(\zeta_{\rm i})$	stored energy of interfacial damage	$\mathbb{R}$	[Pa m]
$\mathfrak{d}$	dissipation energy of damage	$\mathbb{R}$	[Pa]
$\mathfrak{d}_{\rm i}$	dissipation energy of interfacial damage	$\mathbb{R}$	[Pa m]
P	undamaged elasticity domain (plastic yield stress)	$\subset\mathbb{R}^{d\times d}_{\rm dev}$	[Pa]
$P_{\rm i}$	undamaged interfacial plastic yield stress	$\subset\mathbb{R}^{d-1}$	[Pa]
$\alpha(\zeta)$	damage coefficient for plastic yield stress	$\mathbb{R}$	[1]
$\alpha_{\rm i}(\zeta_{\rm i})$	damage coefficient for interfacial plastic yield stress	$\mathbb{R}$	[1]
$\mathfrak{a}$	bulk time-scale-of-healing coefficient	$\mathbb{R}$	[Pa s]
$\mathfrak{a}_{\rm i}$	interfacial time-scale-of-healing coefficient	$\mathbb{R}$	[Pa s m]
$\mathfrak{f}$	damage flow-rule nonlinearity	$\mathbb{R}$	[Pa]
$\mathfrak{f}_{\rm i}$	interfacial damage flow-rule nonlinearity	$\mathbb{R}$	[Pa m]
f	gravity force	$\mathbb{R}^d$	[N m^{-3}]
$\mathfrak{b}$	coefficient for the rate-effect in damage	$\mathbb{R}$	[Pa s]
$\mathfrak{b}_{\rm i}$	coefficient for the rate-effect in interfacial damage	$\mathbb{R}$	[Pa s m]
κ, κ_0	coef. for the scale effect of plasticity and damage	$\mathbb{R}$	[Pa m^2]
κ_1	coefficient for the scale rate effect of damage	$\mathbb{R}$	[Pa m^{r}s^{r-1}]
$\kappa_{\rm i}$	coefficient for the scale effect of interfacial plasticity	$\mathbb{R}$	[Pa m]
$\kappa_{0\rm i}$	coefficient for the scale effect of interfacial damage	$\mathbb{R}$	[Pa m^3]
$\kappa_{1\rm i}$	coef. for the scale rate effect of interfacial damage	$\mathbb{R}$	[Pa m$^{r_{\rm i}+1}$s$^{r_{\rm i}-1}$]

where we used the notation $\mathbb{R}^{d\times d}_{\rm dev}=\{A\in\mathbb{R}^{d\times d}_{\rm sym};\ {\rm trace}\,A=0\}$ for deviatoric matrices with $\mathbb{R}^{d\times d}_{\rm sym}=\{A\in\mathbb{R}^{d\times d};\ A^\top=A\}$, and where the exponents r and $r_{\rm i}$ refer to (2.1e) and (3.1b). We consider a domain $\Omega\subset\mathbb{R}^d$ encompassing the $(d-1)$-dimensional manifold (fault) $\Gamma_{\rm C}$ which divide it into two parts, Ω_1 and Ω_2. Further, $\Gamma_{\rm D}$ is the part of the outer boundary of $\partial\Omega$ where the Dirichlet boundary conditions are imposed, cf. (2.3c) below, and $\Gamma_{\rm N}:=\partial\Omega\backslash\Gamma_{\rm D}$ is the part of the outer boundary where Neumann boundary conditions are imposed.

2. The model in the bulk. We first discuss the model for the bulk surrounding the fault. The basic philosophy is to devise certain minimal amount of internal parameters that, however, still are able to reproduce all the desired phenomena mentioned in Sect.1. In order to capture the propagation of elastic waves in the bulk subject to relatively small attenuation and also the creation of off-fault shear bands and possible creation of a new fault, we have to combine inertia with a visco-elasto-plastic behavior material (of combined Maxwell and Kelvin-Voigt type).

Therefore, we choose the internal parameters in the bulk to be: the *plastic strain* π and the *Maxwellian strain* ε, cf. Fig. 1. Following Frémond's concept [16, 17] used also in geophysics [22, 28–31], in order to capture material degradation during the deformational history, connected with disintegration of the material in the seismic fault zone, we introduce another internal parameter ζ, called *damage*, not depicted on Fig. 1, affecting the elasto-visco-plastic properties. To allow for re-occurrence of earthquakes, it is necessary to allow also for *healing*, i.e. a reverse evolution of damage ζ leading to the reconstruction of the previously damaged material.

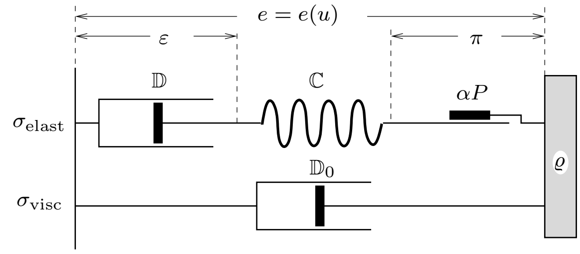

Fig. 1 *Schematic 4-parameter rheological model used in (2.1a-c,e): Maxwell material $(\mathbb{C},\mathbb{D})$ in series with perfectly plastic element P and parallel with a Kelvin-Voigt damper $\mathbb{D}_0$. Damage ζ influencing $\mathbb{C}$, $\mathbb{D}$, and α is not depicted.*

One of the simplest possible scenarios is then to consider a linear response through the Hook-law elastic-moduli tensor $\mathbb{C}$ dependent on damage ζ, viscous response expressed through Maxwell and Kelvin-Voigt viscous-moduli tensors $\mathbb{D}_0$ and $\mathbb{D}$ combined with perfect (no hardening) plasticity with a plastic yield stress dependent on damage ζ (to some extent like used in the Cam-Clay model, cf. e.g. [10, 27, 52], or in the Perzyna model with damage, cf. [49]). Using the dot notation "$(\cdot)^{\bullet}$" for the time derivative, the prime notation "$(\cdot)'$" for the derivative of a smooth function, and "$\partial(\cdot)$" for the subdifferential of a convex possibly nonsmooth function, the dynamic problem then corresponds to the force balance

$$\varrho\ddot{u} - \operatorname{div}\sigma = f, \tag{2.1a}$$

with ϱ the mass density and f a bulk force (here just gravity), and with the rheology expressed as

$$\sigma = \mathbb{D}_0(\zeta)e(\dot{u}) + \mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon), \quad \text{with} \tag{2.1b}$$

$$\dot{\varepsilon} = \mathbb{D}^{-1}(\zeta)\mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon), \tag{2.1c}$$

together with a plastic flow rule (considering a single-threshold linearized plasticity without any hardening)

$$\dot{\pi} \in N_{\alpha(\zeta)P}\Big(\operatorname{dev}\big(\mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon) - \kappa\Delta\pi\big)\Big), \tag{2.1d}$$

where α is a coefficient - α being presumably a monotone function $[0,1]\to[0,1]$ with $\alpha(1)=1$ and where N_K denotes the normal cone to the convex set K, here used for the convex set $K=\alpha(\zeta)P$ which depends on ζ while the surface of the convex set P itself determines the *plastic yield stress* in an undamaged material, and the flow rule for a scalar gradient damage:

$$\mathfrak{f}(\dot{\zeta}) - \mathfrak{c}'(\zeta) \ni -\frac{1}{2}\mathbb{C}'(\zeta)\big(e(u){-}\pi{-}\varepsilon\big){:}\big(e(u){-}\pi{-}\varepsilon\big) + \operatorname{div}\big(\kappa_0\nabla\zeta{+}\kappa_1|\nabla\dot{\zeta}|^{r-2}\nabla\dot{\zeta}\big) \quad \text{with} \quad \mathfrak{f}(\dot{\zeta}) = \begin{cases} \mathfrak{a}\dot{\zeta} & \text{if } \dot{\zeta}>0, \\ [-\mathfrak{d},0] & \text{if } \dot{\zeta}=0, \\ \mathfrak{b}\dot{\zeta}-\mathfrak{d} & \text{if } \dot{\zeta}<0, \end{cases} \tag{2.1e}$$

with $\mathfrak{c}$ being the stored energy for bulk damage, $\mathfrak{d}$ being the dissipation energy for bulk damage, and $\kappa_0,\kappa_1>0$ presumably small coefficients influencing spatial scale of damage profiles. In (2.1e), the notation "$:$" means summation over two indices; later "$\cdot$" will analogously denote summation over one index and "$\vdots$" over three indices. Note that, by using the simple convex-analysis calculus $N_{\alpha P}(\cdot)=\partial\delta_{\alpha P}(\cdot)=\partial\delta_P(\cdot/\alpha)=N_P(\cdot/\alpha)$ with δ_P being the indicator function of P and ∂ denoting the subdifferential, we can write equivalently (2.1d) as $\dot{\pi}\in N_P(\operatorname{dev}((\mathbb{D}(\zeta)\dot{\varepsilon}{-}\kappa\Delta\pi)/\alpha(\zeta)))$. The set-valued nonlinearity $\mathfrak{f}$ has a convex nonsmooth potential $\mathfrak{F}$, i.e. $\mathfrak{f}=\partial\mathfrak{F}$, which we will use later:

$$\mathfrak{F}(\dot{\zeta}) = \frac{\mathfrak{a}}{2}|\dot{\zeta}^+|^2 + \frac{\mathfrak{b}}{2}|\dot{\zeta}^-|^2 - \mathfrak{d}\dot{\zeta}^-\ , \tag{2.2}$$

with $\zeta^+=\max(\zeta,0)$ and $\zeta^-=\min(\zeta,0)$; thus naturally $\mathfrak{F}(\cdot)\geq 0$.

The modelling assumption is that the smooth functions $\mathbb{C}(\cdot)$ and $\mathfrak{c}(\cdot)$ are constant on $(-\infty,0]$ and on $[1,\infty)$, respectively. In particular, $\mathbb{C}'(0)=0$ while $\mathfrak{c}'(0)\geq 0$, and $\mathfrak{c}'(1)=0$ while $\mathbb{C}'(1)\geq 0$. This keeps ζ valued in $[0,1]$ and such constraint need not be explicitly involved in the problem; here also the compatibility of the κ_0- and κ_1-terms with the maximum principle plays an essential role. For this trick, see also [24, Prop. 4.2]. In particular, (2.1e) involves only one set-valued mapping, which facilitates its mathematical analysis.

Note also that, combining (2.1b) with (2.1c), one can express the stress $\sigma=\mathbb{D}_0(\zeta)e(\dot{u}){+}\mathbb{D}(\zeta)\dot{\varepsilon}$ so that for slow processes (when both $\dot{u}$ and $\dot{\varepsilon}$ are small) the stress σ is small and the lithosphere behaves rather like a fluid and never goes into inelastic processes. For this, $\mathbb{D}$ is presumably large to pronounce such aseismic fluidic-like behavior only for large time scales (typical values in Earth mantle are $\sim 10^{19}-10^{24}$ Pa s, depending on the time-scale of the process involved) while $\mathbb{D}_0$ is presumably small rather to "stabilize" mathematically the model and to let the Maxwellian rheology dominant. In any case, the fast (seismic) processes exhibit only relatively small attenuation, expressed for periodic forcing by the so-called "quality factor" Q; $\frac{2\pi}{Q}{:=}\frac{\text{dissipated energy per period}}{\text{stored energy}}$. The typical values in Earth's upper mantle and crust are $\sim 10^2-10^3$, cf. e.g. [40] for a review on Earth's inelasticity. Maxwellian viscoelasticity itself is conventionally considered to be able to capture well such relatively small seismic attenuation and therefore the additional Kelvin-Voigt attenuation (referred to as Jeffrey's rheology, as considered e.g. in [30]) due to $\mathbb{D}_0$ cannot be large and is mostly even neglected in geophysical models.

It is reasonable to assume that damage affects the elastic-plastic properties, typically both the shear and the bulk moduli, cf. [28–31]. Note that we used the so-called gradient theory for damage; for the "static" κ_0-term in (2.1e), we refer also to [30]. In principle, damage affects also viscous properties, cf. e.g. [22], that is why, in full generality, we shall also consider $\mathbb{D}(\zeta)$ and $\mathbb{D}_0(\zeta)$. Only in the final part of Sect. 5, in order to make the convergence analysis tractable, we restrict ourselves to damage-independent $\mathbb{D}$, $\mathbb{D}_0$. Also, if choosing the data reasonably (cf. also Sect. 6 below), we may assume that damage affects faster the plastic activation threshold than the visco/elastic properties. Thus, when damage is triggered, even decaying stresses can still drive the plastic strain to evolve until the "hot" earthquake ends, so that a possible healing can be performed in a truly new configuration, cf. also Remark 2.2 below. In particular, when $\mathfrak{c}'(1)+\mathfrak{d}$ is small (in comparison with $\alpha(1)P \equiv P$), then damage starts first and plasticity only follows. And when $\mathbb{C}(\zeta)/\alpha(\zeta)$ is constant (resp. growing for ζ decaying), even decaying stress in damaging material has enough (resp. even more) strength to evolve plastic strain.

The general perspective of the model is based on the energetics involving the stored energy $\mathscr{E} = \mathscr{E}(q)$, the kinetic energy $\mathscr{M} = \mathscr{M}(\dot{u})$, and the dissipated energy determined by the (pseudo)potential of dissipative forces $\mathscr{R} = \mathscr{R}(q; \dot{q})$.

More specifically, the *bulk* contribution to the *stored energy* discussed in this section is

$$\mathscr{E}_{\rm bulk}(u,\zeta,\pi,\varepsilon) := \int_{\Omega\setminus\Gamma_{\rm C}} \frac{1}{2}\mathbb{C}(\zeta)\big(e(u){-}\pi{-}\varepsilon\big){:}\big(e(u){-}\pi{-}\varepsilon\big) - \mathfrak{c}(\zeta) + \frac{\kappa_0}{2}|\nabla\zeta|^2 + \frac{\kappa}{2}|\nabla\pi|^2\,{\rm d}x, \tag{2.3a}$$

and the outer force $\mathfrak{g}$ acting linearly on u as

$$\big\langle \mathfrak{g}, u\big\rangle := \int_\Omega f{\cdot}u\,{\rm d}x + \int_{\Gamma_{\rm N}} g{\cdot}u\,{\rm d}S, \tag{2.3b}$$

where, beside the gravity force f from (2.1a), we consider the traction force g on a part $\Gamma_{\rm N}$ of the boundary. It is important to specify also the set of the admissible displacements. We consider no cavities on the faults and a prescribed time-dependent motion of parts of the boundary of the considered domain, i.e.

$$[\![u]\!]_{\rm n} = 0 \text{ a.e. on } \Gamma_{\rm C} \quad\text{ and }\quad u|_{\Gamma_{\rm D}} = u_{\rm Dir}(t) \text{ a.e. on } \Gamma_{\rm D}, \tag{2.3c}$$

where $[\![u]\!]_{\rm n}$ is the normal component of the differences of the traces across $\Gamma_{\rm C}$ referring to the unit normal vector ν. Further ingredients are the (pseudo)*potential of dissipative forces*, whose bulk contribution is considered as

$$\mathscr{R}_{\rm bulk}(\zeta;\dot{u},\dot{\zeta},\dot{\pi},\dot{\varepsilon}) := \int_\Omega \frac{1}{2}\mathbb{D}_0(\zeta)e(\dot{u}){:}e(\dot{u}) + \mathfrak{F}(\dot{\zeta}) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}|^r + \alpha(\zeta)\delta_P^*(\dot{\pi}) + \frac{1}{2}\mathbb{D}(\zeta)\dot{\varepsilon}{:}\dot{\varepsilon}\,{\rm d}x, \tag{2.4a}$$

where $\mathfrak{F}$ is from (2.2) and δ_P^* is the Fenchel-Legendre conjugate to the indicator function δ_P of a convex set P determining the yield stress of the undamaged material, and the kinetic energy is

$$\mathscr{M}(\dot{u}) := \int_\Omega \frac{\varrho}{2}|\dot{u}|^2\,{\rm d}x. \tag{2.4b}$$

To facilitate mathematical analysis, it is convenient to make a transformation to a time-constant Dirichlet condition by replacing u with $u{+}u_{\rm D}(t)$ with a suitable extension $u_{\rm D}(t)$ of $u_{\rm Dir}(t)$. Keeping (2.1) unaltered under this substitution, we must modify (2.3) to make $\mathscr{E} = \mathscr{E}(t,u,\zeta,\pi,\varepsilon)$ and $\mathfrak{g} = \mathfrak{g}(t,\zeta)$ time-dependent, namely

$$\mathscr{E}_{\rm bulk}(t,u,\zeta,\pi,\varepsilon) := \int_{\Omega\setminus\Gamma_{\rm C}} \frac{1}{2}\mathbb{C}(\zeta)\big(e(u{+}u_{\rm D}(t)){-}\pi{-}\varepsilon\big){:}\big(e(u{+}u_{\rm D}(t)){-}\pi{-}\varepsilon\big) - \mathfrak{c}(\zeta) + \frac{\kappa_0}{2}|\nabla\zeta|^2 + \frac{\kappa}{2}|\nabla\pi|^2\,{\rm d}x, \tag{2.4c}$$

$$\big\langle \mathfrak{g}(t,\zeta), u\big\rangle = \int_{\Omega\setminus\Gamma_{\rm C}} (f{-}\varrho\,\ddot{u}_{\rm D}(t)){\cdot}u{-}\mathbb{D}_0(\zeta)e(\dot{u}_{\rm D}(t)){:}e(u)\,{\rm d}x + \int_{\Gamma_{\rm N}} g{\cdot}u\,{\rm d}S, \tag{2.4d}$$

$$[\![u]\!]_{\rm n} = 0 \text{ a.e. on } \Gamma_{\rm C} \quad\text{ and }\quad u|_{\Gamma_{\rm D}} = 0 \text{ a.e. on } \Gamma_{\rm D}. \tag{2.4e}$$

Note that $\alpha(\zeta)\delta_P^* = \delta_{\alpha(\zeta)P}^*$ so that the actual activation yield stress in the damaged material is $\alpha(\zeta)P$. Important feature is that $\mathscr{E}$ involves the contribution $\mathfrak{c}$ related with the microcracks and microvoids in the case of damage, which facilitates healing due to the tendency of minimizing the stored energy. For a schematic situation that $\mathfrak{c}(\cdot)$ is affine, the overall *activation energy* for damage is $\mathfrak{c}'{+}\mathfrak{d}$ and the dissipation potential governing (2.1e) is thus the potential $\mathfrak{F}$ of $\mathfrak{f}$ effectively shifted by the affine function $\mathfrak{c}'(\zeta)\dot{\zeta}$, as schematically depicted on Fig. 2. All the coefficients and nonlinearities may depend also on x, which is not explicitly written just for brevity.

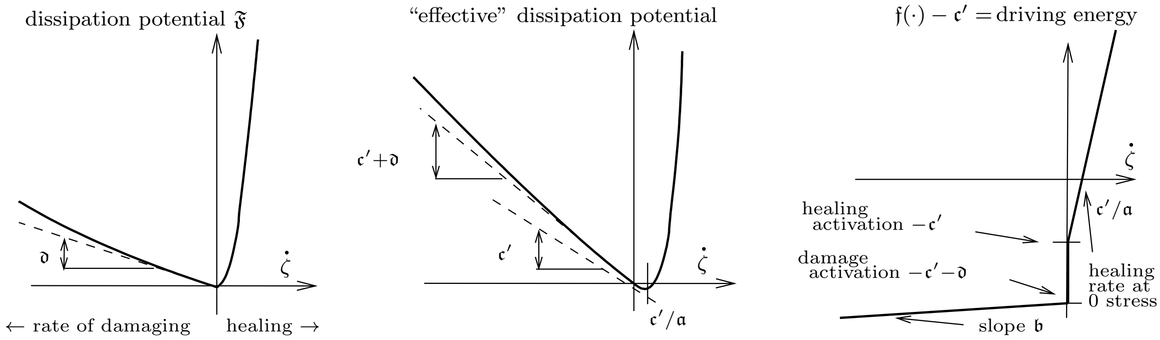

Fig. 2 *Schematic illustration of damage/healling driven by "effective" dissipation potential, its shift by a contribution coming from the stored energy if $\mathfrak{c}(\cdot)$ were affine (middle) and the maximal monotone graph (=its gradient) occurring in the left-hand side of the flow rule* (3.9b) *(right).*

Remark 2.1 (Nonconvex elastic energies). Often, instead of the quadratic form $e\mapsto\mathbb{C}(\zeta)e{:}e$, non-quadratic and even nonconvex potentials are considered to model experimentally observed instabilities, cf. [28, 31]. To put it into a mathematically rigorous frame, one could adopt a concept of the so-called non-simple materials (also called multipolar solids or complex materials), leading to the so-called hyperstresses i.e. the gradient theory for $e(u)$, cf. [39, 51].

Remark 2.2 (Concepts of healing). Reversible damage (or adhesion in Sect. 3) itself (i.e. allowing healing, or so-called rebonding) has been routinely addressed in mathematical literature, cf. [48, 49]. If not combined with any inelastic strain allowing for permanent deformation, healing has a tendency to remember not only the original state of the material but also the original configuration and such models thus have only limited application and, in particular, cannot model re-occurring earthquakes. Thus it appears popular in seismic damage-based models to introduce certain inelastic strain. Often, this strain is controlled directly by damage and, in particular, stops evolving when damage completes (i.e. reaches the constraint, here $\zeta=0$), cf. e.g. [22, Formula (9)], [23, Formula (5)], or [30, Formula (7)]. Therefore, such models can avoid only partly the unwanted remembrance of the past configuration before the damage. To suppress the remembrance of the past configuration completely, we have used the concept of perfect plasticity (combined here with damage).

3. The model on the fault, and its combination with (2.4).

Now the idea is to "translate" the model from the d-dimensional bulk to the fault which is considered as a $(d{-}1)$-dimensional surface. We will do it rather intuitively, as the rigorous passage from a bulk to an interfacial model requires a rather sophisticated scaling and very involved analysis which, so far, was done only for a passage from damage to brittle delamination in [34, 50].

Analogously to the internal parameters π, ζ, and ε, on the faults we introduce internal parameters denoted by $\pi_{\rm i}$, $\zeta_{\rm i}$, and $\varepsilon_{\rm i}$, having the meaning of interfacial plastic-like slip, interfacial damage (called also *delamination*), and interfacial Maxwellian-type slip, respectively. Instead of Frémond's type concept of gradient damage, we use his concept of gradient delamination, cf. [15, 16], for an adhesive-type contact with possible weakening effects and with combination to the plastic and the Maxwellian interfacial slips.

More specifically, we consider the *interfacial* contribution to the *stored energy*

$$\mathscr{E}_{\rm fault}(e_{\rm i},\zeta_{\rm i},\pi_{\rm i},\varepsilon_{\rm i}):=\int_{\Gamma_{\rm C}}\frac12\mathbb{C}_{\rm i}(\zeta_{\rm i})\big(e_{\rm i}{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big){\cdot}\big(e_{\rm i}{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big)-\mathfrak{c}_{\rm i}(\zeta_{\rm i})+\frac{\kappa_{0\rm i}}{2}|\nabla_{\rm S}\zeta_{\rm i}|^2+\frac{\kappa_{\rm i}}{2}|\nabla_{\rm S}\pi_{\rm i}|^2\,{\rm d}S \tag{3.1a}$$

where $\nabla_{\rm S}$ denotes the "surface gradient" (i.e. the tangential derivative defined as $\nabla_{\rm S}v=\nabla v-(\nabla v{\cdot}\nu)\nu$ for v defined in the neighborhood of $\Gamma_{\rm C}$) and where $\mathbb{C}_{\rm i}$ is the matrix of coefficients of elastic adhesive response (dependent on interfacial damage $\zeta_{\rm i}$), $\mathfrak{c}_{\rm i}$ is the stored energy of interfacial damage, and $\mathbb{T}:\Gamma_{\rm C}\to{\rm Lin}(\mathbb{R}^{d-1},\mathbb{R}^d)$ makes the embedding $\mathbb{T}(x)$ of the $(d{-}1)$-dimensional tangent space to $\Gamma_{\rm C}$ at x into $\mathbb{R}^d$ where $e_{\rm i}$ is valued, being defined here simply as the jump in traces of displacements across $\Gamma_{\rm C}$, i.e. $e_{\rm i}:=[\![u]\!]$, where the symbol $[\![\cdot]\!]$ denotes the jump of the bracketed quantity across $\Gamma_{\rm C}$ (the sign orientation given by the convention of the discontinuity $\Gamma_{\rm C}$ normal vector ν pointing from "+" to "−" side of $\Gamma_{\rm C}$). The *interfacial* contribution to the (pseudo)*potential of dissipative forces* is:

$$\mathscr{R}_{\rm fault}(\zeta_{\rm i};\dot\zeta_{\rm i},\dot\pi_{\rm i},\dot\varepsilon_{\rm i}):=\int_{\Gamma_{\rm C}}\mathfrak{F}_{\rm i}(\dot\zeta_{\rm i})+\frac{\kappa_{1\rm i}}{r_{\rm i}}|\nabla_{\rm S}\dot\zeta_{\rm i}|^{r_{\rm i}}+\alpha_{\rm i}(\zeta_{\rm i})\delta^*_{P_{\rm i}}(\dot\pi_{\rm i})+\frac12\mathbb{D}_{\rm i}(\zeta_{\rm i})\dot\varepsilon_{\rm i}{:}\dot\varepsilon_{\rm i}\,{\rm d}S, \tag{3.1b}$$

where $\mathfrak{F}_{\rm i}$ is the primitive function to $\mathfrak{f}_{\rm i}$ from (3.9b) below, $\alpha_{\rm i}:[0,1]\to[0,1]$ is monotone with $\alpha_{\rm i}(1)=1$, and $P_{\rm i}\subset\mathbb{R}^{d-1}$ a convex set determining the yield stress of the undamaged material. Again, our modelling assumption is that the smooth functions $\mathbb{C}_{\rm i}(\cdot)$ and $\mathfrak{c}_{\rm i}(\cdot)$ are constant on $(-\infty,0]$ and on $[1,\infty)$, respectively, which keeps $\zeta_{\rm i}$ valued in $[0,1]$.

For the combination of the interfacial plasticity with adhesive contact see [45, 46] where it was used for a different purpose, namely for modelling of mode-mixity sensitive delamination, and with a different scenario, namely that the interface plasticity with hardening is triggered before the delamination starts.

This adhesive contact with interface plasticity indeed merely copies the philosophy we applied in the bulk, except that we do not consider any analogue of the Kelvin-Voigt viscosity and naturally also no inertia on the surface. This is obvious from the form of (2.3a) versus (3.1a) with $e_{\rm i}$ playing the role differences of displacement in (3.1a) instead of symmetric gradient of displacement in (2.3a). The analogy in (2.4a) versus (3.1b) is straightforward. If $\Gamma_{\rm C}$ and $\Gamma_{\rm D}$ are disjoint, which we will assume for simplicity throughout the whole article, we can also assume $[\![u_{\rm D}(t)]\!]=0$ so that the shift transformation of the Dirichlet data made in (2.4c-e) does not affect $\mathscr{E}_{\rm fault}$, nor $\mathscr{R}_{\rm fault}$ from (3.1).

To merge (2.4) with (3.1), the state of the system is to be considered as the 7-tuple

$$\boldsymbol{q}=(u,\boldsymbol{\zeta},\boldsymbol{\pi},\boldsymbol{\varepsilon})\quad\text{ with }\ \boldsymbol{\zeta}=(\zeta,\zeta_{\rm i}),\ \boldsymbol{\pi}=(\pi,\pi_{\rm i}),\ \boldsymbol{\varepsilon}=(\varepsilon,\varepsilon_{\rm i}). \tag{3.2}$$

Then the overall stored energy $\mathscr{E} = \mathscr{E}(t, \boldsymbol{q})$ and the (pseudo)potential of dissipative forces $\mathscr{R} = \mathscr{R}(\boldsymbol{q}; \dot{\boldsymbol{q}})$ are to be considered as

$$\mathscr{E}(t, \boldsymbol{q}) = \mathscr{E}(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) = \mathscr{E}_{\rm bulk}(t, u, \zeta, \pi, \varepsilon) + \mathscr{E}_{\rm fault}([\![u]\!], \zeta_{\rm i}, \pi_{\rm i}, \varepsilon_{\rm i}), \tag{3.3a}$$

$$\mathscr{R}(\boldsymbol{q}; \dot{\boldsymbol{q}}) = \mathscr{R}(\boldsymbol{\zeta}; \dot{u}, \dot{\boldsymbol{\zeta}}, \dot{\boldsymbol{\pi}}, \dot{\boldsymbol{\varepsilon}}) = \mathscr{R}_{\rm bulk}(\zeta; \dot{u}, \dot{\zeta}, \dot{\pi}, \dot{\varepsilon}) + \mathscr{R}_{\rm fault}(\zeta_{\rm i}; \dot{\zeta}_{\rm i}, \dot{\pi}_{\rm i}, \dot{\varepsilon}_{\rm i}), \tag{3.3b}$$

while the kinetic energy $\mathscr{M} = \mathscr{M}(\dot{u})$ is from (2.4b). We then consider the evolution to be governed formally by

$$\mathscr{M}'\,\ddot{u} + \partial_{\dot{\boldsymbol{q}}}\mathscr{R}(\boldsymbol{\zeta}; \dot{\boldsymbol{q}}) + \mathscr{E}'_{\boldsymbol{q}}(t, \boldsymbol{q}) \ni \mathcal{G}(t, \boldsymbol{q}) \tag{3.4}$$

where $\partial_{\dot{\boldsymbol{q}}}\mathscr{R}$ means a subdifferential of the convex function $\mathscr{R}(\zeta, \zeta_{\rm i}; \cdot)$ and $\mathscr{E}'_{\boldsymbol{q}}$ is the differential of the smooth function $\mathscr{E}(t, \cdot)$ and where the abstract functional $\mathcal{G}(t, \boldsymbol{q})$ is defined by $\langle \mathcal{G}(t, \boldsymbol{q}), \tilde{\boldsymbol{q}} \rangle := \langle \mathfrak{g}(t, \zeta), \tilde{u} \rangle$ with $\mathfrak{g}$ from (2.4d) for $\boldsymbol{q}$ as in (3.2) and analogously $\tilde{\boldsymbol{q}} = (\tilde{u}, \tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\pi}}, \tilde{\boldsymbol{\varepsilon}})$.

The energetics can formally be obtained by testing (3.4) by $\dot{\boldsymbol{q}}$. We define the overall dissipation rate $\Xi = \Xi(\boldsymbol{\zeta}; \dot{\boldsymbol{q}})$ as

$$\begin{aligned}\Xi(\boldsymbol{\zeta}; \dot{\boldsymbol{q}}) = \big\langle \partial_{\dot{\boldsymbol{q}}}\mathscr{R}(\boldsymbol{\zeta}; \dot{\boldsymbol{q}}), \dot{\boldsymbol{q}} \big\rangle = \int_{\Omega\setminus\Gamma_{\rm C}} &\mathbb{D}_0(\zeta)e(\dot{u}){:}e(\dot{u}) + \mathfrak{a}|\dot{\zeta}^+|^2 + \mathfrak{b}|\dot{\zeta}^-|^2 - \mathfrak{d}\dot{\zeta}^- + \kappa_1|\nabla\dot{\zeta}|^r \\ &+ \alpha(\zeta)\delta^*_P(\dot{\pi}) + \mathbb{D}(\zeta)\dot{\varepsilon}{:}\dot{\varepsilon}\,{\rm d}x \\ + \int_{\Gamma_{\rm C}} &\mathfrak{a}_{\rm i}|\dot{\zeta}_{\rm i}^+|^2 + \mathfrak{b}_{\rm i}|\dot{\zeta}_{\rm i}^-|^2 - \mathfrak{d}_{\rm i}\dot{\zeta}_{\rm i}^- + \kappa_{1\rm i}|\nabla_{\rm S}\dot{\zeta}_{\rm i}|^{r_{\rm i}} \\ &+ \alpha_{\rm i}(\zeta_{\rm i})\delta^*_{P_{\rm i}}(\dot{\pi}_{\rm i}) + \mathbb{D}_{\rm i}(\zeta_{\rm i})\dot{\varepsilon}_{\rm i}{:}\dot{\varepsilon}_{\rm i}\,{\rm d}S.\end{aligned} \tag{3.5}$$

Considering the initial conditions

$$u(0) = u_0, \quad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0, \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0, \quad \boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0, \quad \dot{u}(0) = v_0, \tag{3.6}$$

and integrating (3.4) over a time and using the particular homogeneities of the dissipation potentials, it gives formally

$$\underbrace{\mathscr{M}(\dot{u}(t)) + \mathscr{E}(t, \boldsymbol{q}(t))}_{\substack{\text{kinetic + stored energy}\\ \text{at time } t}} + \underbrace{\int_0^t \Xi(\boldsymbol{\zeta}(t); \dot{\boldsymbol{q}}(t)){\rm d}t}_{\substack{\text{dissipated energy over}\\ \text{the time interval } [0,t]}} = \underbrace{\mathscr{M}(v_0) + \mathscr{E}(t, \boldsymbol{q}_0)}_{\substack{\text{kinetic+stored energy}\\ \text{at time } t=0}} + \underbrace{\int_0^t \mathscr{E}'_t(t, \boldsymbol{q}) + \langle \mathfrak{g}(t, \zeta), \dot{\boldsymbol{q}} \rangle {\rm d}t}_{\substack{\text{work done by loading}\\ \text{over time interval } [0,t]}} \tag{3.7}$$

with $\boldsymbol{q}_0 = (u_0, \boldsymbol{\zeta}_0, \boldsymbol{\pi}_0, \boldsymbol{\varepsilon}_0)$. In fact, (3.7) is usually obtained from the sub-differential formulation rather as an inequality only, and the equality in (3.7) needs some data qualification (e.g. to ensure $\mathscr{M}'\,\ddot{u}$ in duality with $\dot{u}$ etc.).

The governing equations/inclusions arising from the abstract inclusion (3.4) with the specific choice (3.3) and (2.4b) in the bulk were already specified in (2.1). Abbreviating the normal and tangential components of the surface traction forces at the two sides of $\Gamma_{\rm C}$ respectively as

$$\sigma_{\rm n}^{\pm} = \nu^{\pm} \cdot \Big(\mathbb{D}_0(\zeta)e(\dot{u}(t, \cdot)) + \mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon)\Big)^{\pm} \nu^{\pm}, \quad \text{and} \tag{3.8a}$$

$$\sigma_{\rm t}^{\pm} = \Big(\mathbb{D}_0(\zeta)e(\dot{u}(t, \cdot)) + \mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon)\Big)^{\pm} \nu^{\pm} - \sigma_{\rm n}^{\pm}\nu^{\pm}, \tag{3.8b}$$

and defining $\nu{:=}\nu^+{=}{-}\nu^-$, the governing equations/inclusions on the faults $\Gamma_{\rm C}$ can now be identified as:

$$[\![\sigma_{\rm n}]\!] = 0, \quad \sigma_{\rm t}^+ = -\sigma_{\rm t}^- = -\mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big), \quad [\![u(t, \cdot)]\!] \cdot \nu = 0, \tag{3.9a}$$

$$\begin{aligned}\mathfrak{f}_{\rm i}(\dot{\zeta}_{\rm i}) - \mathfrak{c}'_{\rm i}(\zeta_{\rm i}) \ni -\frac{1}{2}\mathbb{C}_{\rm i}{}'(\zeta_{\rm i})\big([\![u]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big){\cdot}\big([\![u]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big) \\ + {\rm div}_{\rm S}\big(\kappa_{0\rm i}\nabla_{\rm S}\zeta_{\rm i}{+}\kappa_{1\rm i}|\nabla_{\rm S}\dot{\zeta}_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\dot{\zeta}_{\rm i}\big) \quad \text{with } \ \mathfrak{f}_{\rm i}(\dot{\zeta}_{\rm i}) = \begin{cases}\mathfrak{a}_{\rm i}\dot{\zeta}_{\rm i} & \text{if } \dot{\zeta}_{\rm i} > 0, \\ [-\mathfrak{d}_{\rm i}, 0] & \text{if } \dot{\zeta}_{\rm i} = 0, \\ \mathfrak{b}_{\rm i}\dot{\zeta}_{\rm i} - \mathfrak{d}_{\rm i} & \text{if } \dot{\zeta}_{\rm i} < 0,\end{cases}\end{aligned} \tag{3.9b}$$

$$\dot{\pi}_{\rm i} \in N_{\alpha_{\rm i}(\zeta_{\rm i})P_{\rm i}}\Big(\mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big) - {\rm div}_{\rm S}\nabla_{\rm S}\pi_{\rm i}\Big), \tag{3.9c}$$

$$\dot{\varepsilon}_{\rm i} = \mathbb{D}_{\rm i}^{-1}(\zeta_{\rm i})\mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big), \tag{3.9d}$$

where ${\rm div}_{\rm S} := {\rm trace}(\nabla_{\rm S})$ denotes the $(d{-}1)$-dimensional "surface divergence". This term in (3.9b) follows from the directional-derivative $\int_{\Gamma_{\rm C}}\kappa_{0\rm i}|\nabla_{\rm S}\zeta_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\zeta_{\rm i}{\cdot}\nabla_{\rm S}\tilde{\zeta}_{\rm i}{\rm d}S$ of the potential $\int_{\Gamma_{\rm C}}\frac{1}{r_{\rm i}}\kappa_{0\rm i}|\nabla_{\rm S}\zeta_{\rm i}|^{r_{\rm i}}{\rm d}S$ by applying a Green formula on a curved surface

$$\int_{\Gamma_{\rm C}} w{\cdot}\nabla_{\rm S}v\,{\rm d}S = -\int_{\Gamma_{\rm C}}({\rm div}_{\rm S}\nu)(w{\cdot}\nu)v + {\rm div}_{\rm S}(w_{\rm t})v\,{\rm d}S + \int_{\partial\Gamma_{\rm C}}(w{\cdot}\nu_1)v\,{\rm d}l \tag{3.10}$$

with ν and ν_1 the normal to $\Gamma_{\rm C}$ and $\partial\Gamma_{\rm C}$, respectively and $w_{\rm t}{:=}w{-}(w{\cdot}\nu)\nu$ the tangential component of w. Here (3.10) was used with $w = \kappa_{0\rm i}|\nabla_{\rm S}\zeta_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\zeta_{\rm i} + \kappa_{1\rm i}|\nabla_{\rm S}\dot{\zeta}_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\dot{\zeta}_{\rm i}$ and, as such w is always orthogonal to ν, the term involving the mean curvature of the surface $\Gamma_{\rm C}$, which is $-\frac{1}{2}({\rm div}_{\rm S}\nu)$, vanishes. Also, from the last term in (3.10), one can see the natural "boundary" condition for $\zeta_{\rm i}$ and similar condition arises also for $\pi_{\rm i}$, cf. (3.12b) below. For the tensorial variant of (3.10) used in a similar context in mechanics of the above mentioned non-simple continua of the 2nd-grade, cf. [39, 51].

Together with the Dirichlet condition $u|_{\Gamma_{\rm D}} = u_{\rm Dir}$, cf. (2.4c), and zero traction stress on the remaining boundary, the system (2.1), (3.9), and (3.6) represent the classical formulation of the initial-boundary-value problem governing the model; in fact, still the remaining boundary conditions (3.12) will be formulated below.

In terms of the particular components, we can write (3.4) in a bit more detailed way as

$$\mathscr{M}'\,\ddot{u} + \mathscr{R}'_{\dot{u}}(\zeta; \dot{u}) + \mathscr{E}'_u(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) = \mathfrak{g}(t, \zeta), \tag{3.11a}$$

$$\partial_{\dot{\boldsymbol{\zeta}}}\mathscr{R}(\dot{\boldsymbol{\zeta}}) + \mathscr{E}'_{\boldsymbol{\zeta}}(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) \ni 0, \tag{3.11b}$$

$$\partial_{\dot{\boldsymbol{\pi}}}\mathscr{R}(\boldsymbol{\zeta}; \dot{\boldsymbol{\pi}}) + \mathscr{E}'_{\boldsymbol{\pi}}(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) \ni 0, \tag{3.11c}$$

$$\mathscr{R}'_{\dot{\boldsymbol{\varepsilon}}}(\boldsymbol{\zeta}; \dot{\boldsymbol{\varepsilon}}) + \mathscr{E}'_{\boldsymbol{\varepsilon}}(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) = 0, \tag{3.11d}$$

by using that $\partial_{\dot u}\mathscr{R} = \partial_{\dot u}\mathscr{R}_{\rm bulk}$ is single-valued independent of $\zeta_{\rm i}$, that $\partial_{\dot\zeta}\mathscr{R}$ is independent of $\boldsymbol{\zeta}$, and $\mathscr{M}(\cdot)$ and $\mathscr{E}(t,\cdot,\cdot,\cdot)$ are smooth.

One should also realize that (3.11) involves, beside the bulk system (2.1) transformed by the substitution $u \mapsto u{+}u_{\rm Dir}(t)$ and the boundary conditions (2.4e), also some other boundary conditions, namely

$$\sigma{\cdot}\nu = g \qquad \text{on } \Gamma_{\rm N} := \Gamma\backslash\Gamma_{\rm D}, \quad \Gamma := \partial\Omega, \tag{3.12a}$$

$$\kappa_1|\nabla\dot\zeta|^{r-2}\frac{\partial\dot\zeta}{\partial\nu} + \kappa_0\frac{\partial\zeta}{\partial\nu} = 0 \qquad \text{and} \quad \frac{\partial\pi}{\partial\nu} = 0 \text{ on } \Gamma_{\rm C}\cup\Gamma. \tag{3.12b}$$

Analogous "boundary" conditions are also involved in (3.11) as far as the $(d{-}2)$-dimensional boundaries of $\Gamma_{\rm C}$ concern, namely:

$$\big(\kappa_{0\rm i}\nabla_{\rm S}\zeta_{\rm i} + \kappa_{1\rm i}|\nabla_{\rm S}\dot\zeta_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\dot\zeta_{\rm i}\big){\cdot}\nu_1 = 0 \quad \text{and} \quad \nabla_{\rm S}\pi_{\rm i}{\cdot}\nu_1 = 0 \quad \text{on} \quad \partial\Gamma_{\rm C}, \tag{3.12c}$$

where ν_1 denotes the normal to the $(d{-}2)$-dimensional boundary $\partial\Gamma_{\rm C}$.

Remark 3.1 (*Relation with the frictional models*). The usual frictional Signorini contact can be described by the dissipation rate $\mu\sigma_{\rm n}|\dot\pi_{\rm i}|$ with $\pi_{\rm i} = \mathbb{T}^{-1}[\![u]\!]_{\rm t}$ and the unilateral constraint $[\![u]\!]_{\rm n} \geq 0$, where $[\![\cdot]\!]_{\rm n}$ and $[\![\cdot]\!]_{\rm t}$ refer to the normal and the tangential components of the jump across $\Gamma_{\rm C}$, respectively, $\sigma_{\rm n}$ is the normal force exerted at the contact and μ is the friction coefficient. It is well recognized that this brings serious mathematical difficulties even if μ is constant, in particular conservation of energy in the dynamical case is still an open problem in the multidimensional case. A certain regularization is thus worth considering. One can think either about "penalization" of the constraint $[\![u]\!]_{\rm n} \geq 0$ (by allowing a small penetration of the subdomains in contact) or a penalization of the constraint $\mathbb{T}\pi_{\rm i} = [\![u]\!]_{\rm t}$ (which is, in fact, the adhesive concept chosen here). Indeed, for $\mathbb{C}_{\rm i}$ large (as can be considered even for $\zeta_{\rm i} = 0$) and neglecting also the Maxwellian slip $\varepsilon_{\rm i} = 0$, we have $[\![u]\!] \sim \mathbb{T}\pi_{\rm i}$ so that, considering also $P_{\rm i}$ a ball of the radius $r_{\rm i}$, the dissipation rate $\alpha_{\rm i}(\zeta_{\rm i})\delta^*_{P_{\rm i}}(\dot\pi_{\rm i})$ essentially equals $\alpha_{\rm i}(\zeta_{\rm i})r_{\rm i}|[\![\dot u]\!]|$, which reveals the relation $\alpha_{\rm i}(\zeta_{\rm i})r_{\rm i}{\simeq}\mu\sigma_{\rm n}$. We will also assume that the dominant contribution to the normal force $\sigma_{\rm n}$ in the friction law comes from the lithostatic pressure, and thus can be recovered by merely considering an additional dependence of the friction-like coefficient on the vertical coordinate, i.e. on the depth x_3 as $\alpha_{\rm i} = \alpha_{\rm i}(x_3,\zeta_{\rm i})$. Analogous dependencies may be considered for all other possibly pressure- or normal-stress- dependent coefficients $\mathfrak{c}$, $\mathfrak{c}_{\rm i}$, etc. Also note that, because of the high lithostatic pressures, the fault rupturing does not produce cavities, and we have thus already replaced the Signorini kinematic contact condition $[\![u]\!]_{\rm n} \geq 0$ by $[\![u]\!]_{\rm n} = 0$ in our formulation. An extension of our model in order to capture the dependence of the activation (friction) coefficient $\alpha_{\rm i}$ on the true normal stress instead of just the lithostatic pressure would be possible by considering a suitable penalization of the non-penetration condition $[\![u]\!]_{\rm n} \geq 0$ (allowing only for a small penetration) and imposing an additional dependence $\alpha_{\rm i} = \alpha_{\rm i}([\![u]\!]_{\rm n},\zeta_{\rm i})$. Moreover by adding also a dependence on the tangential slip, i.e. taking $\alpha_{\rm i} = \alpha_{\rm i}([\![u]\!]_{\rm n},[\![u]\!]_{\rm t},\zeta_{\rm i})$, would allow to simultaneously model slip weakening/hardening of the friction coefficient, a phenomenon relevant for seismology, cf. [1, 7, 37].

Remark 3.2 (*Concept of ageing*). In seismology the contact problem between the adjacent lithospheric faults is typically assumed to be of the friction type described in Remark 3.1. Laboratory experiments assert however that additional internal parameters θ_i have to be introduced in order to capture other than static cases (see [32]) leading to the form $\mu = \mu([\![u]\!]_{\rm t},\theta_i)$ or often also $\mu = \mu([\![\dot u]\!]_{\rm t},\theta_i)$. A most popular and successful class of such models introduces only one internal scalar parameter θ called ageing, which reflects the "dynamic age" of the contact (being interpreted as its roughness), and which is assumed to be governed by its own evolution. Combination of the Signorini contact with ageing parameter and its evolution law represent the so-called *rate-and-state models*, cf. e.g. [6, 14, 19, 29]. This additional internal parameter can also accent what is in tribology called *stick-slip* motion. In seismology, a popular and widely used dynamic fault model is that of Dieterich [12, 14] and Ruina [47]. The evolution of the ageing variable and the rate-dependence of the sliding coefficient in these models have been deduced rather intuitively from the sliding experiments measuring the force response to an imposed velocity jump for rock specimens or from experiments measuring the time-dependence of a static friction; see [32] for a review. These empirically-fitted frictional laws often lack thermodynamic reasoning and in some cases may even violate the 2nd law of thermodynamics due to a negative-valued friction coefficient, c.f. [12, 13], which may numerically facilitate rupture initiation but is physically inconsistent and naturally also inappropriate for rigorous mathematical treatment. Various regularizations have thus been suggested, see e.g. [38], but the lack of experimental data for very small sliding velocities however prevents discrimination between the various models. Moreover, the contribution of ageing to energy dissipation rate is traditionally not considered in seismology, which does not possibly cause much error when only mechanical balance of forces is of interest, but definitely comes to play when a full thermomechanical description of the fault is desired, e.g. when frictional heating and thermally induced fluid pressurization of the fault are considered, e.g. [6]. Similarly, also the contribution of ageing to the stored energy is standardly not taken into account, which makes it difficult to view the empirical dynamics of ageing as being driven by the stored-energy gradient like (3.4).

To relate at least vaguely our model to the rate-and-state dependent friction, we may say that the delamination parameter $\zeta_{\rm i}$ is in the position of a certain ageing of the fault. Generalization of our model by allowing for rate-dependency of the dissipation in terms of $\dot\pi_{\rm i}$ and for some more coefficients state dependent, as e.g. also $\mathfrak{d}_{\rm i}{=}\mathfrak{d}_{\rm i}(\zeta_{\rm i})$ or $\mathfrak{a}_{\rm i}{=}\mathfrak{a}_{\rm i}(\zeta_{\rm i})$, may bring our model closer to that friction one from [12, 14, 47]. It is, however, out of the scope of this article. In contrast to that friction model, we have formulated all the dissipative processes in a unified and thermodynamically consistent manner allowing for complete description of all the energetics of the rupture process, we made the model on the fault conceptually consistent with the model in the bulk, we eliminated the phenomenon of artificial remembrance of the previous configuration pointed out in Remark 2.2, and, on top of it, we will show that this model allows for numerically stable and convergent approximation.

4. Semi-implicit time discretization.

For a conceptual numerical algorithm (and also as a theoretical tool to prove existence of a solution, cf. Proposition 5.2 below), we use the semi-implicit time discretisation of (3.11). Due to the inertial term, we consider an equidistant partition of $[0,T]$ with a time step $\tau > 0$. We denote the approximate values of u at time $t = k\tau$ by u^k_τ for $k = 0, ..., T/\tau \in \mathbb{N}$, and similarly for $\boldsymbol{\zeta}$, $\boldsymbol{\pi}$, and $\boldsymbol{\varepsilon}$. The notation of the Lebesgue L^p-spaces and Sobolev $W^{k,p}$-spaces is standard, together with the shorthand notation $W^{k,2} = H^k$ and the corresponding Bochner spaces of Banach-space valued functions on $I = (0,T)$. We consider a fixed time horizon $T > 0$.

The semi-implicit discretisation advantageously decouples the problem and keeps the variational structure. Namely, we consider (3.11) discretised as:

$$\mathscr{M}'\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}+\mathscr{R}'_{\dot u}\Big(\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\mathscr{E}'_u(k\tau,u_\tau^k,\boldsymbol{\zeta}_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k)=\mathfrak{g}(k\tau,\zeta_\tau^{k-1}),\tag{4.1a}$$

$$\partial_{\dot{\boldsymbol{\zeta}}}\mathscr{R}\Big(\frac{\boldsymbol{\zeta}_\tau^k-\boldsymbol{\zeta}_\tau^{k-1}}{\tau}\Big)+\mathscr{E}'_{\boldsymbol{\zeta}}(k\tau,u_\tau^k,\boldsymbol{\zeta}_\tau^k,\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k)\ni 0,\tag{4.1b}$$

$$\partial_{\dot{\boldsymbol{\pi}}}\mathscr{R}\Big(\boldsymbol{\zeta}_\tau^{k-1};\frac{\boldsymbol{\pi}_\tau^k-\boldsymbol{\pi}_\tau^{k-1}}{\tau}\Big)+\mathscr{E}'_{\boldsymbol{\pi}}(k\tau,u_\tau^k,\boldsymbol{\zeta}_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k)\ni 0,\tag{4.1c}$$

$$\mathscr{R}'_{\dot{\boldsymbol{\varepsilon}}}\Big(\boldsymbol{\zeta}_\tau^{k-1};\frac{\boldsymbol{\varepsilon}_\tau^k-\boldsymbol{\varepsilon}_\tau^{k-1}}{\tau}\Big)+\mathscr{E}'_{\boldsymbol{\varepsilon}}(k\tau,u_\tau^k,\boldsymbol{\zeta}_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k)=0.\tag{4.1d}$$

This recursive formula is to be solved for $k=1,...,T/\tau$, starting for $k=1$ by using

$$u_\tau^0=u_0,\qquad u_\tau^{-1}=u_0-\tau v_0,\qquad \boldsymbol{\zeta}_\tau^0=\boldsymbol{\zeta}_0,\qquad \boldsymbol{\pi}_\tau^0=\boldsymbol{\pi}_0,\qquad \boldsymbol{\varepsilon}_\tau^0=\boldsymbol{\varepsilon}_0;\tag{4.2}$$

cf. (3.6). In our isothermal case, we can benefit from a variational structure of the formula (4.1), i.e. we are to solve successively two decoupled minimization problems at each time level:

$$\left\{\begin{array}{ll}\text{minimize} & \tau^2\mathscr{M}\Big(\dfrac{u-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big)\\ & \quad+\tau\mathscr{R}\Big(\boldsymbol{\zeta}_\tau^{k-1};\dfrac{u-u_\tau^{k-1}}{\tau},0,\dfrac{\boldsymbol{\pi}-\boldsymbol{\pi}_\tau^{k-1}}{\tau},\dfrac{\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_\tau^{k-1}}{\tau}\Big)\\ & \quad+\mathscr{E}\big(k\tau,u,\boldsymbol{\zeta}_\tau^{k-1},\boldsymbol{\pi},\boldsymbol{\varepsilon}\big)-\big\langle\mathfrak{g}(k\tau,\zeta_\tau^{k-1}),u\big\rangle\\ \text{subject to} & u\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d),\\ & \boldsymbol{\pi}=(\pi,\pi_{\rm i})\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}),\\ & \boldsymbol{\varepsilon}=(\varepsilon,\varepsilon_i)\in L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1}),\end{array}\right.\tag{4.3a}$$

and, denoting the (unique) solution to (4.3a) by u_τ^k, $\boldsymbol{\pi}_\tau^k$, and $\boldsymbol{\varepsilon}_\tau^k$, further solve:

$$\left\{\begin{array}{ll}\text{minimize} & \tau\mathscr{R}\Big(0;0,\dfrac{\boldsymbol{\zeta}-\boldsymbol{\zeta}_\tau^{k-1}}{\tau},0,0\Big)+\mathscr{E}\big(k\tau,u_\tau^k,\boldsymbol{\zeta},\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k\big)\\ \text{subject to} & \boldsymbol{\zeta}=(\zeta,\zeta_{\rm i})\in W^{1,r}(\Omega\backslash\Gamma_{\rm C})\times W^{1,r_{\rm i}}(\Gamma_{\rm C}),\end{array}\right.\tag{4.3b}$$

whose solution will be denoted by $\boldsymbol{\zeta}_\tau^k$. In fact, if $\mathbb{C}(\cdot)e{:}e$, $\mathbb{C}_{\rm i}(\cdot)u{\cdot}u$, $-\mathfrak{c}(\cdot)$, and $-\mathfrak{c}_{\rm i}(\cdot)$ are not strictly convex, a solution to (4.3b) need not be unique and, in such cases, we just choose one of these solutions for $\boldsymbol{\zeta}_\tau^k$. Obviously, (4.1a,c,d) just represents 1st-order necessary optimality condition for (4.3a), while (4.1b) is the optimality condition for (4.3b).

Let us define the piecewise affine interpolant u_τ by

$$u_\tau(t):=\frac{t-(k-1)\tau}{\tau}u_\tau^k+\frac{k\tau-t}{\tau}u_\tau^{k-1}\quad\text{for }t\in[(k-1)\tau,k\tau]\ \text{ with }k{=}1,...,T/\tau,\tag{4.4a}$$

and the backward and the forward piecewise constant interpolants $\bar u_\tau$ and $\underline{u}_\tau$ by

$$\bar u_\tau(t):=u_\tau^k\qquad\text{for }t\in((k-1)\tau,k\tau],\ \ k=0,...,T/\tau,\text{ and}\tag{4.4b}$$

$$\underline{u}_\tau(t):=u_\tau^{k-1}\qquad\text{for }t\in[(k-1)\tau,k\tau),\ \ k=1,...,T/\tau{+}1.\tag{4.4c}$$

The notation $\boldsymbol{\zeta}_\tau$, $\bar{\boldsymbol{\zeta}}_\tau$, $\underline{\boldsymbol{\zeta}}_\tau$ or $\boldsymbol{\pi}_\tau$, $\bar{\boldsymbol{\pi}}_\tau$, $\underline{\boldsymbol{\pi}}_\tau$, and $\boldsymbol{\varepsilon}_\tau$, $\bar{\boldsymbol{\varepsilon}}_\tau$, $\underline{\boldsymbol{\varepsilon}}_\tau$ is defined analogously. By $\bar u_{{\rm D},\tau}$, we denote the piecewise constant interpolant with values $u_{\rm D}(k\tau)$ on $((k-1)\tau,k\tau)$. Analogously, $\bar{\mathscr{E}}_\tau(t,\boldsymbol{q}):=\mathscr{E}(k\tau,\boldsymbol{q})$ and $\bar{\mathfrak{g}}_\tau(t,\zeta):=\mathfrak{g}(k\tau,\zeta)$ for $t\in((k-1)\tau,k\tau]$, $k=0,...,T/\tau$. Let us summarize the main assumptions we will need:

$$\mathbb{C}(\cdot),\mathbb{C}_{\rm i}(\cdot)\ \text{ continuously differentiable, uniformly positive definite,}\tag{4.5a}$$

$$\forall\, e{\in}\mathbb{R}^{d\times d}_{\rm sym},\ u{\in}\mathbb{R}^d:\quad \mathbb{C}(\cdot)e{:}e\ \text{ and }\ \mathbb{C}_{\rm i}(\cdot)u{\cdot}u\ \text{ are convex,}\tag{4.5b}$$

$$\mathfrak{c}(\cdot),\mathfrak{c}_{\rm i}(\cdot)\ \text{ continuously differentiable and concave,}\tag{4.5c}$$

$$\mathbb{D}(\cdot),\ \mathbb{D}_0(\cdot),\ \mathbb{D}_{\rm i}(\cdot)\ \text{ continuous, uniformly positive definite,}\tag{4.5d}$$

$$f\in L^2(\Omega;\mathbb{R}^d),\quad g\in H^{1/2}(\Gamma_{\rm N};\mathbb{R}^d),\tag{4.5e}$$

$$u_{\rm Dir}\in W^{1,2}(I;H^{3/2}(\Gamma_{\rm D};\mathbb{R}^d))\ \text{has an extension }u_{\rm D}\in W^{2,1}(I;L^2(\Omega;\mathbb{R}^d)),\tag{4.5f}$$

$$u_0\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d),\quad v_0\in L^2(\Omega;\mathbb{R}^d),\quad \boldsymbol{\zeta}_0\in W^{1,r}(\Omega\backslash\Gamma_{\rm C})\times W^{1,r_{\rm i}}(\Gamma_{\rm C}),\tag{4.5g}$$

$$\boldsymbol{\pi}_0\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}),\quad \boldsymbol{\varepsilon}_0\in L^2(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1}).\tag{4.5h}$$

Note that (4.5a) means that only an uncomplete damage is allowed, which seems however to be quite a realistic modelling assumption as the disintegration of the lithosphere is always rather partial even during intensive earthquakes. Note also that (4.5f) allows, e.g. for spatially constant Dirichlet loading with velocities in $W^{1,1}$, which "nearly" allows jumps. In fact, the apriori estimates (4.6) survive under such jumps, which are regimes standardly used for testing geophysical frictional models, cf. e.g. [18, 32].

LEMMA 4.1 (Stability of the time discretisation). *Let* (4.5) *hold. Then the recursive scheme* (4.1) *has a solution and the following a-priori estimates hold:*

$$\big\|u_\tau\big\|_{H^1(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\,\cap\,W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))}\le C,\tag{4.6a}$$

$$\big\|\boldsymbol{\zeta}_\tau\big\|_{L^\infty(I;H^1(\Omega\backslash\Gamma_{\rm C})\times H^1(\Gamma_{\rm C}))\,\cap\,(W^{1,r}(I;W^{1,r}(\Omega))\times W^{1,r_{\rm i}}(I;W^{1,r_{\rm i}}(\Gamma_{\rm C})))}\le C,\tag{4.6b}$$

$$\big\|\boldsymbol{\pi}_\tau\big\|_{L^\infty(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}))\cap W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times L^1(\Gamma_{\rm C};\mathbb{R}^{d-1}))}\le C,\tag{4.6c}$$

$$\big\|\boldsymbol{\varepsilon}_\tau\big\|_{H^1(I;L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1}))}\le C,\tag{4.6d}$$

with C independent of τ provided $\tau>0$ is sufficiently small. Moreover, for any $\tau>0$, the discrete weak formulation of (3.11a) holds, namely

$$\int_0^T\!\Big(\int_{\Omega\setminus\Gamma_{\rm C}}\!\Big(\mathbb{C}(\underline{\zeta}_\tau)\big(e(\bar u_\tau{+}\bar u_{{\rm D},\tau}){-}\bar\pi_\tau{-}\bar\varepsilon_\tau\big)+\mathbb{D}_0(\underline{\zeta}_\tau)e(\dot u_\tau)\Big){:}e(\widetilde u)\\ -\varrho\big[\dot u_\tau\big]_\tau^{\rm int}{\cdot}\dot{\widetilde u}\,{\rm d}x+\int_{\Gamma_{\rm C}}\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big){\cdot}[\![\widetilde u]\!]\,{\rm d}S-\big\langle\bar{\mathfrak g}_\tau(\underline{\zeta}_\tau),\widetilde u\big\rangle\Big){\rm d}t\\ =\int_\Omega\varrho v_0{\cdot}\widetilde u(0)-\varrho\dot u_\tau(T){\cdot}\widetilde u(T)\,{\rm d}x \tag{4.7a}$$

holds for all $\widetilde u\in H^1(I;H^1(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))$, where $[\dot u_\tau]_\tau^{\rm int}$ denotes the piece-wise affine interpolant of the piece-wise constant function $\dot u_\tau$, and moreover (3.11b) holds as a variational inequality, namely

$$\int_{Q\setminus\Sigma_{\rm C}}\mathfrak F(\widetilde\zeta)+\frac12\mathbb{C}'(\bar\zeta_\tau)\big(e(\bar u_\tau){-}\bar\pi_\tau{-}\bar\varepsilon_\tau\big){:}\big(e(\bar u_\tau){-}\bar\pi_\tau{-}\bar\varepsilon_\tau\big)\big(\widetilde\zeta{-}\dot\zeta_\tau\big)\\ -\mathfrak c'(\bar\zeta_\tau)\big(\widetilde\zeta{-}\dot\zeta_\tau\big)+\kappa_0\nabla\bar\zeta_\tau{\cdot}\nabla\big(\widetilde\zeta{-}\dot\zeta_\tau\big)+\frac{\kappa_1}{r}|\nabla\widetilde\zeta|^r\,{\rm d}x{\rm d}t\\ \ge\int_{Q\setminus\Sigma_{\rm C}}\mathfrak F(\dot\zeta_\tau)+\frac{\kappa_1}{r}|\nabla\dot\zeta_\tau|^r\,{\rm d}x{\rm d}t \tag{4.7b}$$

holds for all $\widetilde\zeta\in L^\infty(I;W^{1,r}(\Omega\setminus\Gamma_{\rm C}))$, with $\mathfrak F$ from (2.2), where we denoted $Q{:=}\Omega{\times}(0,T)$, and $\Sigma_{\rm C}{:=}\Gamma_{\rm C}{\times}(0,T)$. Analogously, on the fault

$$\int_{\Sigma_{\rm C}}\mathfrak F_{\rm i}(\widetilde\zeta)+\frac12\mathbb{C}_{\rm i}'(\bar\zeta_{{\rm i},\tau})\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big){\cdot}\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big)\big(\widetilde\zeta{-}\dot\zeta_{{\rm i},\tau}\big)\\ -\mathfrak c_{\rm i}'(\bar\zeta_{{\rm i},\tau})\big(\widetilde\zeta{-}\dot\zeta_{{\rm i},\tau}\big)+\kappa_{0{\rm i}}\nabla_{\rm S}\bar\zeta_{{\rm i},\tau}{\cdot}\nabla_{\rm S}\big(\widetilde\zeta{-}\dot\zeta_{{\rm i},\tau}\big)+\frac{\kappa_{1{\rm i}}}{r_{\rm i}}|\nabla\widetilde\zeta|^{r_{\rm i}}\,{\rm d}S{\rm d}t\\ \ge\int_{\Sigma_{\rm C}}\mathfrak F_{\rm i}(\dot\zeta_{{\rm i},\tau})+\frac{\kappa_{1{\rm i}}}{r_{\rm i}}|\nabla\dot\zeta_{{\rm i},\tau}|^{r_{\rm i}}\,{\rm d}S{\rm d}t \tag{4.7c}$$

holds for all $\widetilde\zeta\in L^\infty(I;W^{1,r_{\rm i}}(\Gamma_{\rm C}))$, and an analogue of the energy balance (3.7), namely

$$\mathscr M(\dot u_\tau(t)){+}\mathscr E(t,\bar u_\tau(t),\bar{\boldsymbol\zeta}_\tau(t),\bar{\boldsymbol\pi}_\tau(t),\bar{\boldsymbol\varepsilon}_\tau(t)){+}\int_0^t\Xi(\underline{\boldsymbol\zeta}_\tau(t);\dot u_\tau(t),\dot{\boldsymbol\zeta}_\tau(t),\dot{\boldsymbol\pi}_\tau(t),\dot{\boldsymbol\varepsilon}_\tau(t))\,{\rm d}t\\ \le\mathscr M(v_0)+\mathscr E(t,u_0,\boldsymbol\zeta_0,\boldsymbol\pi_0,\boldsymbol\varepsilon_0)+\int_0^t\mathscr E_t'(t,\underline u_\tau,\underline{\boldsymbol\zeta}_\tau,\underline{\boldsymbol\pi}_\tau,\underline{\boldsymbol\varepsilon}_\tau)+\big\langle\bar{\mathfrak g}_\tau(t,\underline{\zeta}_\tau),\dot u_\tau\big\rangle\,{\rm d}t, \tag{4.7d}$$

holds at any mesh point $t=k\tau$, $k=1,...,T/\tau$ with Ξ from (3.5), and the so-called discrete semi-stability holds:

$$\bar{\mathscr E}_\tau\big(t,\bar u_\tau(t),\bar{\boldsymbol\zeta}_\tau(t),\bar{\boldsymbol\pi}_\tau(t),\bar{\boldsymbol\varepsilon}_\tau(t)\big)\le\bar{\mathscr E}_\tau\big(t,\bar u_\tau(t),\bar{\boldsymbol\zeta}_\tau(t),\widetilde{\boldsymbol\pi},\bar{\boldsymbol\varepsilon}_\tau(t)\big)\\ +\mathscr R\big(\underline{\boldsymbol\zeta}_\tau(t);0,0,\widetilde{\boldsymbol\pi}{-}\bar{\boldsymbol\pi}_\tau(t),0\big) \tag{4.7e}$$

for all $t{\in}(0,T]$ and all $\widetilde{\boldsymbol\pi}\in H^1(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev}){\times}H^1(\Gamma_{\rm C};\mathbb{R}^{d-1})$.

Proof. The solutions of both minimization problems in (4.3) do exist due to standard compactness/coercivity arguments.

Testing (4.1a) by $\widetilde u(t)$ and writing it in terms of the interpolants and integrate it over I, and using the by-part integration for the inertial term yields (4.7a). Likewise, the weak formulation of (4.1b) yields (4.7b) and (4.7c).

The energy balance is generally obtained, like (3.7), by testing (4.1a), (4.1b), (4.1c), and (4.1d) by $u_\tau^k{-}u_\tau^{k-1}$, $\boldsymbol\zeta_\tau^k{-}\boldsymbol\zeta_\tau^{k-1}$, $\boldsymbol\pi_\tau^k{-}\boldsymbol\pi_\tau^{k-1}$, and $\boldsymbol\varepsilon_\tau^k{-}\boldsymbol\varepsilon_\tau^{k-1}$, respectively. By using the convexity of $\mathscr E(t,\cdot,\boldsymbol\zeta,\cdot,\cdot)$ and 1- and 2-homogeneity of $\mathscr R(\boldsymbol\zeta;\dot u,\dot{\boldsymbol\zeta},\cdot,\dot{\boldsymbol\varepsilon})$ and $\mathscr R(\boldsymbol\zeta;\cdot,\dot{\boldsymbol\zeta},\dot{\boldsymbol\pi},\cdot)$, respectively, we obtain

$$\mathscr M\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big){+}\tau\mathscr R\Big(\boldsymbol\zeta_\tau^{k-1};0,0,\frac{\boldsymbol\pi_\tau^k{-}\boldsymbol\pi_\tau^{k-1}}{\tau},0\Big)\\ +2\tau\mathscr R\Big(\boldsymbol\zeta_\tau^{k-1};\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau},0,0,\frac{\boldsymbol\varepsilon_\tau^k{-}\boldsymbol\varepsilon_\tau^{k-1}}{\tau}\Big)\\ +\mathscr E\big(k\tau,u_\tau^k,\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^k,\boldsymbol\varepsilon_\tau^k\big)-\big\langle\mathfrak g(k\tau,\zeta_\tau^{k-1}),u_\tau^k\big\rangle\\ \le\mathscr M\Big(\frac{u_\tau^{k-1}{-}u_\tau^{k-2}}{\tau}\Big)+\mathscr E\big(k\tau,u_\tau^{k-1},\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^{k-1},\boldsymbol\varepsilon_\tau^{k-1}\big)-\big\langle\mathfrak g(k\tau,\zeta_\tau^{k-1}),u_\tau^{k-1}\big\rangle. \tag{4.8}$$

Still, we execute the announced test of (4.1b). Thus, using the convexity of $\mathscr E(t,u,\cdot,\boldsymbol\pi,\boldsymbol\varepsilon)$ and of $\mathscr R(\boldsymbol\zeta;\dot u,\cdot,\dot{\boldsymbol\pi},\dot{\boldsymbol\varepsilon})$, we obtain

$$\Big\langle\partial_{\dot\zeta}\mathscr R\big(0;0,\frac{\boldsymbol\zeta_\tau^k{-}\boldsymbol\zeta_\tau^{k-1}}{\tau},0,0\big),\boldsymbol\zeta_\tau^k{-}\boldsymbol\zeta_\tau^{k-1}\Big\rangle+\mathscr E\big(k\tau,u_\tau^k,\boldsymbol\zeta_\tau^k,\boldsymbol\pi_\tau^k,\boldsymbol\varepsilon_\tau^k\big)\\ \le\mathscr E\big(k\tau,u_\tau^k,\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^k,\boldsymbol\varepsilon_\tau^k\big). \tag{4.9}$$

Summing (4.8) and (4.9), we can enjoy the cancellation of $\pm\mathscr E(k\tau,u_\tau^k,\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^k,\boldsymbol\varepsilon_\tau^k)$. Using (3.5), we obtain

$$\mathscr M\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big)+\mathscr E\big(k\tau,u_\tau^k,\boldsymbol\zeta_\tau^k,\boldsymbol\pi_\tau^k,\boldsymbol\varepsilon_\tau^k\big)\\ +\tau\Xi\Big(\boldsymbol\zeta_\tau^{k-1};\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau},\frac{\boldsymbol\zeta_\tau^k{-}\boldsymbol\zeta_\tau^{k-1}}{\tau},\frac{\boldsymbol\pi_\tau^k{-}\boldsymbol\pi_\tau^{k-1}}{\tau},\frac{\boldsymbol\varepsilon_\tau^k{-}\boldsymbol\varepsilon_\tau^{k-1}}{\tau}\Big)\\ \le\mathscr M\Big(\frac{u_\tau^{k-1}{-}u_\tau^{k-2}}{\tau}\Big)+\mathscr E\big(k\tau,u_\tau^{k-1},\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^{k-1},\boldsymbol\varepsilon_\tau^{k-1}\big)\\ +\big\langle\mathfrak g(k\tau,\zeta_\tau^{k-1}),u_\tau^k{-}u_\tau^{k-1}\big\rangle\\ =\mathscr M\Big(\frac{u_\tau^{k-1}{-}u_\tau^{k-2}}{\tau}\Big)+\mathscr E\big((k{-}1)\tau,u_\tau^{k-1},\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^{k-1},\boldsymbol\varepsilon_\tau^{k-1}\big)\\ +\int_{(k-1)\tau}^{k\tau}\mathscr E_t'\big(t,u_\tau^{k-1},\boldsymbol\zeta_\tau^{k-1},\boldsymbol\pi_\tau^{k-1},\boldsymbol\varepsilon_\tau^{k-1}\big)+\Big\langle\mathfrak g(k\tau,\zeta_\tau^{k-1}),\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}\Big\rangle\,{\rm d}t. \tag{4.10}$$

Summing inequalities (4.10) for $k = 1, ..., l \in \mathbb{N}$ and referring to the initial conditions (3.6) and to (4.4) gives (4.7d).

Taking (2.4c) into account, we have $\mathscr{E}_t' = \mathscr{E}_t'(t, u, \zeta, \pi, \varepsilon)$ given by

$$\mathscr{E}_t'(t, u, \zeta, \pi, \varepsilon) = \int_{\Omega\setminus\Gamma_C} \mathbb{C}(\zeta)\big(e(u{+}u_D(t)){-}\pi{-}\varepsilon\big){:}e(\dot u_D(t))\,dx. \tag{4.11}$$

By the assumption (4.5f), we have also guaranteed that $\mathfrak{g}(\underline{\zeta}_\tau)$ is a-priori bounded in $L^1(I; L^2(\Omega;\mathbb{R}^d)) \cap L^2(I; W^{1,2}(\Omega;\mathbb{R}^d)^*)$. By Hölder's inequality applied to (4.11) and a discrete Gronwall's inequality applied to (4.10), we then get uniform boundedness of $\mathscr{M}(\dot u_\tau(t))$, of $\mathscr{E}(t, u_\tau(t), \boldsymbol{\zeta}_\tau(t), \boldsymbol{\pi}_\tau(t), \boldsymbol{\varepsilon}_\tau(t))$, and of $\int_0^t \Xi(\underline{\boldsymbol{\zeta}}_\tau(t); \dot u_\tau(t), \dot{\boldsymbol{\zeta}}_\tau(t), \dot{\boldsymbol{\pi}}_\tau(t), \dot{\boldsymbol{\varepsilon}}_\tau(t))\,dt$. In view of (3.5), we get all the estimates (4.6).

Moreover, by using again (4.3a) and compare its value at $(u_\tau^k, \boldsymbol{\pi}_\tau^k, \boldsymbol{\varepsilon}_\tau^k)$ with a value at $(u_\tau^k, \widetilde{\boldsymbol{\pi}}, \boldsymbol{\varepsilon}_\tau^k)$ with a general $\widetilde{\boldsymbol{\pi}}$ and using the 1-homogeneity of $\mathscr{R}(\boldsymbol{\zeta}; \dot u, 0, \cdot, \dot\varepsilon)$ and thus the corresponding triangle inequality, we get

$$\begin{aligned}
\mathscr{E}(k\tau, u_\tau^k, \boldsymbol{\zeta}_\tau^{k-1}, \boldsymbol{\pi}_\tau^k, \boldsymbol{\varepsilon}_\tau^k) &\le \mathscr{E}(k\tau, u_\tau^k, \boldsymbol{\zeta}_\tau^{k-1}, \widetilde{\boldsymbol{\pi}}, \boldsymbol{\varepsilon}_\tau^k) \\
&\quad - \tau\mathscr{R}\Big(\boldsymbol{\zeta}_\tau^{k-1}; \frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}, 0, \frac{\boldsymbol{\pi}_\tau^k{-}\boldsymbol{\pi}_\tau^{k-1}}{\tau}, \frac{\boldsymbol{\varepsilon}_\tau^k{-}\boldsymbol{\varepsilon}_\tau^{k-1}}{\tau}\Big) \\
&\quad + \tau\mathscr{R}\Big(\boldsymbol{\zeta}_\tau^{k-1}; \frac{u_\tau^k{-}u_\tau^{k-1}}{\tau}, 0, \frac{\widetilde{\boldsymbol{\pi}}{-}\boldsymbol{\pi}_\tau^{k-1}}{\tau}, \frac{\boldsymbol{\varepsilon}_\tau^k{-}\boldsymbol{\varepsilon}_\tau^{k-1}}{\tau}\Big) \\
&= \mathscr{E}(k\tau, u_\tau^k, \boldsymbol{\zeta}_\tau^{k-1}, \widetilde{\boldsymbol{\pi}}, \boldsymbol{\varepsilon}_\tau^k) - \mathscr{R}(\boldsymbol{\zeta}_\tau^{k-1}; 0, 0, \boldsymbol{\pi}_\tau^k{-}\boldsymbol{\pi}_\tau^{k-1}, 0) \\
&\quad + \mathscr{R}(\boldsymbol{\zeta}_\tau^{k-1}; 0, 0, \widetilde{\boldsymbol{\pi}}{-}\boldsymbol{\pi}_\tau^{k-1}, 0) \\
&\le \mathscr{E}(k\tau, u_\tau^k, \boldsymbol{\zeta}_\tau^{k-1}, \widetilde{\boldsymbol{\pi}}, \boldsymbol{\varepsilon}_\tau^k) + \mathscr{R}(\boldsymbol{\zeta}_\tau^{k-1}; 0, 0, \widetilde{\boldsymbol{\pi}}{-}\boldsymbol{\pi}_\tau^k, 0)
\end{aligned} \tag{4.12}$$

from which (4.7e) follows. ☐

Remark 4.2 (Damage weakening). The assumption (4.5b,c) can, in fact, be relaxed to bound only the second derivative of $\mathbb{C}(\cdot)$, $\mathbb{C}_i(\cdot)$, $\mathfrak{c}(\cdot)$ and $\mathfrak{c}_i(\cdot)$. This so-called semi-convexity/concavity may be exploited to implement the concept of damage weakening to the stored energy (beside the dissipation energy we used so far). Semi-convexity of $\mathscr{E}(t, u, \cdot, \boldsymbol{\pi}, \boldsymbol{\varepsilon})$ would need a certain regularization and would yield the assertion of Lemma 4.1 only for sufficiently small τ with (4.7d) slightly modified but exhibiting the same asymptotics, cf. [41] for related technicalities in a particular model of an adhesive contact.

5. Convergence analysis. Let us devise a suitable notion of the weak solution to the system (3.11) with the initial conditions (3.6), designed by modifying the concept of so-called *energetic solutions* devised by Mielke at al. [33,35,36] applied here to the "rate-independent part" (3.11c) like it was done in [42,43]. By this way, we can avoid explicit occurrence of the measure $\dot{\boldsymbol{\pi}}$ in the weak formulation and, at the same time, to keep selectivity of such a definition in particular if $\mathscr{E}(t, u, \boldsymbol{\zeta}, \cdot, \boldsymbol{\varepsilon})$ is convex, as it is the case considered here. The mentioned important attribute "selectivity" means that any smooth weak solution is simultaneously the classical one, i.e. it satisfies (3.11) which, in fact, means (2.1), (3.9), and (3.12). For the selectivity of a weak/energetic formulation of such a combination of the rate-dependent part (3.11a,b,d) and the rate-independent part (3.11c) see [42].

Let $\mathrm{BV}(I; X)$ denote the space of functions $I \to X$ with a bounded variation. In this section we assume $r \ge 3$ and $r_i > 2$ (if $d = 3$) or $r > 2$ and $r_i > 1$ (if $d = 2$).

Definition 5.1. *The quadruple* $(u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon})$ *with* $\boldsymbol{\zeta} = (\zeta, \zeta_i)$, $\boldsymbol{\pi} = (\pi, \pi_i)$, *and* $\boldsymbol{\varepsilon} = (\varepsilon, \varepsilon_i)$ *such that*

$$u \in H^1\big(I; H^1(\Omega\setminus\Gamma_C;\mathbb{R}^d)\big) \cap C^1\big(\bar I; (L^2(\Omega;\mathbb{R}^d), \text{weak})\big), \tag{5.1a}$$

$$\boldsymbol{\zeta} \in W^{1,r}(I; W^{1,r}(\Omega\setminus\Gamma_C)) \times W^{1,r_i}(I; W^{1,r_i}(\Gamma_C)), \tag{5.1b}$$

$$\begin{aligned}\boldsymbol{\pi} \in\ & L^\infty\big(I; H^1(\Omega\setminus\Gamma_C;\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_C;\mathbb{R}^{d-1})\big) \\ &\cap \mathrm{BV}\big(I; L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times L^1(\Gamma_C;\mathbb{R}^{d-1})\big),\end{aligned} \tag{5.1c}$$

$$\boldsymbol{\varepsilon} \in H^1\big(I; L^2(\Omega\setminus\Gamma_C;\mathbb{R}^{d\times d}) \times L^2(\Gamma_C;\mathbb{R}^{d-1})\big) \tag{5.1d}$$

is called an energetic solution to (3.11) with the initial conditions (3.6) if:

(i) *the (conventional) weak formulation of (3.11a) holds, i.e.*

$$\begin{aligned}\int_0^T \bigg(\int_{\Omega\setminus\Gamma_C} \Big(\mathbb{C}(\zeta)\big(e(u{+}u_D){-}\pi{-}\varepsilon\big) + \mathbb{D}_0(\zeta)e(\dot u)\Big){:}e(\widetilde u) - \varrho\dot u{\cdot}\dot{\widetilde u}\,dx \\ + \int_{\Gamma_C} \mathbb{C}_i(\zeta_i)\big([\![u]\!]{-}\mathbb{T}(\pi_i{+}\varepsilon_i)\big){\cdot}[\![\widetilde u]\!]\,dS - \big\langle \mathfrak{g}(\zeta), \widetilde u\big\rangle\bigg)dt + \int_\Omega \varrho v_0{\cdot}\widetilde u(0)\,dx = 0\end{aligned} \tag{5.2a}$$

holds for all $\widetilde u \in H^1(I; H^1(\Omega\setminus\Gamma_C;\mathbb{R}^d))$ *with* $\widetilde u(T) = 0$,

(ii) *(3.11b) holds as a variational inequality, i.e.*

$$\begin{aligned}\int_{Q\setminus\Sigma_C} \mathfrak{F}(\widetilde\zeta) + \frac12 \mathbb{C}'(\zeta)\big(e(u){-}\pi{-}\varepsilon\big){:}\big(e(u){-}\pi{-}\varepsilon\big)\big(\widetilde\zeta{-}\dot\zeta\big) - \mathfrak{c}'(\zeta)\big(\widetilde\zeta{-}\dot\zeta\big) \\ + \kappa_0\nabla\zeta{\cdot}\nabla\big(\widetilde\zeta{-}\dot\zeta\big) + \frac{\kappa_1}{r}|\nabla\widetilde\zeta|^r\,dxdt \ge \int_{Q\setminus\Sigma_C} \mathfrak{F}(\dot\zeta) + \frac{\kappa_1}{r}|\nabla\dot\zeta|^r\,dxdt\end{aligned} \tag{5.2b}$$

holds for all $\widetilde\zeta \in L^\infty(I; W^{1,r}(\Omega\setminus\Gamma_C))$, *with* $\mathfrak{F}$ *from (2.2), and analogously on the fault*

$$\begin{aligned}\int_{\Sigma_C} \mathfrak{F}_i(\widetilde\zeta) + \frac12 \mathbb{C}_i'(\zeta_i)\big([\![u]\!]{-}\mathbb{T}(\pi_i{+}\varepsilon_i)\big){:}\big([\![u]\!]{-}\mathbb{T}(\pi_i{+}\varepsilon_i)\big)\big(\widetilde\zeta{-}\dot\zeta_i\big) - \mathfrak{c}_i'(\zeta_i)\big(\widetilde\zeta{-}\dot\zeta_i\big) \\ + \kappa_{0i}\nabla_S\zeta_i{\cdot}\nabla_S\big(\widetilde\zeta{-}\dot\zeta_i\big) + \frac{\kappa_{1i}}{r_i}|\nabla\widetilde\zeta|^{r_i}\,dSdt \ge \int_{\Sigma_C} \mathfrak{F}_i(\dot\zeta_i) + \frac{\kappa_{1i}}{r_i}|\nabla\dot\zeta_i|^{r_i}\,dSdt\end{aligned} \tag{5.2c}$$

holds for all $\widetilde{\zeta}\in L^\infty(I;W^{1,r_{\rm i}}(\Gamma_{\rm C}))$,

(iii) *the energy inequality analogous to (3.7), i.e.*

$$\mathscr{M}(\dot u(t))+\mathscr{E}(t,u(t),\boldsymbol{\zeta}(t),\boldsymbol{\pi}(t),\boldsymbol{\varepsilon}(t))+\int_0^t\Xi(\boldsymbol{\zeta}(t);\dot u(t),\dot{\boldsymbol{\zeta}}(t),\dot{\boldsymbol{\pi}}(t),\dot{\boldsymbol{\varepsilon}}(t))\,{\rm d}t \le\mathscr{M}(v_0)+\mathscr{E}(t,u_0,\boldsymbol{\zeta}_0,\boldsymbol{\pi}_0,\boldsymbol{\varepsilon}_0)+\int_0^t\mathscr{E}_t'(t,u,\boldsymbol{\zeta},\boldsymbol{\pi},\boldsymbol{\varepsilon})+\big\langle\mathfrak{g}(t,\zeta),\dot u\big\rangle\,{\rm d}t \tag{5.2d}$$

holds for all $t\in[0,T]$*, and*

(iv) *the so-called semi-stability holds in the bulk:*

$$\forall t\in\bar I\ \forall\widetilde\pi\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev}):\ \int_{\Omega\backslash\Gamma_{\rm C}}\frac{\kappa}{2}|\nabla\widetilde\pi|^2+\alpha(\zeta(t))\delta_P^*(\widetilde\pi-\pi(t))-\frac{\kappa}{2}|\nabla\pi(t)|^2 -\frac12\mathbb{C}(\zeta(t))\big(\pi(t)+\widetilde\pi+2\varepsilon(t)-2e(u(t)+u_{\rm D}(t))\big){:}(\pi(t)-\widetilde\pi)\,{\rm d}x\ge0, \tag{5.2e}$$

as well as on the fault

$$\forall t\in\bar I\ \forall\widetilde\pi_{\rm i}\in H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}):\ \int_{\Gamma_{\rm C}}\frac{\kappa_{\rm i}}{2}|\nabla_{\rm S}\widetilde\pi_{\rm i}|^2+\alpha_{\rm i}(\zeta_{\rm i}(t))\delta_{P_{\rm i}}^*(\widetilde\pi_{\rm i}-\pi_{\rm i}(t))-\frac{\kappa_{\rm i}}{2}|\nabla_{\rm S}\pi_{\rm i}(t)|^2 -\frac12\mathbb{C}_{\rm i}(\zeta_{\rm i}(t))\big(\mathbb{T}(\pi_{\rm i}(t)+\widetilde\pi_{\rm i}+2\varepsilon_{\rm i}(t))-2[\![u(t)]\!]\big){\cdot}\mathbb{T}(\pi_{\rm i}(t)-\widetilde\pi_{\rm i})\,{\rm d}x\ge0, \tag{5.2f}$$

(v) *also (3.11d) in a classical sense (i.e. (2.1c) holds a.e. on* $Q\backslash\Sigma_{\rm C}$ *and (3.9d) holds a.e. on* $\Sigma_{\rm C}$*), and eventually also*
(vi) *the initial conditions* (3.6) *hold.*

Note that (5.2e) together with (5.2f) bears the abbreviation

$$\forall t\in\bar I\ \forall\widetilde{\boldsymbol{\pi}}\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}):\ \mathscr{E}(t,u(t),\boldsymbol{\zeta}(t),\boldsymbol{\pi}(t),\boldsymbol{\varepsilon}(t))\le\mathscr{E}(t,u(t),\boldsymbol{\zeta}(t),\widetilde{\boldsymbol{\pi}},\boldsymbol{\varepsilon}(t))+\mathscr{R}(\boldsymbol{\zeta}(t);0,0,\widetilde{\boldsymbol{\pi}}-\boldsymbol{\pi}(t),0), \tag{5.3}$$

which is the so-called *semistability*, modifying the concept of the usual global stability [33,35,36] as devised in [42,43].

We will prove the convergence only in a simplified case that the viscous attenuation is not influenced by damage. As this attenuation is anyhow presumably only small in seismic applications we have in mind, this simplification seems reasonably acceptable.

PROPOSITION 5.2. *Let the assumptions of Lemma 4.1 be fulfilled, then there is a subsequence of the time-step parameters* $\tau\to0$ *(not explicitly indexed, without any confusion) and* $(u,\boldsymbol{\zeta},\boldsymbol{\pi},\boldsymbol{\varepsilon})$ *satisfying (5.1) such that*

$$u_\tau\to u \quad \text{weakly* in } H^1(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\cap W^{1,\infty}(\bar I;(L^2(\Omega;\mathbb{R}^d)), \tag{5.4a}$$

$$\boldsymbol{\zeta}_\tau\to\boldsymbol{\zeta} \quad \text{weakly in } W^{1,r}(I;W^{1,r}(\Omega\backslash\Gamma_{\rm C}))\times W^{1,r_{\rm i}}(I;W^{1,r_{\rm i}}(\Gamma_{\rm C})), \tag{5.4b}$$

$$\boldsymbol{\pi}_\tau\to\boldsymbol{\pi} \quad \text{weakly* in } L^\infty(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1})), \tag{5.4c}$$

$$\bar{\boldsymbol{\pi}}_\tau(t)\to\boldsymbol{\pi}(t) \quad \text{weakly in } H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1})\quad \forall t\in\bar I, \tag{5.4d}$$

$$\boldsymbol{\varepsilon}_\tau\to\boldsymbol{\varepsilon} \quad \text{weakly in } H^1(I;L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1})), \tag{5.4e}$$

$$\bar{\boldsymbol{\varepsilon}}_\tau(t)\to\boldsymbol{\varepsilon}(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1})\quad \forall t\in\bar I. \tag{5.4f}$$

Moreover, if $\mathbb{D}(\cdot)$*,* $\mathbb{D}_0(\cdot)$*, and* $\mathbb{D}_{\rm i}(\cdot)$ *are constant, we have the strong convergence of elastic stresses in the bulk and on the interface:*

$$\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau+u_{\rm D})-\bar\pi_\tau-\bar\varepsilon_\tau)\to\mathbb{C}(\zeta)(e(u+u_{\rm D})-\pi-\varepsilon)\ \text{ in } L^p(I;L^2(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d})), \tag{5.5a}$$

$$\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})([\![\bar u_\tau]\!]-\mathbb{T}(\bar\pi_{{\rm i}\tau}+\bar\varepsilon_{{\rm i}\tau}))\to\mathbb{C}_{\rm i}(\zeta_{\rm i})([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i}))\ \text{ in } L^p(I;L^2(\Gamma_{\rm C};\mathbb{R}^d)), \tag{5.5b}$$

for any $1\le p<\infty$ *and any such a quadruple* $(u,\boldsymbol{\zeta},\boldsymbol{\pi},\boldsymbol{\varepsilon})$ *is an energetic solution in accord to Definition 5.1.*

We should comment the main features of the proof. The gradient of the plastic variable $\boldsymbol{\pi}$ is used not because of a compactness in semistability (although we could alternatively use it to modify Step 6 below, too) but for proving the strong convergence of the driving force for damage evolution, cf. (5.6) with (5.9) below. Here one should emphasize that no regularity like [24] seems possible to be used because we consider the dynamical case. Also, we need $\nabla\dot{\boldsymbol{\zeta}}$ estimated to facilitate the limit passage (5.14). On the other hand, the weak L^2-convergence of ε suffices for the limit passage in (5.17) and thus we do not need any gradient of ε.

Proof of Proposition 5.2. For lucidity, we split the proof into the seven steps.

Step 1: Selection of converging subsequences: By the estimates (4.6) and Banach's selection principle, we can select a subsequence converging weakly* as specified in (5.4a-c,e). The $W^{1,1}$-estimate (4.6c) furthermore yields the BV-information in (5.1c) and the convergence (5.4d) and (5.4f) by Helly's selection principle.

Step 2: Improved convergence (5.5): We show the strong convergence of $e(\bar u_\tau+\bar u_{{\rm D},\tau})-\bar\pi_\tau-\bar\varepsilon_\tau$ by using uniform monotonicity of $\mathscr{E}(t,\cdot,\boldsymbol{\zeta},\cdot,\cdot)$. For simplicity, we perform the calculations for $u_{\rm D}=0$, the general case being just a rather straightforward but technical modification. We write the mentioned monotonicity between the approximate solution and its limit from Step 1. Further we use (4.7a) tested by $\widetilde u=u_\tau-u$, and (4.1c) tested by $\bar{\boldsymbol{\pi}}_\tau-\boldsymbol{\pi}$, and also (4.1d) tested by $\bar{\boldsymbol{\varepsilon}}_\tau-\boldsymbol{\varepsilon}$. By this way, we obtain

$$
\begin{aligned}
(5.6)\qquad &\int_{Q\setminus\Sigma_C}\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau-u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon):(e(\bar u_\tau-u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon)\\
&\quad+\kappa|\nabla\bar\pi_\tau-\nabla\pi|^2\,\mathrm{d}x\mathrm{d}t\\
&+\int_{\Sigma_C}\Big(\mathbb{C}_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\big([\![\bar u_\tau-u]\!]-\mathbb{T}(\bar\pi_{\mathrm{i}\tau}-\pi_\mathrm{i}+\bar\varepsilon_{\mathrm{i}\tau}-\varepsilon_\mathrm{i})\big)\cdot\big([\![\bar u_\tau-u]\!]-\mathbb{T}(\bar\pi_{\mathrm{i}\tau}+\pi_\mathrm{i}-\bar\varepsilon_{\mathrm{i}\tau}+\varepsilon_\mathrm{i})\big)\\
&\quad+\kappa_\mathrm{i}|\nabla_\mathrm{S}\bar\pi_{\mathrm{i}\tau}-\nabla_\mathrm{S}\pi_\mathrm{i}|^2\Big)\mathrm{d}S\mathrm{d}t=\\
&=\int_{Q\setminus\Sigma_C}\Big(\mathbb{D}_0e(\dot u_\tau):e(u-\bar u_\tau)+\mathbb{D}\dot\varepsilon_\tau:(\varepsilon-\bar\varepsilon_\tau)+\alpha(\underline{\zeta}_\tau)\bar\xi_\tau:(\pi-\bar\pi_\tau)\\
&\quad+\varrho\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\cdot(\dot u_\tau-\dot u)-\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau-u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon):(e(u)-\pi-\varepsilon)\\
&\quad-\kappa\nabla(\bar\pi_\tau-\pi):\nabla\pi-\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau-u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon):e(u_\tau-\bar u_\tau)\Big)\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\langle\bar{\mathfrak g}_\tau(\underline{\zeta}_\tau),u_\tau-u\rangle\,\mathrm{d}t-\int_\Omega\rho\dot u_\tau(T)(u_\tau(T)-u(T))\,\mathrm{d}x\\
&+\int_{\Sigma_C}\Big(\mathbb{D}_\mathrm{i}\dot\varepsilon_{\mathrm{i}\tau}:(\varepsilon_\mathrm{i}-\bar\varepsilon_{\mathrm{i}\tau})+\alpha_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\bar\xi_{\mathrm{i},\tau}\cdot(\pi_\mathrm{i}-\bar\pi_{\mathrm{i}\tau})-\kappa_\mathrm{i}\nabla_\mathrm{S}(\bar\pi_{\mathrm{i}\tau}-\pi_\mathrm{i})\cdot\nabla_\mathrm{S}\pi_\mathrm{i}\\
&\quad-\mathbb{C}_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\big([\![\bar u_\tau-u]\!]-\mathbb{T}(\bar\pi_{\mathrm{i}\tau}-\pi_\mathrm{i}+\bar\varepsilon_{\mathrm{i}\tau}-\varepsilon_\mathrm{i})\big)\cdot\big([\![u]\!]-\mathbb{T}(\pi_\mathrm{i}+\varepsilon_\mathrm{i})\big)\\
&\quad-\mathbb{C}_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\big([\![\bar u_\tau-u]\!]-\mathbb{T}(\bar\pi_{\mathrm{i}\tau}-\pi_\mathrm{i}+\bar\varepsilon_{\mathrm{i}\tau}-\varepsilon_\mathrm{i})\big)\cdot[\![u-\bar u_\tau]\!]\Big)\,\mathrm{d}S\mathrm{d}t\to0
\end{aligned}
$$

with some $\bar\xi_\tau\in\partial\delta^*_P(\dot{\bar\pi}_\tau)$ and $\bar\xi_{\mathrm{i},\tau}\in\partial\delta^*_{P_\mathrm{i}}(\dot{\bar\pi}_{\mathrm{i}\tau})$. From the convergence in (5.6), by uniform positive definiteness of $\mathbb{C}$ and $\mathbb{C}_\mathrm{i}$, we obtain the strong convergence like (5.5) but in $L^2(Q\backslash\Sigma_C;\mathbb{R}^{d\times d})$ and $L^2(\Sigma_C;\mathbb{R}^{d-1})$, respectively. Interpolating it with the respective bounds in $L^\infty(I;L^2(\Omega\backslash\Gamma_C;\mathbb{R}^{d\times d}))$ and $L^\infty(I;L^2(\Gamma_C;\mathbb{R}^{d-1}))$, which follows from cf. (4.6a,c,d), we obtain the claimed strong convergence (5.5).

To prove the claimed convergence in (5.6), we use

$$
\begin{aligned}
(5.7)\qquad \limsup_{\tau\to0}\int_{Q\setminus\Sigma_C}&\mathbb{D}_0e(\dot u_\tau):e(u-\bar u_\tau)\,\mathrm{d}x\mathrm{d}t\le\int_{\Omega\setminus\Gamma_C}\frac12\mathbb{D}_0e(u_0):e(u_0)\,\mathrm{d}x\\
&-\liminf_{\tau\to0}\int_{\Omega\setminus\Gamma_C}\frac12\mathbb{D}_0e(u_\tau(T)):e(u_\tau(T))\,\mathrm{d}x+\lim_{\tau\to0}\int_{Q\setminus\Sigma_C}\mathbb{D}_0e(\dot u_\tau):e(u)\,\mathrm{d}x\mathrm{d}t\\
&\le\int_{\Omega\setminus\Gamma_C}\frac12\mathbb{D}_0e(u_0):e(u_0)-\frac12\mathbb{D}_0e(u(T)):e(u(T))\,\mathrm{d}x+\int_{Q\setminus\Sigma_C}\mathbb{D}_0e(\dot u):e(u)\,\mathrm{d}x\mathrm{d}t=0
\end{aligned}
$$

where we used $u_\tau(T)\to u(T)$ weakly in $H^1(\Omega\backslash\Gamma_C;\mathbb{R}^d)$ and $\dot u_\tau\to\dot u$ weakly in $L^2(I;H^1(\Omega\backslash\Gamma_C;\mathbb{R}^d))$; here we used the assumption $\mathbb{D}_0$ independent of ζ. Further,

$$
\begin{aligned}
(5.8)\qquad \limsup_{\tau\to0}\int_Q\mathbb{D}\dot\varepsilon_\tau:(\varepsilon-\bar\varepsilon_\tau)\,\mathrm{d}x\mathrm{d}t&\le\int_\Omega\frac12\mathbb{D}\varepsilon_0:\varepsilon_0\,\mathrm{d}x-\liminf_{\tau\to0}\int_\Omega\frac12\mathbb{D}\varepsilon_\tau(T):\varepsilon_\tau(T)\,\mathrm{d}x\\
+\lim_{\tau\to0}\int_Q\mathbb{D}\dot\varepsilon_\tau:\varepsilon\,\mathrm{d}x\mathrm{d}t&\le\int_\Omega\frac12\mathbb{D}\varepsilon_0:\varepsilon_0-\frac12\mathbb{D}\varepsilon(T):\varepsilon(T)\,\mathrm{d}x+\int_Q\mathbb{D}\dot\varepsilon:\varepsilon\,\mathrm{d}x\mathrm{d}t=0
\end{aligned}
$$

where we used $\varepsilon_\tau(T)\to\varepsilon(T)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ and $\dot\varepsilon_\tau\to\dot\varepsilon$ weakly in $L^2(Q;\mathbb{R}^{d\times d})$; here we used the assumption $\mathbb{D}$ is independent of ζ. By analogous arguments, also $\int_{\Sigma_C}\mathbb{D}_\mathrm{i}\dot\varepsilon_{\mathrm{i}\tau}:(\varepsilon_\mathrm{i}-\bar\varepsilon_{\mathrm{i}\tau})\,\mathrm{d}S\mathrm{d}t\to0$. Moreover, we use the (generalized) Aubin-Lions' theorem which yields $\bar\pi_\tau\to\pi$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ so that

$$
(5.9)\qquad \int_Q\alpha(\underline{\zeta}_\tau)\bar\xi_\tau:(\pi-\bar\pi_\tau)\,\mathrm{d}x\mathrm{d}t\to0
$$

because $\alpha(\underline{\zeta}_\tau)\bar\xi_\tau$ is bounded in $L^\infty(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. By analogous arguments, using boundedness of $\alpha_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\bar\xi_{\mathrm{i},\tau}$ in $L^\infty(\Sigma_C;\mathbb{R}^{d-1})$, we have also $\int_{\Sigma_C}\alpha_\mathrm{i}(\underline{\zeta}_{\mathrm{i},\tau})\bar\xi_{\mathrm{i},\tau}\cdot(\pi_\mathrm{i}-\bar\pi_{\mathrm{i}\tau})\mathrm{d}S\mathrm{d}t\to0$. Eventually, after some algebra,

$$
\begin{aligned}
(5.10)\qquad \limsup_{\tau\to0}\int_Q\varrho\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\cdot(\dot u_\tau-\dot u)\,\mathrm{d}x\mathrm{d}t&=\limsup_{\tau\to0}\int_Q\varrho\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\cdot\dot u_\tau\,\mathrm{d}x\mathrm{d}t-\lim_{\tau\to0}\int_Q\varrho\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\cdot\dot u\,\mathrm{d}x\mathrm{d}t\\
&\le\lim_{\tau\to0}\int_Q\varrho\big|\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\big|^2-\frac\tau4\big|\dot u_\tau(T)\big|^2+\frac\tau4|v_0|^2\,\mathrm{d}x\mathrm{d}t-\lim_{\tau\to0}\int_Q\varrho\big[\dot u_\tau\big]^{\mathrm{int}}_\tau\cdot\dot u\,\mathrm{d}x\mathrm{d}t=0
\end{aligned}
$$

where we used $[\dot u_\tau]^{\mathrm{int}}_\tau\to\dot u$ strongly in $L^2(Q;\mathbb{R}^d)$, which follows by the (generalized) Aubin-Lions' theorem from (5.4a) when taking into account also an estimate $\|\dot u_\tau\|_{BV(I;H^1(\Omega\backslash\Gamma_C;\mathbb{R}^d)^*)}\le C$ implied by (4.6) through (4.1a). Also we used that $\tau|\dot u_\tau(T)|^2\to0$ in $L^1(\Omega)$ since $\dot u_\tau(T)$ is bounded in $L^2(\Omega;\mathbb{R}^d)$ by (4.6a). Also, by strong convergence $u_\tau(T)\to u(T)$ in $L^2(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ and again boundedness of $\dot u_\tau(T)$ in $L^2(\Omega\backslash\Gamma_C;\mathbb{R}^d)$, we obtain

$$
(5.11)\qquad \int_\Omega\rho\dot u_\tau(T)(u_\tau(T)-u(T))\,\mathrm{d}x\to0.
$$

For the remaining terms in (5.6) converge to 0 by the weak convergence of $e(\bar u_\tau)-\bar\pi_\tau-\varepsilon_\tau\to e(u)-\pi-\varepsilon$ and $[\![\bar u_\tau]\!]-\mathbb{T}(\bar\pi_{\mathrm{i}\tau}+\varepsilon_{\mathrm{i}\tau})\to[\![u]\!]-\mathbb{T}(\pi_\mathrm{i}+\varepsilon_\mathrm{i})$, and by the strong convergence $e(u_\tau-\bar u_\tau)\to0$ due to the estimate

$$
(5.12)\qquad \big\|e(u_\tau-\bar u_\tau)\big\|_{L^2(Q\setminus\Sigma_C;\mathbb{R}^{d\times d})}=3^{-1/2}\tau\big\|e(\dot u_\tau)\big\|_{L^2(Q\setminus\Sigma_C;\mathbb{R}^{d\times d})}\to0.
$$

Step 3: Limit passage to (5.2a)*:* By (generalized) Aubin-Lions' theorem, $\underline{\boldsymbol{\zeta}}_\tau$ converges strongly to $\boldsymbol{\zeta}$ and thus also $\mathbb{C}(\underline{\zeta}_\tau)$ and $\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})$ converge strongly in the corresponding L^p-spaces, $p<\infty$. Then the convergence in (4.7a) towards (5.2a) is easy.

Step 4: Limit passage to (5.2b) *and* (5.2c)*:* Like in Step 3, we have strong convergence of $\mathbb{C}'(\underline{\zeta}_\tau)$ and $\mathfrak{c}'\underline{\zeta}_\tau)$ in the corresponding L^p-spaces, $p<\infty$. Combining it with (5.5a), we obtain

$$\mathbb{C}'(\underline{\zeta}_\tau)\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big){:}\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\to\mathbb{C}'(\zeta)\big(e(u)-\pi-\varepsilon\big){:}\big(e(u)-\pi-\varepsilon\big) \tag{5.13}$$

strongly in $L^p(I;L^1(\Omega\backslash\Gamma_{\rm C}))$ for any $1\le p<\infty$. Using still $\dot\zeta_\tau\to\dot\zeta$ weakly in $L^r(I;W^{1,r}(\Omega\backslash\Gamma_{\rm C}))$, we have the convergence of the term $\mathbb{C}'(\underline{\zeta}_\tau)\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\,{:}\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\dot\zeta_\tau$ occurring in (4.7b), namely

$$\mathbb{C}'(\underline{\zeta}_\tau)\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big){:}\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\dot\zeta_\tau\to\mathbb{C}'(\zeta)\big(e(u)-\pi-\varepsilon\big){:}\big(e(u)-\pi-\varepsilon\big)\dot\zeta \tag{5.14}$$

weakly in $L^q(I;L^1(\Omega))$ for any $1\le q<r$; here we employed the assumption $r\ge 3$ (if $d=3$) or $r>2$ (if $d=2$), and thus the embedding of $W^{1,r}(\Omega\backslash\Gamma_{\rm C})\subset L^\infty(\Omega)$. The resting terms in (4.7b) can be treated by weak lower semicontinuity combined with by-part integration.

Analogous arguments based on the embedding $W^{1,r_{\rm i}}(\Gamma_{\rm C})\subset L^\infty(\Gamma_{\rm C})$ lead to the limit passage to the interfacial flow rule (5.2c) for $\zeta_{\rm i}$, provided $r_{\rm i}>2$ (if $d=3$) and $r_{\rm i}>1$ (if $d=2$).

Step 5: Limit passage in the energy balance (4.7d)*:* By already proved convergences and by the weak lower semicontinuity, (5.2d) easily follows from (4.7d).

Step 6: Limit passage in the semistability (4.7e) *towards* (5.2e) *and* (5.2f)*:* Let us consider $t\in I$ and a general $\widetilde{\boldsymbol{\pi}}=(\widetilde\pi,\widetilde\pi_{\rm i})$ and put

$$\widetilde{\boldsymbol{\pi}}_\tau=(\widetilde\pi_\tau,\widetilde\pi_{{\rm i},\tau})\quad\text{ with }\quad\widetilde\pi_\tau:=\bar\pi_\tau(t)-\pi(t)+\widetilde\pi\quad\text{ and }\quad\widetilde\pi_{{\rm i},\tau}:=\bar\pi_{{\rm i},\tau}(t) \tag{5.15}$$

in place of $\widetilde{\boldsymbol{\pi}}$ into (4.7e). Thus (4.7e) turns into

$$\begin{aligned}0\le\int_{\Omega\backslash\Gamma_{\rm C}}&\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}){-}\widetilde\pi_\tau{-}\bar\varepsilon_\tau(t)\big){:}\big(e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}(t)){-}\widetilde\pi_\tau{-}\bar\varepsilon_\tau(t)\big)\\&-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}(t)){-}\bar\pi_\tau(t){-}\bar\varepsilon_\tau(t)\big){:}\big(e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}(t)){-}\bar\pi_\tau(t){-}\bar\varepsilon_\tau(t)\big)\\&-\frac\kappa2|\nabla\bar\pi_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta^*_P(\widetilde\pi_\tau{-}\bar\pi_\tau(t))+\frac\kappa2|\nabla\widetilde\pi_\tau|^2\,{\rm d}x.\end{aligned} \tag{5.16}$$

Realizing that $\widetilde\pi_\tau-\bar\pi_\tau(t)=\widetilde\pi-\pi(t)$, hence independent of τ, we can rewrite and converge (5.16) as

$$\begin{aligned}0\le\lim_{\tau\to0}\int_{\Omega\backslash\Gamma_{\rm C}}&\mathbb{C}(\underline{\zeta}_\tau(t))\Big(\frac{\widetilde\pi_\tau{+}\bar\pi_\tau(t)}{2}{-}e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}(t)){+}\bar\varepsilon_\tau(t)\Big){:}\big(\widetilde\pi_\tau{-}\bar\pi_\tau(t)\big)\\&+\alpha(\underline{\zeta}_\tau(t))\delta^*_P(\widetilde\pi_\tau{-}\bar\pi_\tau(t))+\frac\kappa2\nabla\big(\widetilde\pi_\tau{+}\bar\pi_\tau(t)\big)\vdots\nabla\big(\widetilde\pi_\tau{-}\bar\pi_\tau(t)\big)\,{\rm d}x\\=\lim_{\tau\to0}\int_{\Omega\backslash\Gamma_{\rm C}}&\mathbb{C}(\underline{\zeta}_\tau(t))\Big(\frac{\widetilde\pi_\tau{+}\bar\pi_\tau(t)}{2}{-}e(\bar u_\tau(t){+}\bar u_{{\rm D},\tau}(t)){+}\bar\varepsilon_\tau(t)\Big){:}\big(\widetilde\pi{-}\pi(t)\big)\\&+\alpha(\underline{\zeta}_\tau(t))\delta^*_P(\widetilde\pi{-}\pi(t))+\frac\kappa2\nabla\big(\widetilde\pi_\tau{+}\bar\pi_\tau(t)\big)\vdots\nabla\big(\widetilde\pi{-}\pi(t)\big)\,{\rm d}x\\=\int_{\Omega\backslash\Gamma_{\rm C}}&\mathbb{C}(\zeta(t))\Big(\frac{\widetilde\pi{+}\pi(t)}{2}{-}e(u(t){+}u_{\rm D}(t)){+}\varepsilon(t)\Big){:}\big(\widetilde\pi{-}\pi(t)\big)\\&+\alpha(\zeta(t))\delta^*_P(\widetilde\pi{-}\pi(t))+\frac\kappa2\big|\nabla\widetilde\pi\big|^2-\frac\kappa2\big|\nabla\pi(t)\big|^2\,{\rm d}x.\end{aligned} \tag{5.17}$$

Note that the convergence of the integrands in (5.17) has been weak in $L^1(\Omega)$; note that we used both (5.4d) and (5.4f), as well as that $\widetilde\pi_\tau\to\widetilde\pi$ weakly in $H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})$. In the limit, we thus have obtained (5.2e).

Moreover, let us put

$$\widetilde{\boldsymbol{\pi}}_\tau=(\widetilde\pi_\tau,\widetilde\pi_{{\rm i},\tau})\quad\text{ with }\quad\widetilde\pi_\tau:=\bar\pi_\tau(t)\quad\text{ and }\quad\widetilde\pi_{{\rm i},\tau}:=\bar\pi_{{\rm i},\tau}(t)-\pi_{\rm i}(t)+\widetilde\pi_{\rm i} \tag{5.18}$$

in place of $\widetilde{\boldsymbol{\pi}}$ into (4.7e). Thus (4.7e) turns into

$$\begin{aligned}0\le\int_{\Gamma_{\rm C}}&\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\widetilde\pi_{\rm i}{+}\bar\varepsilon_{{\rm i},\tau})\big){\cdot}\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\pi_{\rm i}{+}\varepsilon_{\rm i})\big)+\frac{\kappa_{1\rm i}}{2}|\nabla_{\rm S}\widetilde\pi_{\rm i}|^2\\&-\frac12\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big){\cdot}\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big)-\frac{\kappa_{1\rm i}}{2}|\nabla_{\rm S}\bar\pi_{{\rm i},\tau}|^2\\&+\alpha_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\delta^*_{P_{\rm i}}(\widetilde\pi_{{\rm i},\tau}{-}\bar\pi_{{\rm i}\tau})\,{\rm d}S\end{aligned} \tag{5.19}$$

and proceeding analogously as in (5.17), we eventually obtain also (5.2f).

Step 7: Limit passage in (3.11d)*:* Eventually, (4.1d) yields the equation $\mathscr{R}'_{\dot{\boldsymbol\varepsilon}}(\underline{\boldsymbol\zeta}_\tau;\dot{\boldsymbol\varepsilon}_\tau)+[\bar{\mathscr{E}}_\tau]'_{\boldsymbol\varepsilon}(\bar u_\tau,\underline{\boldsymbol\zeta}_\tau,\bar{\boldsymbol\pi}_\tau,\bar{\boldsymbol\varepsilon}_\tau)=0$. This involves semilinear equations

$$\mathbb{D}\dot\varepsilon=\mathbb{C}(\underline{\zeta}_\tau)\big(e(\bar u_\tau){-}\bar\pi_\tau{-}\bar\varepsilon_\tau\big)\quad\text{ and }\quad\mathbb{D}_{\rm i}\dot\varepsilon_{\rm i}=\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau]\!]{-}\mathbb{T}(\bar\pi_{{\rm i},\tau}{+}\bar\varepsilon_{{\rm i},\tau})\big), \tag{5.20}$$

cf. (2.1c) and (3.9d), which bears easily the limit passage towards (3.11d). □

REMARK 5.3 (Spatial discretization by FEM). Computer implementation of the model needs a spatial discretization. In polygonal domains, the simplest way is by simplicial triangulation and P1-finite elements for u, $\boldsymbol\zeta$, and $\boldsymbol\pi$, while $\boldsymbol\varepsilon$ bears the P0-elements approximation. The gradient of π is, in fact, needed only for using the compactness to prove (5.9) and analogously the surface gradient of $\pi_{\rm i}$ is needed for using the compactness in its interfacial variant. Thus, one can alternatively consider the nonlocal "$W^{\alpha,2}$-fractional gradient"; i.e. instead of $\int_{\Omega\backslash\Gamma_C}\frac{\kappa}{2}|\nabla\pi|^2 \mathrm{d}x$ in (2.4c), for a fixed parameter $0<\alpha<1$, one can consider

$$\sum_i \frac{\kappa}{4}\int_{\Omega_i}\int_{\Omega_i}\frac{|\pi(x)-\pi(\tilde{x})|^2}{|x-\tilde{x}|^{d+2\alpha}}\,\mathrm{d}\tilde{x}\mathrm{d}x \tag{5.21}$$

where Ω_i denotes connected components of $\Omega\backslash\Gamma_C$. If $\alpha<1/2$, the P0-elements can be used for spatial discretization of π. Similar observation concerns the term $\int_{\Gamma_C}\frac{\kappa_i}{2}|\nabla_S\pi_i|^2\mathrm{d}S$ in (3.1a). On the other hand, we need "full" gradient of ζ (or even of $\dot{\zeta}$) to control (5.14). Thus, P1-elements can be used for u and ζ while, under this modification of the above theory, the other variables $\boldsymbol{\pi}$ and $\boldsymbol{\varepsilon}$ bear the P0-elements approximation. For an efficient wavelet-type numerical implementation of (an equivalent modification) of the double integral form (5.21) see [3, Sect. 3.3].

6. Illustrative computational experiments: a single d-o-f test. The purpose of this section is to demonstrate the capacity of the above model to describe the one (and perhaps the most important) phenomenon of re-occurring spontaneous ruptures of faults and subsequent healing during motion of lithospheric plates with a constant velocity (assumed sufficiently fast to eliminate fluidic behavior which would suppress inelastic response). For this, we neglect most of the other aspects of the model. In particular, we neglect all inertial/inelastic/viscous effects in the bulk (which will then be considered purely elastic), and also the Maxwellian rheology both in the bulk and on the fault, thus we set $\varepsilon=0$, $\pi=0$, $\zeta=0$, and $\varepsilon_i=0$. The semi-implicit discretisation (4.1) now takes $\mathscr{M}=0$ and $\mathscr{R}'_{\dot{u}}=0$ in (4.1a), while (4.1d) is not considered at all.

To test the very basic desired stick-slip behavior of the adhesive contact with interface plasticity and healing, we performed an essentially 0-dimensional test, which is a standard approach in seismic modelling for testing basic validity of any new model. To this goal, we consider the ansatz that $e(u)$ is constant on each particular subdomain, here Ω_1 and Ω_2, and π_i and ζ_i are constant along Γ_C. Thus, in particular, $u|_{\Omega_1}$ and $u|_{\Omega_2}$ are affine. We further consider a symmetrical geometry as depicted on Fig. 3. To that (piece-wise) constant ansatz of $e(u)$, π_i, and ζ_i, and symmetry of the geometry, we also assume symmetry of the Dirichlet loading, as on Fig. 3, and still consider an ansatz that the solution inherits the symmetry of the geometry and loading. Thus essentially we have only one degree of freedom as far as "observable" parameters concerns, namely u, which is why in seismic literature on fault friction such test is also called "single-degree-of-freedom slider" or "spring-slider" experiment, while there are other 2 degrees of freedom in internal parameters π_i and ζ_i.

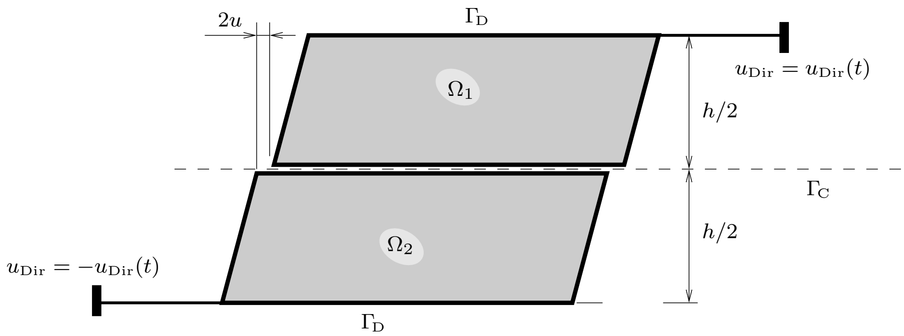

Fig. 3 *A single-degree-of-freedom slider, having 1 d-o-f observable parameter u (other 2 d-o-f are in internal parameters π_i and ζ_i).*

We use only test (dimensionless) constants without any special relevance to real lithospheric models and the following (intentionally very simple) nonlinearities: $P_i=[-1,1]$, $\alpha_i(\zeta_i):=\alpha_{i0}+\alpha_{i1}\zeta_i$ with $\alpha_{i1}=1$ and $\alpha_{i0}=10^{-4}$, $\mathfrak{c}_i(\zeta_i):=\mathfrak{c}_0\zeta_i$ with $\mathfrak{c}_0$ specified later, $\mathbb{C}_i(\zeta_i):=\mathbb{C}_{i0}+\mathbb{C}_{i1}\zeta_i$ with $\mathbb{C}_{i0}=0.1$ and $\mathbb{C}_{i1}=1$, $\mathfrak{b}_i=0.1$, $\mathfrak{a}_i=20$, and $\mathfrak{d}_i=0$; in fact, the value α_{i0} ranging $[0,...,10^{-3}]$ was tested, giving essentially the same results. Note that, for simplicity, we considered both nonlinearities $\mathfrak{c}_i(\cdot)$ and $\mathbb{C}_i(\cdot)$ affine and the constraints $0\le\zeta_i(t)\le1$ have been simply implemented into the optimization routine.

Note that, in view of (3.9c), we obtain weakening effects (in interfacial plastic flow) for $\alpha_{i1}/\mathbb{C}_{i1}>\alpha_{i0}/\mathbb{C}_{i0}$, which is indeed always satisfied for our parameter choices. For $\alpha_{i0}=0$ we got a frictionless model when complete delamination takes place. Note also that in this simple affine setting for $\mathbb{C}_i$, both minimization problems described above are linear-quadratic problems.

The bulk stored energy $\mathscr{E}_{\mathrm{bulk}}(t,u)$ after the mentioned shift of Dirichlet condition and counting a unit length of the specimen from Fig. 3 is $\frac{1}{2}h\mathbb{C}|u-u_{\mathrm{Dir}}(t)|^2$. We consider linearly increasing prescribed horizontal shift $u_{\mathrm{Dir}}(t)=7.10^{-5}t$ over the time interval $t\in[0,T]$ with $T=8.10^7$. Except for Fig. 6(right), we consider $h\mathbb{C}=10^{-4}$.

We performed the experiments for $\mathfrak{c}_0$ varying. The results of the simulations are depicted on Fig. 4. The response of u was nearly the same as π_i, which is why we did not depict it. In a detailed view, as on Fig. 5, one can indeed see the scenario during the rupture: at the beginning, the interface damage ζ_i starts falling down, then the interface plastic slip is activated to evolve and simultaneously the stored elastic energy is released and the stress is relaxed so that, eventually the healing (increase of ζ_i) can evolve, elastic energy starts again being stored, and new rupture thus starts preparing.

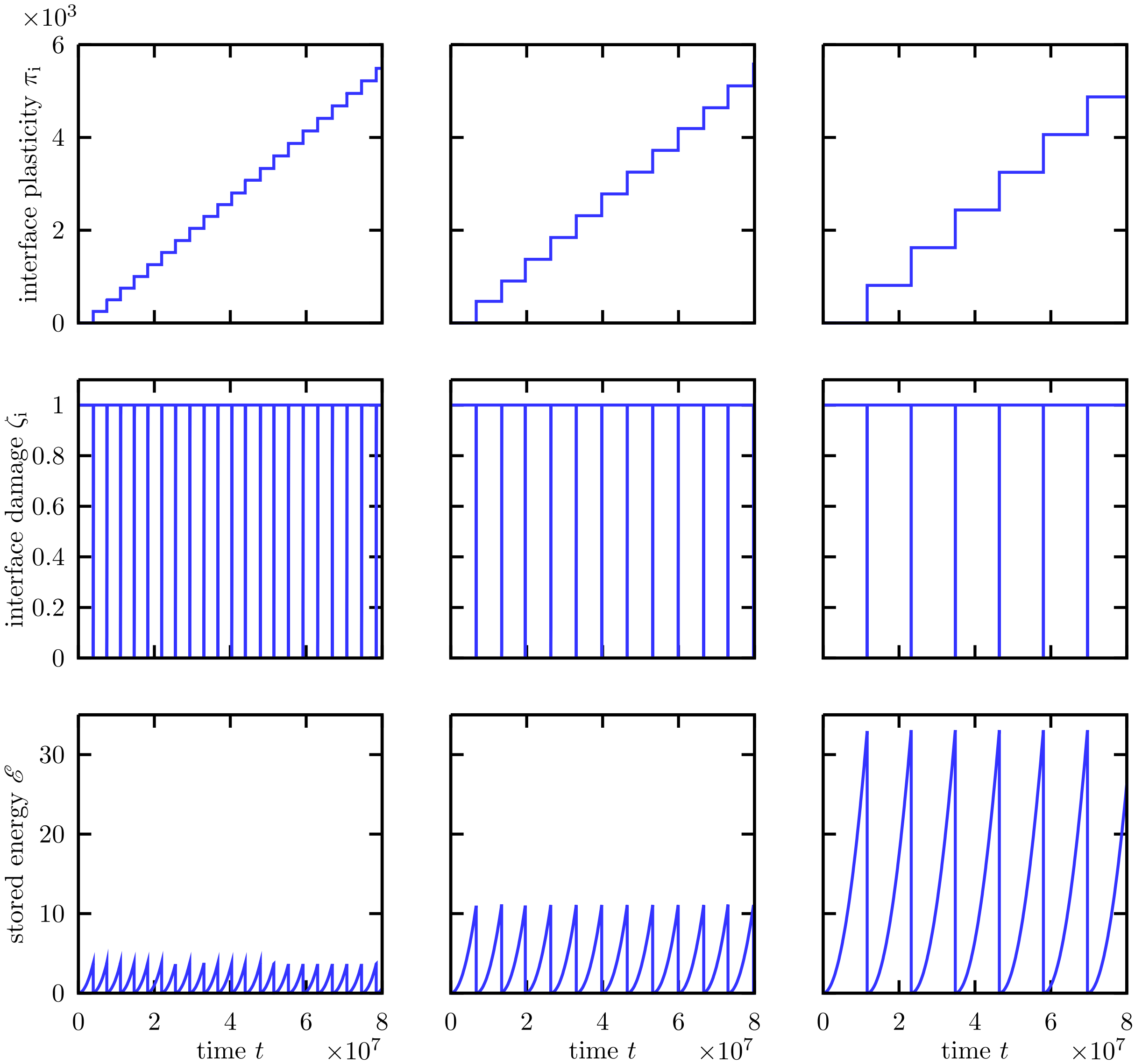

Fig. 4 *Oscillatory response of ζ_i, π_i, and $\mathscr{E}$ in time on the linearly increasing load u_Dir displayed for three different values of $\mathfrak{c}_0$, namely (from left to to right) $\mathfrak{c}_0 = 3.10^{-4}$, 9.10^{-4}, and 27.10^{-4}.*

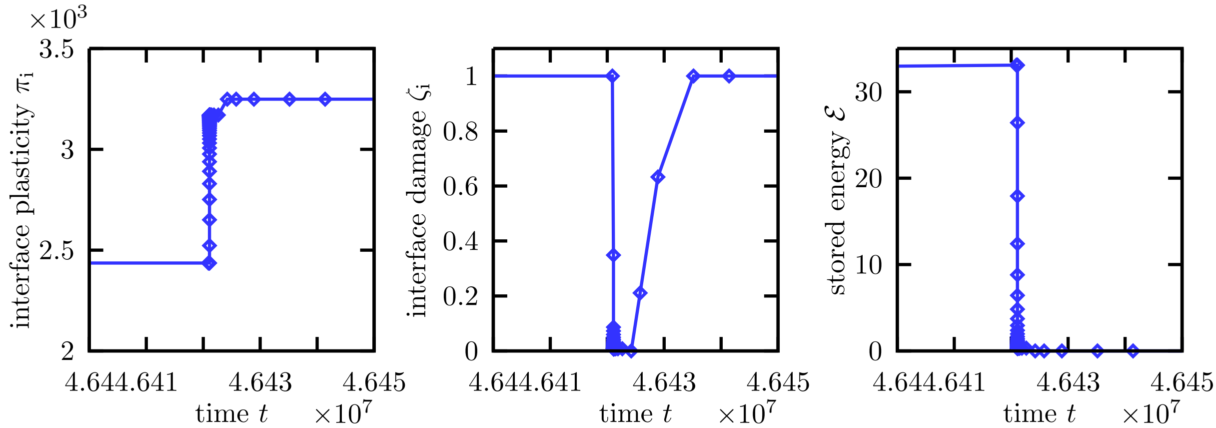

Fig. 5 *Time-zoom of π_i, ζ_i, and $\mathscr{E}$ during one particular (namely the fourth) "earthquake" from Fig. 4(right).*

The energy released during each particular earthquake can be measured by evaluating the difference of the stored energy immediately before and after this earthquake. We can see, as expected, that frequency of occurrence of earthquakes decays proportionally to $1/\mathfrak{c}_0$ while released energy increases proportionally to $\mathfrak{c}_0^2$. Thus considering for simplicity just uniform distribution of the values of activation parameters in a considered seismically active region and neglecting all dynamical coupling phenomena (dynamic earthquake triggering etc.), we would obtain similar linear relationship between logarithm of released energy and occurrence frequency in the region as observed in nature and known as Gutenberg-Richter's law [20]. This linear relationship between logarithms of released energy and interval of rupture reoccurrence in our 1-dof-slider experiment is depicted on Fig. 6(left).

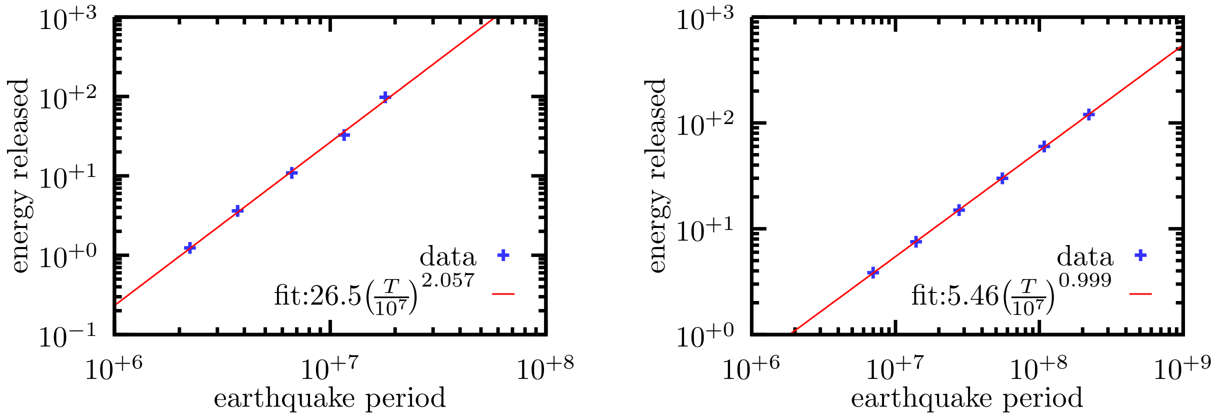

Fig. 6 *Variation of stored energy versus periods between particular earthquakes:*
Left: the activation energy $\mathfrak{c}_0$ *(=fault fracture toughness) varies as* $3^i.10^{-4}$ *for* $i = 0, ..., 4$*; the slope is close to 2.*
Right: the plate height h *varies such that* $h\mathbb{C} = 2^i.10^{-6}$ *for* $i = 0, ..., 5$.

For comparison, we also varied the height h of the plates, cf. Fig. 3. The released energy is then expected to be proportional to their height while the frequency of earthquakes is inversely proportional (i.e. the slope is ~ 1 in the logarithmic scale), as indeed seen from Fig. 6(right) calculated for $\mathfrak{c}_0 = 10^{-4}$ fixed.

It is important to realize that the desired oscillatory behaviour of the model requires certain tuning of the parameters. In particular, in our case, we consider $[\alpha_{\rm i}/\mathbb{C}_{\rm i}](\cdot)$ nondecreasing, and then we can see that healing is prevented for $\zeta_{\rm i} = 0$ if $\mathfrak{c}_0 < \frac{1}{2}\mathbb{C}_{\rm i\,1}(\alpha_{\rm i\,0}/\mathbb{C}_{\rm i\,0})^2$ because then, for $\zeta_{\rm i} = 0$, the minimizer of $\mathscr{E}(t, \cdot, z)$ is always 0. Similarly, interface damage prevented for $\zeta_{\rm i} = 1$ if $\mathfrak{c}_0 > \frac{1}{2}\mathbb{C}_{\rm i\,1}(\alpha_{\rm i\,0}{+}\alpha_{\rm i\,1})^2/(\mathbb{C}_{\rm i\,0}{+}\mathbb{C}_{\rm i\,1})^2$ because then, for $\zeta_{\rm i} = 1$, the minimizer of $\mathscr{E}(t, \cdot, z)$ is always 1. Written more generally, we need

$$\frac{\mathbb{C}_{\rm i}'(0)}{2}\Big(\frac{\alpha_{\rm i}(0)}{\mathbb{C}_{\rm i}(0)}\Big)^2 < \mathfrak{c}_0 < \frac{\mathbb{C}_{\rm i}'(1)}{2}\Big(\frac{\alpha_{\rm i}(1)}{\mathbb{C}_{\rm i}(1)}\Big)^2 \tag{6.1}$$

which, in our case, means that

$$\frac{\mathbb{C}_{\rm i\,1}}{2}\Big(\frac{\alpha_{\rm i\,0}}{\mathbb{C}_{\rm i\,0}}\Big)^2 < \mathfrak{c}_0 < \frac{\mathbb{C}_{\rm i\,1}}{2}\Big(\frac{\alpha_{\rm i\,0}{+}\alpha_{\rm i\,1}}{\mathbb{C}_{\rm i\,0}{+}\mathbb{C}_{\rm i\,1}}\Big)^2. \tag{6.2}$$

This condition essentially determined the range of $\mathfrak{c}_0$ we used for Fig. 6(left).

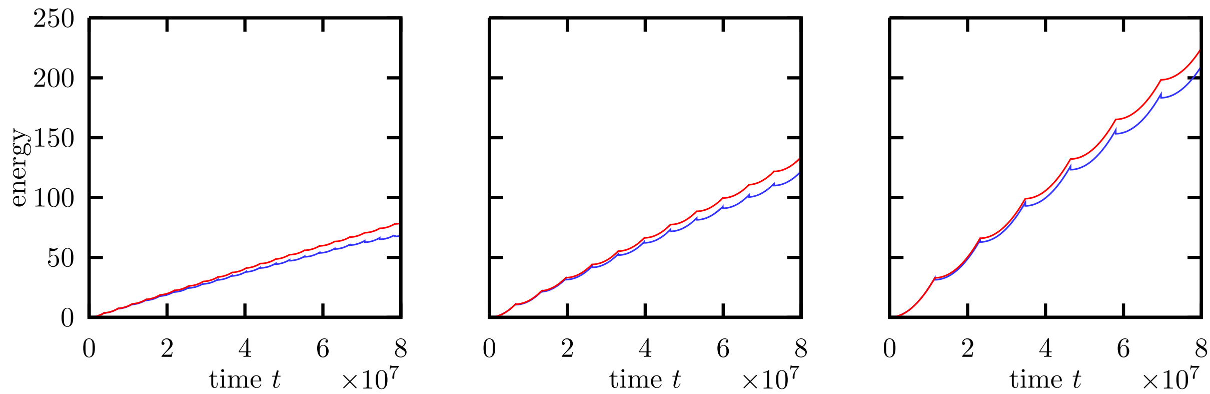

Fig. 7 *The energies on the left- and the right-hand sides in* (4.7d) *as functions of time (i.e. the lower and the upper curve, respectively) for the three values of* $\mathfrak{c}_{\rm i}$ *used also in Fig. 4; the refinement/coarsening of time step* τ *during earthquakes/healing periods, respectively, was chosen just to control this difference and keep it reasonably small.*

As mentioned in Sect. 1, the problem is obviously multiscaled in time in the sense that earthquakes dynamics is much faster than the slow dynamics of healing/waiting period. Numerically, it ultimately calls for an adaptive variation of the time step. Here, we used a physically motivated strategy based on checking the difference in the energy balance (4.7d). More specifically, when the rate of the difference of the left- and the right-hand sides in (4.7d) exceeded a prescribed tolerance, the time step was shortened, otherwise it was gradually enlarged. The slowly diverging bounds in (4.7d) are depicted on Fig. 7 for one particular case corresponding to Fig. 4(right). The non-uniform time discretisation automatically refining during jumps can also be seen from Fig. 5.

Acknowledgements. The authors are thankful to dr. František Gallovič, to doc. Ctirad Matyska, as well as to two annonymous referees for many useful comments. T.R. and R.V. acknowledge the hospitality of Universidad de Sevilla, where this work has partly been accomplished, covered by Junta de Andalucía and by Gobierno de España, Ministerio de Educación, Cultura y Deporte (project SAB2010-0082). T.R. acknowledges the institutional support RVO: 61388998 (ČR) as partial support from the grants 201/09/0917 and 201/10/0357 (GA ČR). R.V. acknowledges a partial support also from the grant VEGA 1/0201/11. O.S. acknowledges support by the Nečas Center for Mathematical Modelling, project LC06052 (MŠMT ČR).

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