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GLOBAL ILL-POSEDNESS OF THE ISENTROPIC SYSTEM OF GAS DYNAMICS

ELISABETTA CHIODAROLI, CAMILLO DE LELLIS AND ONDŘEJ KREML

ABSTRACT. We consider the isentropic compressible Euler system in 2 space dimensions with pressure law $p(\rho) = \rho^2$ and we show the existence of classical Riemann data, i.e. pure jump discontinuities across a line, for which there are infinitely many admissible bounded weak solutions (bounded away from the void). We also show that some of these Riemann data are generated by a 1-dimensional compression wave: our theorem leads therefore to Lipschitz initial data for which there are infinitely many global bounded admissible weak solutions.

1. Introduction

Consider the isentropic compressible Euler equations of gas dynamics in two space dimensions. This system consists of 3 scalar equations, which state the conservation of mass and linear momentum. The unknowns are the density ρ and the velocity v. The resulting Cauchy problem takes the form:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \;=\; 0 \\ \rho(\cdot, 0) \;=\; \rho^0 \\ v(\cdot, 0) \;=\; v^0 \,. \end{cases} \tag{1.1}$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas under consideration and it is assumed to satisfy $p' > 0$ (this hypothesis guarantees also the hyperbolicity of the system on the regions where ρ is positive). A common choice is the polytropic pressure law $p(\rho) = \kappa\rho^\gamma$ with constants $\kappa > 0$ and $\gamma > 1$. The classical kinetic theory of gases predicts exponents $\gamma = 1 + \frac{2}{d}$, where d is the degree of freedom of the molecule of the gas. Here we will be concerned mostly with the particular choice $p(\rho) = \rho^2$. However several of our technical statements hold under the general assumption $p' > 0$ and the specific choice $p(\rho) = \rho^2$ is relevant only to some portions of our proofs.

It is well-known that, even starting from extremely regular initial data, the system (1.1) develops singularities in finite time. In the mathematical literature a lot of effort has been devoted to understanding how solutions can be continued after the appearance of the first singularity, leading to a quite mature theory in one space dimension (we refer the reader to the monographs [1],[7] and [18]). In this paper we show that, in more than one space dimension, the most popular concept of an admissible solution fails to yield uniqueness even under very strong assumptions on the initial data. In particular we consider bounded weak solutions of (1.1), satisfying (1.1) in the usual distributional sense (we refer to Definition

3.1 for the precise formulation), and we call them admissible if they satisfy the following additional inequality in the sense of distibutions (usually called *entropy inequality*, although for the specific system (1.1) it is rather a weak form of energy balance):

$$\partial_t \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2}\right) + \mathrm{div}_x \left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho)\right) v\right] \le 0 \qquad (1.2)$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$. Indeed, admissible solutions are required to satisfy a slightly stronger condition, i.e. a form of (1.2) which involves also the initial data, see Definition 3.2. For all solutions considered in this paper, ρ will always be bounded away from 0, i.e. $\rho \ge c_0$ for some positive constant c_0.

We denote the space variable as $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \qquad (1.3)$$

where $\rho_\pm, v_\pm$ are constants. It is well-known that for some special choices of these constants there are solutions of (1.1) which are *rarefaction waves*, i.e. self-similar solutions depending only on t and x_2 which are locally Lipschitz for positive t and constant on lines emanating from the origin (see [7, Section 7.6] for the precise definition). Reversing their order (i.e. exchanging $+$ and $-$) the very same constants allow for a *compression wave* solution, i.e. a solution on $\mathbb{R}^2\times]-\infty, 0[$ which is locally Lipschitz and converges, for $t \uparrow 0$, to the jump discontinuity of (1.3). When this is the case we will then say that the data (1.3) are *generated by a classical compression wave*.

We are now ready to state the main theorem of this paper

Theorem 1.1. *Assume $p(\rho) = \rho^2$. Then there are data as in* (1.3) *for which there are infinitely many bounded admissible solutions (ρ, v) of* (1.1) *on $\mathbb{R}^2 \times [0, \infty[$ with $\inf \rho > 0$. Moreover, these data are generated by classical compression waves.*

It follows from the usual treatment of the 1-dimensional Riemann problem that for the data of Theorem 1.1 uniqueness holds if the admissible solutions are also required to be self-similar, i.e. of the form $(\rho, v)(x, t) = \left(r\left(\frac{x_2}{t}\right), w\left(\frac{x_2}{t}\right)\right)$ and to have locally bounded variation (see Proposition 8.1). Note that such solutions must be discontinuous, because the data of Theorem 1.1 are generated by compression waves. We in fact conjecture that this is the case for *any* initial data (1.3) allowing the nonuniqueness property of Theorem 1.1: however this fact does not seem to follow from the usual weak-strong uniqueness (as for instance in [7, Theorem 5.3.1]) because the Lipschitz constant of the classical solution blows up as $t \downarrow 0$. Related results in one space dimension are contained in the work of DiPerna [15] and in the works of Chen and Frid [3], [4].

As an obvious corollary of Theorem 1.1 we arrive at the following statement.

Corollary 1.2. *There are Lipschitz initial data (ρ^0, v^0) for which there are infinitely many bounded admissible solutions (ρ, v) of* (1.1) *on $\mathbb{R}^2 \times [0, \infty[$ with $\inf \rho > 0$. These solutions are all locally Lipschitz on a finite interval on which they all coincide with the unique classical solution.*

We note in passing that, although the last statement of the corollary can be directly proved following the details of our construction, it is also a consequence of the admissibility condition, the Lipschitz regularity of the compression wave (before the singular time is reached) and the well-known weak-strong uniqueness of [7, Theorem 5.3.1].

1.1. h-principle and the Euler equations.

The proof of Theorem 1.1 relies heavily on the works of the second author and László Székelyhidi, who in the paper [10] introduced methods from the theory of differential inclusions to explain the existence of compactly supported nontrivial weak solutions of the *incompressible* Euler equations (discovered in the pioneering work of Scheffer [19]; see also [20]). It was already observed by the same pair of authors that these methods could be applied to the compressible Euler equations and lead to the ill-posedness of bounded admissible solutions, see [11]. However, the data of [11] were extremely irregular and raised the question whether the ill-posedness was due to the irregularity of the data, rather than to the irregularity of the solution.

A preliminary answer was provided in the work [5] where the first author showed that data with very regular densities but irregular velocities still allow for nonuniqueness of admissible solutions. The present paper gives a complete answer, since we show that even for some smooth initial data nonuniqueness of bounded admissible solutions arises after the first blow-up time. It remains however an open question how irregular such solutions have to be in order to display the pathological behaviour of Theorem 1.1. One could speculate that, in analogy to what has been shown recently for the incompressible Euler equations, even a "piecewise Hölder regularity" might not be enough; see [13], [14], [16], [2] and in particular [8].

This paper draws also heavily from the work [22] where Székelyhidi coupled the methods introduced in [10]-[11] with a clever construction to produce rather surprising irregular solutions of the incompressible Euler equations with vortex-sheet initial data. This work of Székelyhidi was in turn motivated by the so-called Muskat problem (see [6], [23] and [21]; we moreover refer to [12] for a rather detailed survey). Indeed the basic idea of looking for piecewise constant subsolutions as defined in Section 3 stems out of several conversations with Székelyhidi and have been inspired by a remark of Shnirelman upon the proof of [22].

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2. Ideas of the proof and plan of the paper

2.1. Subsolutions.

Especially relevant for us is the appropriate notion of *subsolution*, which allows to use the methods of [10]-[11] to solve the equations *and* impose a certain specific initial data. We give here a brief description of the concept of subsolution relevant to us and refer to [12] for the motivation behind it and its links to existing literature in physics and mathematics.

Consider first some data as in (1.3). We then partition the upper half space $\{t > 0\}$ in regions contained between half-planes meeting all at the line $\{t = x_2 = 0\}$, see Definition 3.3 and cf. Figure 1. We then define the density function $\rho = \overline{\rho}$ to be constant in each region: this density function will indeed give the final ρ for all the solutions we construct and it is therefore required to take the constant values $\rho_\pm$ in the outermost regions $P_\pm$.

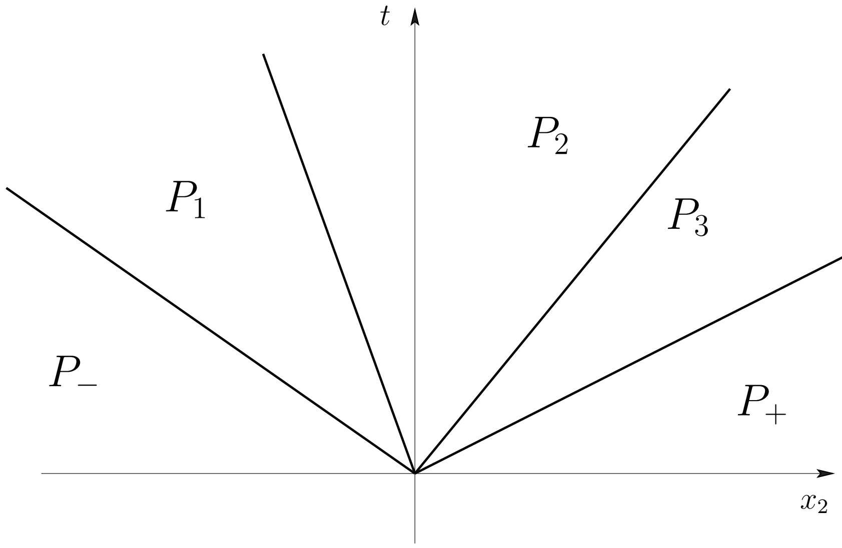

FIGURE 1. A "fan partition" in five regions.

We then solve the compressible Euler equations (1.1) in each region $P_1, \ldots, P_N$ using the methods of [11]. Indeed observe that in each such region the density is constant and thus it suffices to to construct solutions of the *incompressible* Euler equations with *constant pressure*. Employing the methods of [11] we can also impose that the modulus of the velocity is constant (in each region): its square will be denoted by C_i. In [11] such solutions are constructed adding oscillations to an appropriate *subsolution*, which consists of a pair $\overline{v}, \overline{u}$ of smooth functions, the first taking values in $\mathbb{R}^2$ and the second taking values in the space of symmetric, trace-free 2×2 matrices. These functions satisfy the linear system of PDEs

$$\begin{cases} \partial_t \overline{v} + \mathrm{div}_x \overline{u} = 0 \\ \\ \mathrm{div}_x \overline{v} = 0\,. \end{cases}$$

and a suitable relaxation of the nonlinear constraints $u = v \otimes v - \frac{|v|^2}{2}\mathrm{Id}$.

In our particular case we will choose our subsolutions to be *constant* on each region P_i: the corresponding values will be denoted by (ρ_i, v_i, u_i) and the corresponding globally defined (piecewise constant) functions $(\overline{\rho}, \overline{v}, \overline{u})$ will be called *fan subsolutions* of the compressible Euler equations. We then wish to choose our subsolution so that, after solving (1.1) in each region P_i with the methods of [11], the resulting globally defined (ρ, v)

are admissible *global* solutions of (1.1). This leads to a suitable system of PDEs for the piecewise constant functions $(\overline{\rho}, \overline{v}, \overline{u})$ which are summarized in the Definitions 3.4 and 3.5. In Section 3 we then briefly recall the notions of the papers [10]-[11] and in Section 4 we describe how to suitably modify the arguments there to reduce the proof of Theorem 1.1 to the existence of the "fan subsolutions" of Definitions 3.4 and 3.5: the precise statement of this reduction is given in Proposition 3.6.

2.2. **The algebraic system.** In Section 5, by making some specific choices, the existence of such subsolution is reduced to finding an array of real numbers satisfying some algebraic identities and inequalities, see Proposition 5.1. Indeed, since the functions $(\overline{\rho}, \overline{v}, \overline{u})$ assume constant values in each region of the fan decomposition, these conditions are nothing but suitable "Rankine-Hugoniot type" identitites and inequalities. Although at this stage all computations can be carried in general, we restrict our attention to a fan decomposition which consists of only three regions. Therefore, the resulting solutions provided by Proposition 3.6 (and therefore also those of Theorem 1.1) will take the constant values $(\rho_\pm, v_\pm)$ outside a "wedge" of the form $P_1 = \{\nu_- t < x_2 < \nu_+ t\}$: inside this wedge the solutions will instead behave in a very chaotic way.

Thus far, all the statements can be carried out for a general pressure law p. In the case $p(\rho) = \rho^2$ we also compute explicitely the well-known conditions that must be imposed on the velocities $v_\pm$ and $\rho_\pm$ so that the corresponding data (1.3) are generated by a compression wave: this gives then an additional constraint. Observe that for such data the "classical solution" will be a simple shock wave traveling at a certain speed, whereas the nonstandard solutions of Theorem 1.1 "open up" the singularity and fill the corresponding region P_1 with many oscillations.

Coming back to the algebraic constrains of Proposition 5.1, although there seems to be a certain abundance of solutions to this set of identities and inequalities, currently we do not have an efficient and general method for finding them. We propose two possible ways in the Sections 6 and 7. That of Section 6 is the most effective and produces the initial data of Theorem 1.1 which are generated by a compression wave. That of Section 7 is an alternative strategy, where, instead of making a precise choice of the pressure law p, we exploit it as an extra degree of freedom: as a result this method gives data as in Theorem 1.1 but with a different pressure law, which is essentially a suitable smoothing of the step-function. We also do not know whether any of these data are generated by compression waves.

2.3. **Classical Riemann problem.** Finally in Section 8 we show that the self-similar solutions to (1.1)-(1.3) are unique: this follows from classical considerations but since we have not been able to find a precise reference, we include the argument for completeness.

3. Subsolutions

3.1. **Weak and admissible solutions of (1.1).** We recall here the usual definitions of weak and admissible solutions to (1.1).

Definition 3.1. By a *weak solution* of (1.1) on $\mathbb{R}^2\times[0,\infty[$ we mean a pair $(\rho,v)\in L^\infty(\mathbb{R}^2\times[0,\infty[)$ such that the following identities hold for every test functions $\psi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[)$, $\phi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[)$:

$$\int_0^\infty\int_{\mathbb{R}^2}\left[\rho\partial_t\psi+\rho v\cdot\nabla_x\psi\right]dxdt+\int_{\mathbb{R}^2}\rho^0(x)\psi(x,0)dx \;=\; 0 \tag{3.1}$$

$$\int_0^\infty\int_{\mathbb{R}^2}\left[\rho v\cdot\partial_t\phi+\rho v\otimes v:D_x\phi+p(\rho)\,\mathrm{div}_x\,\phi\right]+\int_{\mathbb{R}^2}\rho^0(x)v^0(x)\cdot\phi(x,0)dx \;=\; 0. \tag{3.2}$$

Definition 3.2. A bounded weak solution (ρ,v) of (1.1) is admissible if it satisfies the following inequality for every nonnegative test function $\varphi\in C_c^\infty(\mathbb{R}^2\times[0,\infty[)$:

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^2}&\left[\left(\rho\varepsilon(\rho)+\rho\frac{|v|^2}{2}\right)\partial_t\varphi+\left(\rho\varepsilon(\rho)+\rho\frac{|v|^2}{2}+p(\rho)\right)v\cdot\nabla_x\varphi\right]\\&+\int_{\mathbb{R}^2}\left(\rho^0(x)\varepsilon(\rho^0(x))+\rho^0(x)\frac{|v^0(x)|^2}{2}\right)\varphi(x,0)\,dx\;\geq\;0\,.\end{aligned}\tag{3.3}$$

3.2. **Subsolutions.** To begin with, we state more precisely the definition of subsolution in our context. Here $\mathcal{S}_0^{2\times2}$ denotes the set of symmetric traceless 2×2 matrices and Id is the identity matrix. We first introduce a notion of good partition for the upper half-space $\mathbb{R}^2\times]0,\infty[$.

Definition 3.3 (Fan partition)**.** A *fan partition* of $\mathbb{R}^2\times]0,\infty[$ consists of finitely many open sets $P_-,P_1,\dots,P_N,P_+$ of the following form

$$P_-=\{(x,t):t>0\quad\text{and}\quad x_2<\nu_-t\} \tag{3.4}$$

$$P_+=\{(x,t):t>0\quad\text{and}\quad x_2>\nu_+t\} \tag{3.5}$$

$$P_i=\{(x,t):t>0\quad\text{and}\quad \nu_{i-1}t<x_2<\nu_it\} \tag{3.6}$$

where $\nu_-=\nu_0<\nu_1<\ldots<\nu_N=\nu_+$ is an arbitrary collection of real numbers.

The next two definitions are then motivated by the discussion of Section 2.1. However at the present stage it is not completely clear why the relevant partial differential equations (and inequalities!) for the piecewise constant solutions are given by (3.8), (3.9) and (3.10): their role will become transparent in the next subsection when we prove Proposition 3.6.

Definition 3.4 (*Fan Compressible* subsolutions)**.** A *fan subsolution* to the compressible Euler equations (1.1) with initial data (1.3) is a triple $(\overline{\rho},\overline{v},\overline{u}):\mathbb{R}^2\times]0,\infty[\to(\mathbb{R}^+,\mathbb{R}^2,\mathcal{S}_0^{2\times2})$ of piecewise constant functions satisfying the following requirements.

(i) There is a fan partition $P_-,P_1,\dots,P_N,P_+$ of $\mathbb{R}^2\times]0,\infty[$ such that

$$(\overline{\rho},\overline{v},\overline{u})=\sum_{i=1}^N(\rho_i,v_i,u_i)\mathbf{1}_{P_i}+(\rho_-,v_-,u_-)\mathbf{1}_{P_-}+(\rho_+,v_+,u_+)\mathbf{1}_{P_+}$$

where ρ_i,v_i,u_i are constants with $\rho_i>0$ and $u_\pm=v_\pm\otimes v_\pm-\frac{1}{2}|v_\pm|^2\mathrm{Id}$;

(ii) For every $i \in \{1, \ldots, N\}$ there exists a positive constant C_i such that

$$v_i \otimes v_i - u_i < \frac{C_i}{2}\mathrm{Id}\,. \tag{3.7}$$

(iii) The triple $(\overline{\rho}, \overline{v}, \overline{u})$ solves the following system in the sense of distributions:

$$\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\,\overline{v}) \;=\; 0 \tag{3.8}$$

$$\partial_t(\overline{\rho}\,\overline{v}) + \mathrm{div}_x\,(\overline{\rho}\,\overline{u}) + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\left(\sum_i C_i \rho_i \mathbf{1}_{P_i} + \overline{\rho}|\overline{v}|^2 \mathbf{1}_{P_+ \cup P_-}\right)\right) = 0 \tag{3.9}$$

Definition 3.5 (Admissible fan subsolutions)**.** A fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ is said to be *admissible* if it satisfies the following inequality in the sense of distributions

$$\partial_t\left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \mathrm{div}_x\left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho})\right)\overline{v}\right] + \partial_t\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\mathbf{1}_{P_+\cup P_-}\right) + \mathrm{div}_x\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v}\mathbf{1}_{P_+\cup P_-}\right)$$
$$+ \sum_{i=1}^N \left[\partial_t\left(\rho_i\,\frac{C_i}{2}\,\mathbf{1}_{P_i}\right) + \mathrm{div}_x\left(\rho_i\,\overline{v}\,\frac{C_i}{2}\,\mathbf{1}_{P_i}\right)\right] \;\leq\; 0\,. \tag{3.10}$$

It is possible to generalize these notions in several directions, e.g. allowing partitions with more general open sets and functions v_i, u_i and ρ_i which vary (for instance continuously) in each element of the partition. It is not difficult to extend the conclusions of the next subsection to such settings. However we have chosen to keep the definitions to the minimum needed for our proof of Theorem 1.1.

3.3. **Reduction to admissible fan subsolutions.** Using the techniques introduced in [10]-[11] we then reduce Theorem 1.1 to finding an admissible fan subsolution. The precise statement is given in the following proposition.

Proposition 3.6. *Let p be any C^1 function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ of* (1.1) *with initial data* (1.3)*. Then there are infinitely many bounded admissible solutions (ρ, v) to* (1.1)-(1.3) *such that $\rho = \overline{\rho}$.*

The core of the proof is in fact a corresponding statement for subsolutions of the *incompressible Euler equations* which is essentially contained in the proofs of [10]-[11]. However, since our assumptions and conlusions are slightly different, we state them in the next lemma.

Lemma 3.7. *Let $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ and $C > 0$ be such that $\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C}{2}\mathrm{Id}$. For any open set $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ there are infinitely many maps $(\underline{v}, \underline{u}) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ with the following property*

(i) *$\underline{v}$ and $\underline{u}$ vanish identically outside Ω;*
(ii) *$\mathrm{div}_x\,\underline{v} = 0$ and $\partial_t \underline{v} + \mathrm{div}_x\,\underline{u} = 0$;*
(iii) *$(\tilde{v} + \underline{v}) \otimes (\tilde{v} + \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C}{2}\mathrm{Id}$ a.e. on Ω.*

The proof is a minor variant of the ones given in [10]-[11] but since none of the statements present in the literature matches exactly the one of Lemma 3.7 we give some of the details in the next Section, referring to precise lemmas in the papers [10]-[11]. For the moment we show how Proposition 3.6 derives from Lemma 3.7.

Proof of Proposition 3.6. We apply Lemma 3.7 in each region $\Omega = P_i$ and we call $(\underline{v}_i, \underline{u}_i)$ any pair of maps given by such Lemma. Hence we set

$$v := \overline{v} + \sum_{i=1}^{N} \underline{v}_i \tag{3.11}$$

$$u := \overline{u} + \sum_{i=1}^{N} \underline{u}_i \tag{3.12}$$

whereas $\rho = \overline{\rho}$ (as claimed in the statement of the Proposition!). We next show that the pair (ρ, v) is an admissible weak solution of (1.1)-(1.3). First observe that $\mathrm{div}_x(\rho_i \underline{v}_i) = 0$ since ρ_i is a constant. But since $\underline{v}_i$ is supported in P_i and $\rho = \overline{\rho} \equiv \rho_i$ on P_i, we then conclude $\mathrm{div}_x(\overline{\rho}\underline{v}_i) = 0$. Thus we have

$$\begin{aligned}\partial_t \rho + \mathrm{div}_x(\rho v) &= \partial_t \overline{\rho} + \mathrm{div}_x\left(\overline{\rho v} + \sum_i \overline{\rho}\underline{v}_i\right)\\ &= \partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho v}) + \sum_i \mathrm{div}_x(\overline{\rho}\underline{v}_i) = \partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho v}) = 0\end{aligned} \tag{3.13}$$

in the sense of distributions. Moreover, observe that

$$v \otimes v = \begin{cases} v_+ \otimes v_+ & \text{on } P_+\\ v_- \otimes v_- & \text{on } P_-\\ (v_i + \underline{v}_i) \otimes (v_i + \underline{v}_i) = u_i + \underline{u}_i + \frac{C_i}{2}\mathrm{Id} & \text{on } P_i \end{cases}$$

and

$$\overline{u} = \begin{cases} v_+ \otimes v_+ - \frac{1}{2}|v_+|^2 \mathrm{Id} & \text{on } P_+\\ v_- \otimes v_- - \frac{1}{2}|v_-|^2 \mathrm{Id} & \text{on } P_-\\ u_i & \text{on } P_i\,. \end{cases}$$

Moreover, on each region P_i we have

$$\rho v \otimes v = \rho\left(v \otimes v - \frac{|v|^2}{2}\mathrm{Id}\right) + \frac{\rho C_i}{2}\mathrm{Id} = \overline{\rho u} + \rho_i \underline{u}_i + \frac{C_i \rho_i}{2}\mathrm{Id}\,.$$

Hence, we can write

$$
\begin{aligned}
\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] &= \partial_t \left(\overline{\rho v} + \sum_i \rho_i \underline{v}_i\right) + \operatorname{div}_x \left(\overline{\rho u} + \sum_i \rho_i \underline{u}_i\right) \\
&\quad + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\sum_i C_i \rho_i \mathbf{1}_{P_i} + \frac{1}{2}|v_-|^2 \rho_- \mathbf{1}_{P_-} + \frac{1}{2}|v_+|^2 \rho_+ \mathbf{1}_{P_+}\right) \\
=& \partial_t \left(\overline{\rho v}\right) + \operatorname{div}_x \left(\overline{\rho u}\right) + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\sum_i C_i \rho_i \mathbf{1}_{P_i} + \frac{1}{2}|v_-|^2 \rho_- \mathbf{1}_{P_-} + \frac{1}{2}|v_+|^2 \rho_+ \mathbf{1}_{P_+}\right) \\
&\quad + \sum_i \rho_i \underbrace{\partial_t \underline{v}_i + \operatorname{div}_x \underline{u}_i}_{=0} .
\end{aligned} \tag{3.14}
$$

Therefore, by Definition 3.4 we conclude $\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] = 0$.

Next, we compute

$$
\begin{aligned}
&\partial_t \left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho\right) + \operatorname{div}_x \left(\left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho + p(\rho)\right) v\right) \\
=& \partial_t \left(\overline{\rho}\varepsilon(\overline{\rho}) + \sum_i \frac{1}{2} C_i \rho_i \mathbf{1}_{P_i} + \frac{|v_-|^2}{2}\rho_- \mathbf{1}_{P_-} + \frac{|v_+|^2}{2}\rho_+ \mathbf{1}_{P_+}\right) \\
&+ \operatorname{div}_x \left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}) + \sum_i \frac{1}{2} C_i \rho_i \mathbf{1}_{P_i} + \frac{|v_-|^2}{2}\rho_- \mathbf{1}_{P_-} + \frac{|v_+|^2}{2}\rho_+ \mathbf{1}_{P_+}\right)\left(\overline{v} + \sum_i \underline{v}_i\right)\right]
\end{aligned}
$$

Using the condition (3.10) we therefore conclude

$$
\begin{aligned}
&\partial_t \left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho\right) + \operatorname{div}_x \left(\left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho + p(\rho)\right) v\right) \\
\leq& \sum_i \operatorname{div}_x \Big[\underline{v}_i \underbrace{\Big(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}) + \sum_i \frac{1}{2} C_i \rho_i \mathbf{1}_{P_i} + \frac{|v_-|^2}{2}\rho_- \mathbf{1}_{P_-} + \frac{|v_+|^2}{2}\rho_+ \mathbf{1}_{P_+}\Big)}_{=:\varrho}\Big]
\end{aligned} \tag{3.15}
$$

in the sense of distributions. Observe however that the function ϱ is constant on each P_i, on which $\underline{v}_i$ is supported. Thus

$$
\partial_t \left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho\right) + \operatorname{div}_x \left(\left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho + p(\rho)\right) v\right) \leq \sum_i \varrho \operatorname{div}_x \underline{v}_i = 0 . \tag{3.16}
$$

So far we have shown that (3.1), (3.2) and (3.3) hold whenever the corresponding test functions are supported in $\mathbb{R}^2 \times]0, \infty[$. However observe that, since as $\tau \downarrow 0$ the Lebesgue measure of $P_i \cap \{t = \tau\}$ converges to 0, the maps $\rho(\cdot, \tau)$ and $v(\cdot, \tau)$ converge to the maps ρ^0 and v^0 of (1.3) strongly in L^1_{loc}. This easily implies (3.1), (3.2) and (3.3) in their full generality. For instance, assume $\psi \in C^\infty_c(\mathbb{R}^2 \times]-\infty, \infty[)$ and consider a smooth cut-off function ϑ of time only which vanishes identically on $]-\infty, \varepsilon]$ and equals 1 on $]\delta, \infty[$, where $0 < \varepsilon < \delta$. We know therefore that (3.1) holds for the test function $\psi\vartheta$, which implies that

$$\int_0^\infty \int_{\mathbb{R}^2} \vartheta\, [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi]\, dx dt + \int_0^\delta \int_{\mathbb{R}^2} \vartheta'(t) \rho(x,t) \psi(x,t) dx dt = 0\,.$$

Fix δ and choose a sequence of ϑ converging uniformly to the function

$$\eta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq \delta \\ \frac{t}{\delta} & \text{if } 0 \leq t \leq \delta \end{cases}$$

and such that their derivatives ϑ' converge pointwise to $\frac{1}{\delta} \mathbf{1}_{]0,\delta[}$. We then conclude

$$\int_0^\infty \int_{\mathbb{R}^2} \eta\, [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi]\, dx dt + \frac{1}{\delta} \int_0^\delta \int_{\mathbb{R}^2} \rho(x,t) \psi(x,t) dx dt = 0\,.$$

Letting $\delta \downarrow 0$ we conclude (3.1).

The remaining conditions (3.2) and (3.3) are achieved with analogous arguments, which we leave to the reader. □

4. Proof of Lemma 3.7

4.1. **Functional set-up.** We define X_0 to be the space of $(\underline{v}, \underline{u}) \in C^\infty_c(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ which satisfy (ii) and the pointwise inequality $(\tilde{v}+\underline{v}) \otimes (\tilde{v}+\underline{v}) - (\tilde{u}+\underline{u}) < \frac{C}{2}\mathrm{Id}$. We then take the closure X of X_0 in the L^∞ weak* topology and recall that, since X is a bounded (weakly*) closed subset of L^∞ such topology is metrizable on X, giving a complete metric space (X, d). Observe that any element in X satisfies (i) and (ii) and we want to show that on a residual set (in the sense of Baire category) (iii) holds. We then define for any $N \in \mathbb{N} \setminus \{0\}$ the map I_N as follows: to $(\underline{v}, \underline{u})$ we associate the corresponding restrictions of these maps to $B_N(0) \times]-N, N[$. We then consider I_N as a map from (X, d) to Y, where Y is the space $L^\infty(B_N(0) \times]-N, N[, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ endowed with the *strong* L^2 topology. Arguing as in [10, Lemma 4.5] it is easily seen that I_N is a Baire-1 map and hence, from a classical theorem in Baire category, its points of continuity are a residual set in X. We claim that

(Con) if $(\underline{v}, \underline{u})$ is a point of continuity of I_N, then (iii) holds a.e. on $B_N(0) \times]-N, N[$.

(Con) implies then (iii) for those maps at which *all* I_N are continuous (which is also a residual set).

The proof of (Con) is achieved as in [10, Lemma 4.6] showing that:

(Cl) If $(\underline{v}, \underline{u}) \in X_0$, then there is a sequence $(v_k, u_k) \subset X_0$ converging weakly* to $(\underline{v}, \underline{u})$ for which

$$\liminf_k \|\tilde{v} + v_k\|_{L^2(\Gamma)} \geq \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} + \beta \left(C|\Gamma| - \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} \right)^2 ,$$

where $\Gamma = B_N(0) \times]-N, N[$ and β depends only on Γ.

Indeed assuming that (Cl) holds, fix then a point $(\underline{v}, \underline{u}) \in X$ where I_N is continuous and assume by contradiction that (iii) does not hold on Γ. By definition of X there is a sequence $(\underline{v}_k, \underline{u}_k) \subset X_0$ converging weakly* to $(\underline{v}, \underline{u})$. Since the latter is a point of continuity for I_N, we then have that $\underline{v}_k \to \underline{v}$ strongly in $L^2(\Gamma)$. We apply (Cl) to each $(\underline{v}_k, \underline{u}_k)$ and find a sequence $\{(v_{k,j}, u_{k,j})\}$ such that

$$\liminf_j \|\tilde{v} + v_{k,j}\|_{L^2(\Gamma)} \geq \|\tilde{v} + \underline{v}_k\|^2_{L^2(\Gamma)} + \beta \left(C|\Gamma| - \|\tilde{v} + \underline{v}_k\|^2_{L^2(\Gamma)} \right)^2$$

and $(v_{k,j}, u_{k,j}) \rightharpoonup^\star (\underline{v}_k, \underline{u}_k)$. A standard diagonal argument then allows to conclude the existence of a sequence $(v_{k,j(k)}, u_{k,j(k)})$ which converges weakly* to $(\underline{v}, \underline{u})$ and such that

$$\liminf_k \|\tilde{v} + v_{k,j(k)}\|_{L^2(\Gamma)} \geq \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} + \beta \left(C|\Gamma| - \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} \right)^2 > \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} .$$

However this contradicts the assumption that $(\underline{v}, \underline{u})$ is a point of continuity for I_N.

In order to construct the sequence of (Cl) we appeal to the following Proposition and Lemma.

Proposition 4.1 (Localized plane waves)**.** *Consider a segment $\sigma = [-p, p] \subset \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$, where $p = \lambda[(a, a \otimes a) - (b, b \otimes b)]$ for some $\lambda > 0$ and $a \neq \pm b$ with $|a| = |b| = \sqrt{C}$. Then there exists a pair $(v, u) \in C_c^\infty(B_1(0)\times] - 1, 1[)$ which solves*

$$\begin{cases} \partial_t v + \mathrm{div}_x u = 0 \\ \\ \mathrm{div}_x v = 0 \end{cases} \tag{4.1}$$

and such that

(i) *The image of (v, u) is contained in an ε-neighborhood of σ and $\int (v, u)\, dx\, dt = 0$;*
(ii) *$\int |v(x, t)|\, dx\, dt \geq \alpha\lambda|b - a|$, where α is a positive constant depending only on C.*

In order to the state the next lemma, it is convenient to introduce the following notation.

Definition 4.2. Let $C > 0$ be the positive constant of Lemma 3.7. We let $\mathcal{U}$ be the subset of $\mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ consisting of those pairs (a, A) such that $a \otimes a - A < \frac{C}{2}\mathrm{Id}$.

Lemma 4.3 (Geometric lemma)**.** *There exists a geometric constant c_0 with the following property. Assume $(a, A) \in \mathcal{U}$. Then there is a segment σ as in Proposition 4.1 with $(a, A) + \sigma \subset \mathcal{U}$ and $\lambda|b - a| \geq c_0(C - |a|^2)$.*

We are now ready to prove (Cl). Let $(\underline{v}, \underline{u}) \in X_0$. Consider any point $(x_0, t_0) \in \Gamma$ and observe that $(\tilde{v}, \tilde{u}) + (\underline{v}, \underline{u})$ takes values in $\mathcal{U}$. Let therefore σ be as in Lemma 4.3 when $(a, A) = (\tilde{v}, \tilde{u}) + (\underline{v}(x_0, t_0), \underline{u}(x_0, t_0))$ and choose $r > 0$ so that $(\tilde{v}, \tilde{u}) + (\underline{v}(x, t), \underline{u}(x, t)) + \sigma \subset \mathcal{U}$ for any $(x, t) \in B_r(x_0)\times]t_0 - r, t_0 + r[$. For any $\varepsilon > 0$ consider a pair (v, u) as in Proposition 4.1 and define $(v_{x_0,t_0,r}, u_{x_0,t_0,r})(x, t) := (v, u)\left(\frac{x - x_0}{r}, \frac{t - t_0}{r}\right)$. Observe that $(\underline{v}, \underline{u}) + (v_{x_0,t_0,r}, u_{x_0,t_0,r}) \in X_0$ provided ε is sufficiently small, and moreover

$$\int |v_{x_0,t_0,r}| \geq c_0\alpha\lambda(C - |\tilde{v} + \underline{v}(x_0, t_0)|^2)r^3 . \tag{4.2}$$

By continuity there exists r_0 such that the conclusion above holds for every $r < r_0$ and every (x, t) with $B_r(x)\times]t - r, t + r[\subset \Gamma$. Fix now $k \in \mathbb{N}$ with $\frac{1}{k} < r_0$. Set $r := \frac{1}{k}$ and find a finite number of points (x_j, t_j) such that the sets $B_r(x_j)\times]t_j - r, t_j + r[$ are pairwise disjoint, contained in Γ and satisfy

$$\sum_j \left(C - |\tilde{v} + \underline{v}(x_j, t_j)|^2\right) r^3 \geq \bar{c}\left(C|\Gamma| - \int_\Gamma |\tilde{v} + \underline{v}(x,t)|^2\, dx\, dt\right), \tag{4.3}$$

where $\bar{c}$ is a suitable geometric constant. We then define

$$(v_k, u_k) := (\underline{v}, \underline{u}) + \sum_j (v_{x_j,t_j,r}, u_{x_j,t_j,r}) .$$

Since the supports of the $(v_{x_j,t_j,r}, u_{x_j,t_j,r})$ are pairwise disjoint, (v_k, u_k) belongs to X_0 as well. Moreover, using the property that $\int (v_{x_j,t_j,r}, u_{x_j,t_j,r}) = 0$, it is immediate to check that $(v_k, u_k) \rightharpoonup^\star (\underline{v}, \underline{u})$ in L^∞. On the other hand it also follows from (4.2) and (4.3) that

$$\|v_k - \underline{v}\|_{L^1(\Gamma)} \geq c_1\left(C|\Gamma| - \int_\Gamma |\tilde{v} + \underline{v}|^2\right)$$

where the constant c_1 is only geometric. Using the weak* convergence of (v_k, u_k) to $(\underline{v}, \underline{u})$ we can then conclude

$$\begin{aligned}
\liminf_k \|\tilde{v} + v_k\|^2_{L^2(\Gamma)} &= \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} + \liminf_k \|v_k - \underline{v}\|^2 \\
&\geq \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} + |\Gamma|\left(\liminf_k \|v_k - \underline{v}\|_{L^1}\right)^2 \\
&\geq \|\tilde{v} + \underline{v}\|^2_{L^2(\Gamma)} + c_1^2\|\Gamma\|\left(C|\Gamma| - \int_\Gamma |\tilde{v} + \underline{v}|^2\right)^2 ,
\end{aligned}$$

which concludes the proof of the claim (Cl).

4.2. **Proof of Proposition 4.1 and of Lemma 4.3.**

Proof of Proposition 4.1. Consider the 3×3 matrices

$$U_a = \begin{pmatrix} a \otimes a & a \\ a & 0 \end{pmatrix} \qquad \text{and} \qquad U_b = \begin{pmatrix} b \otimes b & b \\ b & 0 \end{pmatrix}$$

Apply [11, Proposition 4] with $n = 2$ to U_a and U_b and let $A(\partial)$ be the corresponding linear differential operator and $\eta \in \mathbb{R}^2_x \times \mathbb{R}_t$ the corresponding vector. Let φ be a cut-off function which is identically equal to 1 in $B_{1/2}(0)\times] - \frac{1}{2}, \frac{1}{2}[$, is compactly supported in $B_1(0)\times] - 1, 1[$ and takes values in $[-1, 1]$. For N very large, whose choice will be specified later, we consider the function

$$\phi(x,t) = -\lambda N^{-3} \sin(N\eta \cdot (x,t))\varphi(x,t) =: \kappa(x,t)\varphi(x,t)$$

and we let $U(x,t) := A(\partial)(\phi)$. According to [11, Proposition 4], $U : \mathbb{R}^2 \times \mathbb{R} \to \mathcal{S}^{3\times 3}$ is divergence free and trace-free and moreover $U_{33} = 0$. Note also that $\int_{B_1(0)\times]-1,1[} U(x,t) = 0$. Define

$$v(x,t) := (U_{31}(x,t), U_{32}(x,t)) \qquad u(x,t) := \begin{pmatrix} U_{11}(x,t) & U_{12}(x,t) \\ U_{21}(x,t) & U_{22}(x,t) \end{pmatrix} .$$

It then follows easily that (v,u) satisfies (4.1) and that it is supported in $B_1(0)\times]-1,1[$. Also, since $A(\partial)$ is a 3rd order homogeneous linear differential operator with constant coefficients, $\|U-\varphi A(\partial)(\kappa)\|_0 \leq C\lambda N^{-1}$, where C depends only on the cut-off function φ: in particular we can assume that $\|U-\varphi A(\partial)(\kappa)\|_0 < \varepsilon$. On the other hand [11, Proposition 4] clearly implies that

$$\varphi A(\partial)(\kappa) = \lambda(U_a - U_b)\varphi \cos(N(x,t)\cdot\eta)\,.$$

we therefore conclude that U takes values in an ε-neighborhood of the segment $[-\lambda(U_a - U_b), \lambda(U_a-U_b)]$. This obviously implies that (v,u) takes values in an ε-neighborhood of the segment σ. Finally, Let $B^3_{1/2}$ be the 3-dimensional space-time ball in $\mathbb{R}^2\times\mathbb{R}$, centered at 0 and with radius $\frac{1}{2}$. Observe that

$$\int |v(x,t)| \geq \int_{B^3_{1/2}} \lambda|a-b||\cos(N(x,t)\cdot\eta)|\,dx\,dt = \lambda|a-b|\int_{B^3_{1/2}} |\cos(Nt|\eta|)|\,dx\,dt\,.$$

Moreover,

$$\lim_{N\uparrow\infty}\int_{B^3_{1/2}} |\cos(Nt|\eta|)|\,dx\,dt = \bar\alpha$$

for some positive geometric constant $\bar\alpha$. □

Proof of Lemma 4.3. Consider the set

$$K_{\sqrt{C}} := \left\{(v,u)\in\mathbb{R}^2\times\mathcal{S}_0^{2\times2} : u = v\otimes v - \frac{C}{2}\mathrm{Id}\,, |v|^2 = C\right\}\,.$$

It then follows from [11, Lemma 3] that $\mathcal{U}$ is the interior of the convex hull of $K_{\sqrt{C}}$. The existence of the claimed segment σ is then a corollary of [11, Lemma 6], since $\lambda|b-a|$ is indeed comparable (up to a geometric constant) to the length of σ. □

5. A set of algebraic identities and inequalities

In this paper we actually look at fan subsolutions with a fan partition consisting of only three sets, namely P_-, P_1 and P_+.

We introduce therefore the real numbers $\alpha,\beta,\gamma,\delta,v_{-1},v_{-2},v_{+1},v_{+2}$ such hat

$$v_1 = (\alpha,\beta), \tag{5.1}$$
$$v_- = (v_{-1}, v_{-2}) \tag{5.2}$$
$$v_+ = (v_{+1}, v_{+2}) \tag{5.3}$$
$$u_1 = \begin{pmatrix}\gamma & \delta\\ \delta & -\gamma\end{pmatrix}\,. \tag{5.4}$$

Proposition 5.1. *Let $N=1$ and P_-,P_1,P_+ be a fan partition as in Definition 3.3. The constants $v_1,v_-,v_+,u_1,\rho_-,\rho_+,\rho_1$ as in* (5.1)-(5.4) *define an admissible fan subsolution as in Definitions 3.4-3.5 if and only if the following identities and inequalities hold:*

- *Rankine-Hugoniot conditions on the left interface:*

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1\beta \tag{5.5}$$
$$\nu_-(\rho_- v_{-1} - \rho_1\alpha) = \rho_- v_{-1}v_{-2} - \rho_1\delta \tag{5.6}$$
$$\nu_-(\rho_- v_{-2} - \rho_1\beta) = \rho_- v_{-2}^2 + \rho_1\gamma + p(\rho_-) - p(\rho_1) - \rho_1\frac{C_1}{2}\,; \tag{5.7}$$

- *Rankine-Hugoniot conditions on the right interface:*

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta - \rho_+ v_{+2} \tag{5.8}$$
$$\nu_+(\rho_1\alpha - \rho_+ v_{+1}) = \rho_1\delta - \rho_+ v_{+1}v_{+2} \tag{5.9}$$
$$\nu_+(\rho_1\beta - \rho_+ v_{+2}) = -\rho_1\gamma - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+) + \rho_1\frac{C_1}{2}\,; \tag{5.10}$$

- *Subsolution condition:*

$$\alpha^2 + \beta^2 < C_1 \tag{5.11}$$
$$\left(\frac{C_1}{2} - \alpha^2 + \gamma\right)\left(\frac{C_1}{2} - \beta^2 - \gamma\right) - (\delta - \alpha\beta)^2 > 0\,; \tag{5.12}$$

- *Admissibility condition on the left interface:*

$$\begin{aligned}&\nu_-(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1)) + \nu_-\left(\rho_-\frac{|v_-|^2}{2} - \rho_1\frac{C_1}{2}\right)\\ &\le [(\rho_-\varepsilon(\rho_-) + p(\rho_-))v_{-2} - (\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta] + \left(\rho_- v_{-2}\frac{|v_-|^2}{2} - \rho_1\beta\frac{C_1}{2}\right)\;;\end{aligned} \tag{5.13}$$

- *Admissibility condition on the right interface:*

$$\begin{aligned}&\nu_+(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+)) + \nu_+\left(\rho_1\frac{C_1}{2} - \rho_+\frac{|v_+|^2}{2}\right)\\ &\le [(\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta - (\rho_+\varepsilon(\rho_+) + p(\rho_+))v_{+2}] + \left(\rho_1\beta\frac{C_1}{2} - \rho_+ v_{+2}\frac{|v_+|^2}{2}\right)\;.\end{aligned} \tag{5.14}$$

Proof. Observe that the triple $(\overline{\rho}, \overline{v}, \overline{u})$ does not depend on the variable x_1. We will therefore consider it as a map defined on the t, x_2 plane. The various conditions and inequalities follow from straightforward computations, recalling that the maps $\overline{\rho}, \overline{v}$ and $\overline{u}$ are constant in the regions P_-, P_1 and P_+ shown in Figure 2. In particular

- The identities (5.5) and (5.8) are equivalent to the continuity equation (3.8), in particular they derive from the corresponding "Rankine-Hugoniot" type conditions at the interfaces between P^- and P_1 (the *left interface*) and P_1 and P_+ (the *right interface*), respectively.

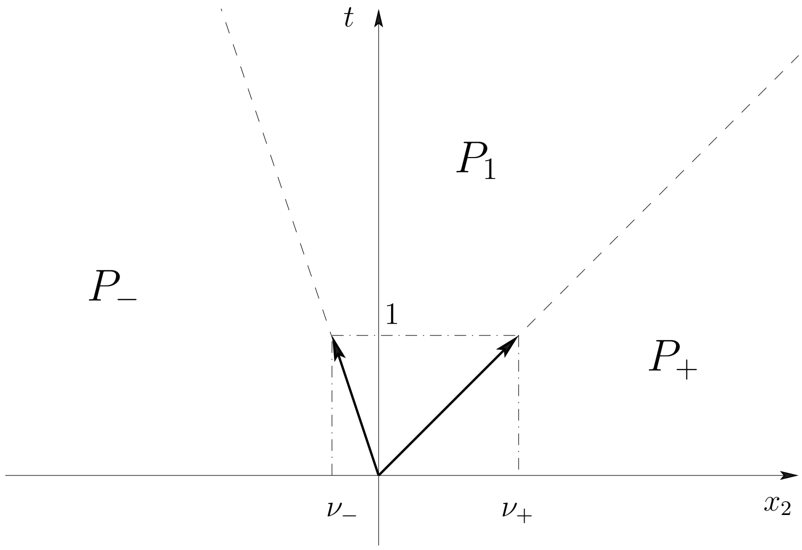

FIGURE 2. The fan partition in three regions.

- The identitities (5.6) and (5.9) are the Rankine-Hugoniot conditions at the left and right interfaces resulting from the first component of the momentum equation (3.9); similarly (5.7) and (5.10) correspond to the Rankine-Hugoniot conditions at the left and right interfaces for the second component of the momentum equation (3.9).
- The inequalities (5.11) and (5.12) are derived applying the usual criterion that the matrix
$$M := \frac{C_1}{2}\mathrm{Id} - v_1 \otimes v_1 + u_1 \tag{5.15}$$
is positive definite if and only if $\operatorname{tr} M$ and $\det M$ are both positive.
- Finally, the conditions (5.13) and (5.14) derive from the admissibility condition (3.10), again considering, respectively, the corresponding inequalities at the left and right interfaces.

□

6. FIRST METHOD: DATA GENERATED BY COMPRESSION WAVES FOR $p(\rho) = \rho^2$

In this section we show how to find solutions of the algebraic constraints in Proposition 5.1 when $p(\rho) = \rho^2$ with pairs $(\rho_\pm, v_\pm)$ which can be connected by a compression wave, thereby showing Theorem 1.1. We start by recalling the following fact, which can be easily derived using (by now) standard theory of hyperbolic conservation laws in one space dimension.

Lemma 6.1. *Let $0 < \rho_- < \rho_+$, $v_+ = (-\frac{1}{\rho_+}, 0)$ and $v_- = (-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}))$. Then there is a pair $(\rho, v) \in W^{1,\infty}_{loc} \cap L^\infty(\mathbb{R}^2 \times]-\infty, 0[, \mathbb{R}^+ \times \mathbb{R}^2)$ such that*

(i) $\rho_+ \geq \rho \geq \rho_- > 0$;

(ii) *The pair solves the hyperbolic system*

$$\left\{\begin{array}{rcl}\partial_t \rho + \mathrm{div}_x(\rho v) &=& 0\\ \partial_t(\rho v) + \mathrm{div}_x\left(\rho v\otimes v\right) + \nabla_x[p(\rho)] &=& 0\end{array}\right. \tag{6.1}$$

with $p(\rho) = \rho^2$ in the classical sense (pointwise a.e. and distributionally);

(iii) *for $t\uparrow 0$ the pair $(\rho(\cdot,t), v(\cdot,t))$ converges pointwise a.e. to (ρ^0, v^0) as in* (1.3)*;*

(iv) $(\rho(\cdot,t), v(\cdot,t))\in W^{1,\infty}$ *for every $t<0$.*

As already mentioned, the proof is a very standard application of the one-dimensional theory for the so-called Riemann problem. However, we give the details for the reader's convenience.

Proof. We look for solutions (ρ, v) with the claimed properties which are independent of the x_1 variable. Moreover we observe that, since we will produce classical $W^{1,\infty}_{loc}$ solutions, the admissibility condition (3.3) will be automatically satisfied as an equality because

$$\left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho, \left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho + p(\rho)\right) v\right)$$

is an entropy-entropy flux pair for the system (6.1) (cf. [7, Sections 3.2, 3.3.6, 4.1]). We then introduce the unknowns

$$(m_1(x_2,t), m_2(x_2,t)) = m(x_2,t) := v(x_2,t)\rho(x_2,t)$$

and hence rewrite the system as

$$\left\{\begin{array}{rcl}\partial_t\rho + \partial_{x_2} m_2 &=& 0\\ \partial_t m_1 + \partial_{x_2}\left(\frac{m_1 m_2}{\rho}\right) &=& 0\\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + \rho^2\right) &=& 0\end{array}\right. \tag{6.2}$$

Observe that if (ρ, m) is a solution of (6.2) then so is

$$(\tilde\rho(x_2,t), \tilde m(x_2,t)) := (\rho(-x_2,-t), m(-x_2,-t))\,.$$

Moreover, if (ρ, m) is locally Lipschitz and hence satisfies the admissibility condition with *equality*, so does $(\tilde\rho, \tilde m)$. We have therefore reduced ourselves to finding classical $W^{1,\infty}_{loc}$ solutions on $\mathbb{R}\times]0,\infty[$ of (6.2) with initial data

$$\rho_0(x) := \begin{cases}\rho_R & \text{if } x_2>0,\\ \rho_L & \text{if } x_2<0,\end{cases} \tag{6.3}$$

and

$$m_0(x) := \begin{cases} m_R := \left(-\frac{\rho_R}{\rho_L}, 2\sqrt{2}\rho_R(\sqrt{\rho_L}-\sqrt{\rho_R})\right) & \text{if } x_2>0,\\ m_L := (-1,0) & \text{if } x_2<0,\end{cases} \tag{6.4}$$

where $\rho_+ = \rho_L > \rho_R = \rho_- > 0$, $m_L = v_+\rho_+$ and $m_R = v_-\rho_-$.

The problem amounts now in showing that, under our assumptions, there is a classical rarefaction wave solving, forward in time, the system (6.2) with initial data (ρ_0, m_0) as in

(6.3) and (6.4). We set therefore $p(\rho) = \rho^2$ and we look for a locally Lipschitz self-similar solution (ρ, m) to the Riemann problem (6.2)-(6.3)-(6.4):

$$(\rho, m)(x_2, t) = (R, M)\left(\frac{x_2}{t}\right), \quad -\infty < x_2 < \infty, \quad 0 < t < \infty\,. \tag{6.5}$$

Thus (R, M) are locally Lipschitz functions on $(-\infty, \infty)$ which satisfy the ordinary differential equations

$$\begin{aligned}
&\frac{d}{d\xi}\left[M_2(\xi) - \xi R(\xi)\right] + R(\xi) = 0\\
&\frac{d}{d\xi}\left[\frac{M_1(\xi)M_2(\xi)}{R(\xi)} - \xi M_1(\xi)\right] + M_1(\xi) = 0\\
&\frac{d}{d\xi}\left[\frac{M_2(\xi)^2}{R(\xi)} + p(R(\xi)) - \xi M_2(\xi)\right] + M_2(\xi) = 0\,.
\end{aligned}$$

Before analyzing our specific Riemann problem, we review some general notions for system (6.2) (referring the reader to the monographs [7] and [18]). If we introduce the state vector $U := (\rho, m_1, m_2)$, we can recast the system (6.2) in the general form

$$\partial_t U + \partial_{x_2} F(U) = 0,$$

where

$$F(U) := \begin{pmatrix} m_2 \\ \frac{m_1 m_2}{\rho} \\ \frac{{m_2}^2}{\rho} + p(\rho) \end{pmatrix}.$$

By definition (cf. [7]) the system (6.2) is hyperbolic since the Jacobian matrix $DF(U)$

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-m_1 m_2}{\rho^2} & \frac{m_2}{\rho} & \frac{m_1}{\rho} \\ \frac{-{m_2}^2}{\rho^2} + p'(\rho) & 0 & \frac{2m_2}{\rho} \end{pmatrix}$$

has real eigenvalues

$$\lambda_1 = \frac{m_2}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m_2}{\rho}, \qquad \lambda_3 = \frac{m_2}{\rho} + \sqrt{p'(\rho)} \tag{6.6}$$

and 3 linearly independent eigenvectors

$$R_1 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} + \sqrt{p'(\rho)} \end{pmatrix}. \tag{6.7}$$

The eigenvalue λ_i of DF, $i = 1, 2, 3$, is called the *i-characteristic speed* of the system (6.2). On the part of the state space of our interest, with $\rho > 0$, the system (6.2) is indeed strictly hyperbolic. Finally, one can easily verify that the functions

$$w_3 = \frac{m_2}{\rho} + \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \qquad w_2 = \frac{m_1}{\rho}, \qquad w_1 = \frac{m_2}{\rho} - \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau \tag{6.8}$$

are, respectively, (1- and 2-), (1- and 3-), (2- and 3-) Riemann invariants of the system (6.2) (for the relevant definitions see [7]).

In order to characterize rarefaction waves of the reduced system (6.2), we can refer to Theorem 7.6.6 from [7]: every i- Riemann invariant is constant along any i- rarefaction wave curve of the system (6.2) and conversely the i- rarefaction wave curve, through a state $(\overline{\rho},\overline{m})$ of genuine nonlinearity of the i- characteristic family, is determined implicitly by the system of equations $w_i(\rho,m) = w_i(\overline{\rho},\overline{m})$ for every i- Riemann invariant w_i. As an application of this theorem, we obtain that (ρ_R, m_R) lies on the 1- rarefaction wave through (ρ_L, m_L). Indeed, the 1- rarefaction wave of the system (6.2) through the point (ρ_L, m_L) is determined in terms of the Riemann invariants w_3 and w_2 by the equations

$$m_1 = -\frac{\rho}{\rho_L}, \qquad m_2 = \rho \int_\rho^{\rho_L} \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \tag{6.9}$$

with $\rho < \rho_L$. In the case of pressure law $p(\rho) = \rho^2$, the equations (6.9) read as

$$m_1 = -\frac{\rho}{\rho_L}, \qquad m_2 = 2\sqrt{2}\rho\,(\sqrt{\rho_L} - \sqrt{\rho})\,. \tag{6.10}$$

Clearly, the constant state (ρ_R, m_R), as defined by (6.3)–(6.4), satisfies the equations (6.10). Since, according to Theorem 7.6.5 in [7], there exists a unique 1-rarefaction wave through (ρ_L, m_L), we have shown the existence of our desired self-similar locally Lipschitz solution.

Observe that, by construction, $\rho_+ = \rho_L \geq \rho \geq \rho_R = \rho_- > 0$, thereby showing (i). The claim (iv) follows easily because there exists a constant $C > 0$ such that, for every positive time t, the pair (ρ, m) takes the constant value (ρ_R, m_R) for $x_2 \geq Ct$ and (ρ_L, m_L) for $x_2 \leq -Ct$. □

We next show the existence of a solution of the algebraic constraints of Proposition 5.1 such that in addition $(\rho_\pm, v_\pm)$ satisfy the identities of Lemma 6.1.

Lemma 6.2. *Let $p(\rho) = \rho^2$. There exist $\rho_\pm, v_\pm$ satisfying the assumptions of Lemma 6.1 and $\rho_1, C_1, v_1, u_1, \nu_\pm$ satisfying the algebraic identities and inequalities* (5.5)-(5.14).

Proof. Taking into account that $p(\rho) = \rho^2$ and therefore $\varepsilon(\rho) = \rho$, we substitute the identities of Lemma 6.1 into the unknowns of Proposition 5.1 and reduce (5.8)-(5.10) to

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta \tag{6.11}$$

$$\nu_+(\rho_1\alpha + 1) = \rho_1\delta \tag{6.12}$$

$$\nu_+\rho_1\beta = -\rho_1\gamma + \rho_1^2 - \rho_+^2 + \rho_1\frac{C_1}{2}\,. \tag{6.13}$$

Similarly, we reduce (5.5)-(5.7) to

$$\nu_-(\rho_- - \rho_1) = 2\sqrt{2}\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1\beta \tag{6.14}$$

$$\nu_-(-\frac{\rho_-}{\rho_+} - \rho_1\alpha) = -2\sqrt{2}\frac{\rho_-}{\rho_+}(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1\delta \tag{6.15}$$

$$\nu_-(2\sqrt{2}\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1\beta) = 8\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-})^2 + \rho_1\gamma + \rho_-^2 - \rho_1^2 - \rho_1\frac{C_1}{2}\,. \tag{6.16}$$

The identities of Lemma 6.1 do not influence the form of (5.11)-(5.12). Instead, plugging them into (5.13)-(5.14) the latter are reduced to

$$\nu_-\left(\rho_-^2-\rho_1^2+\frac{\rho_-}{2\rho_+^2}+4\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2-\frac{C_1\rho_1}{2}\right)$$
$$\leq\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})\left(4\rho_-+\frac{1}{\rho_+^2}+8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2\right)-2\rho_1^2\beta-\frac{\beta C_1\rho_1}{2} \tag{6.17}$$
$$\nu_+\left(\rho_1^2-\rho_+^2+\frac{C_1\rho_1}{2}-\frac{1}{2\rho_+}\right)\leq 2\rho_1^2\beta+\frac{C_1\rho_1\beta}{2}\,. \tag{6.18}$$

We next make the choice $\nu_+=\beta=\delta=0$ and hence (6.11), (6.12) and (6.18) are automatically satisfied. The remaining constraints above then become

$$0=-\rho_1\gamma+\rho_1^2-\rho_+^2+\rho_1\frac{C_1}{2} \tag{6.19}$$
$$\nu_-(\rho_--\rho_1)=2\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-}) \tag{6.20}$$
$$\nu_-(-\frac{\rho_-}{\rho_+}-\rho_1\alpha)=-2\sqrt{2}\frac{\rho_-}{\rho_+}(\sqrt{\rho_+}-\sqrt{\rho_-}) \tag{6.21}$$
$$\nu_-(2\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-}))=8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2+\rho_1\gamma+\rho_-^2-\rho_1^2-\rho_1\frac{C_1}{2} \tag{6.22}$$

and

$$\nu_-\left(\rho_-^2-\rho_1^2+\frac{\rho_-}{2\rho_+^2}+4\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2-\frac{C_1\rho_1}{2}\right)$$
$$\leq\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})\left(4\rho_-+\frac{1}{\rho_+^2}+8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2\right)\,. \tag{6.23}$$

Moreover, (5.11) and (5.12) become

$$\alpha^2<C_1 \tag{6.24}$$
$$0<\left(\frac{C_1}{2}-\alpha^2+\gamma\right)\left(\frac{C_1}{2}-\gamma\right)\,. \tag{6.25}$$

Summarizing we are looking for real numbers $\nu_-<0, 0<\rho_-<\rho_+, \rho_1, \alpha, \gamma$ and C_1 satisfying the set of identities and inequalities (6.19)-(6.25).

We next choose $\rho_-=1<4=\rho_+$ and simplify further (6.19)-(6.23) as

$$\frac{C_1\rho_1}{2}+\rho_1^2-\rho_1\gamma-16=0 \tag{6.26}$$
$$\nu_-(1-\rho_1)=2\sqrt{2} \tag{6.27}$$
$$\nu_-\left(\frac{1}{4}+\alpha\rho_1\right)=\frac{\sqrt{2}}{2} \tag{6.28}$$
$$9+\rho_1\gamma-\rho_1^2-\frac{C_1\rho_1}{2}=2\sqrt{2}\nu_- \tag{6.29}$$

$$\nu_- \left(5 + \frac{1}{32} - \rho_1^2 - \frac{C_1 \rho_1}{2}\right) \leq \sqrt{2}\left(12 + \frac{1}{16}\right) . \tag{6.30}$$

We now observe that (6.27) and (6.28) imply $\alpha = -\frac{1}{4}$ and (6.26)-(6.29) imply $\nu_- = -\frac{7}{2\sqrt{2}}$. Therefore, our constraints further simplify to looking for ρ_1, γ, C_1 such that

$$\frac{1}{16} < C_1 \tag{6.31}$$

$$0 < \left(\frac{C_1}{2} - \frac{1}{16} + \gamma\right)\left(\frac{C_1}{2} - \gamma\right) \tag{6.32}$$

$$0 = \frac{C_1 \rho_1}{2} + \rho_1^2 - \rho_1 \gamma - 16 \tag{6.33}$$

$$8 = -7(1 - \rho_1) \tag{6.34}$$

$$48 + \frac{1}{4} \geq -7\left(5 + \frac{1}{32} - \rho_1^2 - \frac{C_1 \rho_1}{2}\right) . \tag{6.35}$$

From (6.34) we derive $\rho_1 = \frac{15}{7}$ and inserting this into (6.33) we infer $\frac{C_1}{2} - \gamma = \frac{559}{105}$. In turn this last identity reduces (6.32) to the inequality

$$C_1 > \frac{1}{16} + \frac{559}{105} . \tag{6.36}$$

The remaining constraints (6.33) and (6.35) simplify to:

$$\frac{C_1}{2} - \gamma = \frac{559}{105} \tag{6.37}$$

$$48 + \frac{1}{4} + 35 + \frac{7}{32} - \frac{225}{7} \geq \frac{15 C_1}{2} . \tag{6.38}$$

We therefore see that γ can be obtained from C_1 through (6.37). Hence the existence of the desired solution is equivalent to the existence of a C_1 satisfying (6.36) and (6.38). Such C_1 exists if and only if

$$\frac{15}{2}\left(\frac{1}{16} + \frac{559}{105}\right) < 48 + \frac{1}{4} + 35 + \frac{7}{32} - \frac{225}{7} ,$$

which can be trivially checked to hold. □

Theorem 1.1 and Corollary 1.2 now easily follow.

Proofs of Theorem 1.1 and Corollary 1.2. Let $p(\rho) = \rho^2$ and consider the $\rho_\pm, v_\pm$ given by Lemma 6.2. Applying Propositions 5.1 and 3.6 we know that there are infinitely many admissible solutions of (1.1)-(1.3) as claimed in the Theorem.

Let now (ρ_f, v_f) be any such solution and let (ρ_b, v_b) be the locally Lipschitz solutions of (1.1) given by Lemma 6.1. It is straightforward to check that, if we define

$$(\rho, v)(x,t) := \begin{cases} (\rho_f, v_f)(x,t) & \text{if } t \geq 0 \\ (\rho_b, v_b)(x,t) & \text{if } t \leq 0 , \end{cases} \tag{6.39}$$

then the pair (ρ, v) is a bounded admissible solution of (1.1) on the entire space-time $\mathbb{R}^2 \times \mathbb{R}$ with density bounded away from 0. Moreover $(\rho(\cdot, t), v(\cdot, t))$ is a bounded Lipschitz function for every $t < 0$. In particular we can define $(\tilde{\rho}, \tilde{v})(x, t) = (\rho, v)(x, t-1)$ and observe that, no matter which of the infinitely many solutions (ρ_f, v_f) given by Theorem 1.1 we choose, the corresponding $(\tilde{\rho}, \tilde{v})$ defined above is an admissible solution as in Corollary 1.2 for the bounded and Lipschitz initial data $(\rho^0, v^0) = (\rho_b, v_b)(\cdot, -1)$. □

Remark 6.3. In fact, it is not difficult to see by a simple continuity argument that the conclusions of Theorem 1.1 and Corollary 1.2 hold even for general pressure laws $p(\rho) = \rho^\gamma$ with γ in some neighborhood of 2.

7. Second method: further Riemann data for different pressures

In this section we describe a second method for producing solutions to the algebraic set of equations and inequalities of Proposition 5.1. Unlike the method given in the previous section, we do not know whether this one produces Riemann data generated by a compression wave: we can only show that this is not the case for the ones which we have computed explicitely. Moreover we do not fix the pressure law but we exploit it as an extra degree of freedom. On the other hand the reader can easily check that the method below gives a rather large set of solutions (i.e. open) compared to the one of Lemma 6.2 (where we do not know whether one can perturb the choice $\nu_+ = 0$).

Lemma 7.1. *Set $v_\pm = (\pm 1, 0)$. Then there exist $\nu_\pm$, $\rho_\pm$, ρ_1, $\alpha, \beta, \gamma, \delta$, C_1 and a smooth pressure p with $p' > 0$ for which the algebraic identities and inequalities* (5.5)-(5.14) *are satisfied.*

7.1. Part I of the proof of Lemma 7.1: reduction of the admissibility conditions.

We rewrite the conditions (5.5)-(5.10)

$$\nu_-(\rho_1 - \rho_-) = \rho_1\beta \tag{7.1}$$

$$\nu_-(\rho_- + \rho_1\alpha) = \rho_1\delta \tag{7.2}$$

$$\rho_1\frac{C_1}{2} - \rho_1\gamma + p(\rho_1) - p(\rho_-) = \nu_-\rho_1\beta \tag{7.3}$$

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta \tag{7.4}$$

$$\nu_+(\rho_1\alpha - \rho_+) = \rho_1\delta \tag{7.5}$$

$$\rho_1\frac{C_1}{2} - \rho_1\gamma + p(\rho_1) - p(\rho_+) = \nu_+\rho_1\beta\,. \tag{7.6}$$

The conditions (5.11) and (5.12) are not affected by our choice. The conditions (5.13) and (5.14) become

$$\nu_-\left(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1) + \frac{\rho_-}{2} - \rho_1\frac{C_1}{2}\right) + \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1}{2}\right) \le 0 \tag{7.7}$$

$$\nu_+\left(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+) + \rho_1\frac{C_1}{2} - \frac{\rho_+}{2}\right) - \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1}{2}\right) \le 0\,. \tag{7.8}$$

Plugging (7.1) and (7.4) into, respectively, (7.7) and (7.8) we obtain

$$\nu_-\left(\rho_-\varepsilon(\rho_-)-\rho_1\varepsilon(\rho_1)-\rho_1\frac{C_1-1}{2}\right)+\beta\left(\rho_1\varepsilon(\rho_1)+p(\rho_1)+\rho_1\frac{C_1-1}{2}\right)\le 0 \tag{7.9}$$

$$\nu_+\left(\rho_1\varepsilon(\rho_1)-\rho_+\varepsilon(\rho_+)+\rho_1\frac{C_1-1}{2}\right)-\beta\left(\rho_1\varepsilon(\rho_1)+p(\rho_1)+\rho_1\frac{C_1-1}{2}\right)\le 0\,. \tag{7.10}$$

We next rely on the following

Lemma 7.2. *Assume that*

$$\nu_- < 0 < \nu_+\,, \tag{7.11}$$

$$\rho_- < \rho_+\,. \tag{7.12}$$

Then, there exist pressure functions $p \in C^\infty([0,+\infty[)$ *with* $p' > 0$ *on* $]0,+\infty[$ *such that the admissibility conditions* (7.9)-(7.10) *for a subsolution are implied by the following system of inequalities:*

$$(p(\rho_+)-p(\rho_1))\,(\rho_+-\rho_1) > \frac{C_1-1}{2}\rho_+\rho_1 \tag{7.13}$$

$$(p(\rho_1)-p(\rho_-))\,(\rho_1-\rho_-) > \frac{C_1-1}{2}\rho_-\rho_1. \tag{7.14}$$

Proof. First, let us define $g(\rho) := \rho\varepsilon(\rho)$. In view of the relation $p(\rho) = \rho^2\varepsilon'(\rho)$, we obtain

$$g'(\rho) = \varepsilon(\rho) + \frac{p(\rho)}{\rho}.$$

Thus, by vitue of (7.1) and (7.4), respectively, we can rewrite (7.9) and (7.10) as follows:

$$\nu_-(g(\rho_-)-g(\rho_1))+\nu_-(\rho_1-\rho_-)g'(\rho_1)-\nu_-\rho_-\frac{C_1-1}{2}\le 0 \tag{7.15}$$

$$\nu_+(g(\rho_1)-g(\rho_+))+\nu_+(\rho_+-\rho_1)g'(\rho_1)+\nu_+\rho_+\frac{C_1-1}{2}\le 0\,. \tag{7.16}$$

From the hypothesis (7.11) we can further reduce (7.15)-(7.16) to

$$-(g(\rho_1)-g(\rho_-))+(\rho_1-\rho_-)g'(\rho_1)\ge\frac{C_1-1}{2}\rho_- \tag{7.17}$$

$$(g(\rho_+)-g(\rho_1))-(\rho_+-\rho_1)g'(\rho_1)\ge\frac{C_1-1}{2}\rho_+\,. \tag{7.18}$$

Moreover, we observe from (7.1)-(7.4) that

$$\nu_+(\rho_+-\rho_1) = -\nu_-(\rho_1-\rho_-).$$

Hence, in view of (7.11)-(7.12), we must have

$$\rho_- < \rho_1 < \rho_+\,. \tag{7.19}$$

Let us note that

$$(g(\sigma)-g(s))-(\sigma-s)g'(s) = \int_s^\sigma\int_s^\tau g''(r)\,dr\,d\tau$$

for every $s < \sigma$. On the other hand, by simple algebra, we can compute $g''(r) = p'(r)/r$. Hence, the following equalities hold for every $s < \sigma$:

$$(g(\sigma) - g(s)) - (\sigma - s)g'(s) = \int_s^\sigma \int_s^\tau \frac{p'(r)}{r} dr d\tau$$

and

$$(g(s) - g(\sigma)) + (\sigma - s)g'(\sigma) = \int_s^\sigma \int_\tau^\sigma \frac{p'(r)}{r} dr d\tau.$$

As a consequence, and in view of (7.19), we can rewrite (7.17) and (7.18) equivalently as

$$\int_{\rho_-}^{\rho_1} \int_\tau^{\rho_1} \frac{p'(r)}{r} dr d\tau \geq \frac{C_1 - 1}{2} \rho_- \,, \tag{7.20}$$

$$\int_{\rho_1}^{\rho_+} \int_{\rho_1}^\tau \frac{p'(r)}{r} dr d\tau \geq \frac{C_1 - 1}{2} \rho_+ \,. \tag{7.21}$$

Now, we introduce two new variables q_- and q_+ defined by

$$q_- := p(\rho_1) - p(\rho_-)\,,$$

$$q_+ := p(\rho_+) - p(\rho_1)\,.$$

Proving Lemma 7.2 is then equivalent to showing the existence of a pressure law p satisfying $p(\rho_+) - p(\rho_1) = q_+$, $p(\rho_1) - p(\rho_-) = q_-$ and for which the inequalities (7.20)-(7.21) hold.

First, introducing $f := p'$, we define the set of functions

$$\mathcal{L} := \left\{ f \in C^\infty(]0,\infty[,]0,\infty[) \,:\, \int_{\rho_-}^{\rho_1} f = q_- \text{ and } \int_{\rho_1}^{\rho_+} f = q_+ \right\}$$

and the two functionals defined on $\mathcal{L}$

$$L^+(f) := \int_{\rho_1}^{\rho_+} \int_{\rho_1}^\tau \frac{f(r)}{r} dr d\tau,$$

$$L^-(f) := \int_{\rho_-}^{\rho_1} \int_\tau^{\rho_1} \frac{f(r)}{r} dr d\tau.$$

Therefore, a sufficient condition for finding a pressure function p with the above properties is that

$$l^+ := \sup_{f \in \mathcal{L}} L^+(f) > \frac{C_1 - 1}{2} \rho_+$$

and

$$l^- := \sup_{f \in \mathcal{L}} L^-(f) > \frac{C_1 - 1}{2} \rho_-.$$

Let us generalize the space $\mathcal{L}$ as follows. We introduce

$$\mathcal{M}^+ := \{\text{positive Radon measures } \mu \text{ on } [\rho_1, \rho_+] : \mu([\rho_1, \rho_+]) = q_+\}\,,$$

$$\mathcal{M}^- := \{\text{positive Radon measures } \mu \text{ on } [\rho_-, \rho_1] : \mu([\rho_-, \rho_1]) = q_-\}\,.$$

For consistency, we extend the functionals L^+ and L^- defined on $\mathcal{L}$ to new functionals L_+ and L_- respectively defined on $\mathcal{M}^+$ and on $\mathcal{M}^-$:

$$L_+(\mu) := \int_{\rho_1}^{\rho_+} \int_{\rho_1}^{\tau} \frac{1}{r} d\mu(r) d\tau \qquad \text{for } \mu \in \mathcal{M}^+,$$

$$L_-(\mu) := \int_{\rho_-}^{\rho_1} \int_{\tau}^{\rho_1} \frac{1}{r} d\mu(r) d\tau \qquad \text{for } \mu \in \mathcal{M}^-.$$

Upon setting

$$m^+ := \max_{\mu \in \mathcal{M}^+} L_+(\mu)$$

and

$$m^- := \max_{\mu \in \mathcal{M}^-} L_-(\mu),$$

it is clear that

$$l^+ \leq m^+ \quad \text{and} \quad l^- \leq m^-.$$

Moreover, let us note that the the maxima $m^p m$ are achieved due to the compactness of $\mathcal{M}^\pm$ with respect to the weak* topology. By a simple Fubini type argument, we write

$$L_+(\mu) = \int_{\rho_1}^{\rho_+} \frac{\rho_+ - r}{r}\, d\mu(r).$$

Hence, defining the function $h \in C([\rho_1, \rho_+])$ as $h(r) := (\rho_+ - r)/r$ allows us to express the action of the linear functional L_+ as a duality pairing; more precisely we have:

$$L_+(\mu) =< h, \mu > \qquad \text{for } \mu \in \mathcal{M}^+.$$

Analogously, if we define $g \in C([\rho_-, \rho_1])$ as $g(r) := (r - \rho_-)/r$, we can express L_- as a duality pairing as well:

$$L_-(\mu) =< g, \mu > \qquad \text{for } \mu \in \mathcal{M}^-.$$

By standard functional analysis, we know that $m^\pm$ must be achieved at the extreme points of $\mathcal{M}^\pm$. The extreme points of $\mathcal{M}^\pm$ are the single-point measures, i.e. weighted Dirac masses. For $\mathcal{M}^+$ the set of extreme points is then given by $E_+ := \{q_+\delta_\sigma \text{ for } \sigma \in [\rho_1, \rho_+]\}$ while for $\mathcal{M}^-$ the set of extreme points is then given by $E_- := \{q_-\delta_\sigma \text{ for } \sigma \in [\rho_-, \rho_1]\}$ In order to find $m^\pm$, it is sufficient to find the maximum value of $L_\pm$ on $E_\pm$. Clearly, we obtain

$$m^+ = \max_{\sigma \in [\rho_1, \rho_+]} \left\{ q_+ \frac{\rho_+ - \sigma}{\sigma} \right\} = q_+ \frac{\rho_+ - \rho_1}{\rho_1}$$

and

$$m^- = \max_{\sigma \in [\rho_-, \rho_1]} \left\{ q_- \frac{\sigma - \rho_-}{\sigma} \right\} = q_- \frac{\rho_1 - \rho_-}{\rho_1}.$$

Furthermore, given the explicit form of the maximum points, it is rather easy to show that for every $\varepsilon > 0$ there exists a function $f \in \mathcal{L}$ such that

$$L^+(f) > q_+ \frac{\rho_+ - \rho_1}{\rho_1} - \varepsilon$$

and

$$L_-(f) > q_- \frac{\rho_1 - \rho_-}{\rho_1} - \varepsilon.$$

Such a function f is the derivative of the desired pressure function p. □

7.2. **Part II of the proof of Lemma 7.1.** We now choose $\rho_1 = 1$. Applying Lemma 7.2 we set $q_\pm := \pm(p(\rho_\pm) - p(\rho_1)) = \pm(p(\rho_\pm) - p(1))$ and hence reduce our problem to finding real numbers $\rho_\pm, \nu_\pm, q_\pm, \alpha, \beta, \gamma, \delta, C_1$ satisfiyng

$$\nu_- < 0 < \nu_+ , \quad 0 < \rho_- < 1 < \rho_+ , \quad q_\pm > 0 \tag{7.22}$$

$$\nu_-(1-\rho_-) = \beta \tag{7.23}$$
$$\nu_-(\rho_- + \alpha) = \delta \tag{7.24}$$
$$\frac{C_1}{2} - \gamma + q_- = \nu_- \beta \tag{7.25}$$
$$\nu_+(1-\rho_+) = \beta \tag{7.26}$$
$$\nu_+(\alpha - \rho_+) = \delta \tag{7.27}$$
$$\frac{C_1}{2} - \gamma - q_+ = \nu_+ \beta , \tag{7.28}$$

$$q_-(1-\rho_-) > \frac{C_1 - 1}{2}\rho_- \tag{7.29}$$
$$q_+(\rho_+ - 1) > \frac{C_1 - 1}{2}\rho_+ \tag{7.30}$$

and (5.11)-(5.12).

Next, using (7.22), (7.23) and (7.26) we rewrite (7.29)-(7.30) as

$$-\beta q_- > \frac{C_1 - 1}{2}(-\nu_-\rho_-) \tag{7.31}$$
$$-\beta q_+ > \frac{C_1 - 1}{2}\nu_+\rho_+ . \tag{7.32}$$

In order to simplify our computations we then introduce the new variables

$$\overline{\beta} = -\beta , \; \overline{\delta} = -\delta , \; \overline{C} = \frac{C_1}{2} , \; \nu^- = -\nu_- , \; r_+ = \rho_+\nu_+ \text{ and } r_- = \rho_-\nu^- = -\rho_-\nu_- . \tag{7.33}$$

Therefore, our conditions become

$$q_\pm, r_\pm, \nu_+, \nu^- > 0 \tag{7.34}$$

$$\nu^- - r_- = \overline{\beta} \tag{7.35}$$

$$r_+ - \nu_+ = \overline{\beta} \tag{7.36}$$

$$r_- + \alpha\nu^- = \overline{\delta} \tag{7.37}$$

$$r_+ - \alpha\nu_+ = \overline{\delta} \tag{7.38}$$

$$\overline{C} - \gamma + q_- = \nu^-\overline{\beta} \tag{7.39}$$

$$\overline{C} - \gamma - q_+ = -\nu_+\overline{\beta} \tag{7.40}$$

$$\overline{\beta}q_- > \left(\overline{C} - \frac{1}{2}\right) r_- \tag{7.41}$$

$$\overline{\beta}q_+ > \left(\overline{C} - \frac{1}{2}\right) r_+ \,. \tag{7.42}$$

Moreover (5.11)-(5.12) become

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{7.43}$$

$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) - (\overline{\delta} - \alpha\overline{\beta})^2 > 0\,. \tag{7.44}$$

We assume $\alpha^2 \neq 1$ and solve for ν_-, ν^+ and $r_\pm$ in (7.35)-(7.38) to achieve

$$\nu^- = \frac{\overline{\delta} + \overline{\beta}}{1+\alpha}\,, \quad \nu_+ = \frac{\overline{\delta} - \overline{\beta}}{1-\alpha} \quad \text{and} \quad r_\pm = \frac{\overline{\delta} - \alpha\overline{\beta}}{1 \mp \alpha}\,. \tag{7.45}$$

Observe that

$$r_+ r_- = \frac{(\overline{\delta} - \alpha\overline{\beta})^2}{1-\alpha^2}\,.$$

Hence, if we assume $\alpha^2 < 1$ and $\overline{\delta} > \overline{\beta} > 0$, we see that the $\nu^+, \nu_-, r_\pm$ as defined in the formulas (7.45) satisfy the inequalities in (7.34). Hence, inserting (7.45) we look for solutions of the set of identities and inequalities

$$\alpha^2 < 1\,,\ \overline{\delta} > \overline{\beta} > 0\,,\ q_\pm > 0 \tag{7.46}$$

$$\overline{C} - \gamma + q_- = \frac{\overline{\delta} + \overline{\beta}}{1+\alpha}\overline{\beta} \tag{7.47}$$

$$\overline{C} - \gamma - q_+ = -\frac{\overline{\delta} - \overline{\beta}}{1-\alpha}\overline{\beta} \tag{7.48}$$

$$\overline{\beta}q_- > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1+\alpha} \tag{7.49}$$

$$\overline{\beta}q_+ > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1-\alpha} \tag{7.50}$$

combined with (7.43) and (7.44). Observe that, if we assume in addition that $\bar{C} > \frac{1}{2}$, then $\alpha^2 < 1$, $\overline{\delta} > \overline{\beta} > 0$ and (7.49)-(7.50) imply the positivity of $q_\pm$. We can therefore solve the equations (7.47)-(7.48) for $q_\pm$ and insert the corresponding values in the remaining inequalities (7.49) and (7.50). Summarizing, we are looking for $\alpha, \overline{\beta}, \gamma, \overline{\delta}, \overline{C}$ satisfying the following inequalities

$$\alpha^2 < 1\,,\ \overline{\delta} > \overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{7.51}$$

$$\overline{\beta}\left[\overline{\beta}\frac{\overline{\delta}+\overline{\beta}}{1+\alpha} - \overline{C} + \gamma\right] > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1+\alpha} \tag{7.52}$$

$$\overline{\beta}\left[\overline{\beta}\frac{\overline{\delta}-\overline{\beta}}{1-\alpha} + \overline{C} - \gamma\right] > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1-\alpha} \tag{7.53}$$

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{7.54}$$

$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) - (\overline{\delta} - \alpha\overline{\beta})^2 > 0\,. \tag{7.55}$$

We next introduce the variable $\lambda = \overline{\delta} - \alpha\overline{\beta}$ and rewrite our inequalities as

$$\alpha^2 < 1\,,\ \lambda > (1-\alpha)\overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{7.56}$$

$$\overline{\beta}(1+\alpha)(\overline{\beta}^2 - \overline{C} + \gamma) > \left(\overline{C} - \overline{\beta}^2 - \frac{1}{2}\right)\lambda \tag{7.57}$$

$$\overline{\beta}(1-\alpha)(-\overline{\beta}^2 + \overline{C} - \gamma) > \left(\overline{C} - \overline{\beta}^2 - \frac{1}{2}\right)\lambda \tag{7.58}$$

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{7.59}$$

$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) > \lambda^2\,. \tag{7.60}$$

Observe that, if we require $\alpha, \overline{\beta}, \gamma$ and $\overline{C}$ to satisfy the following inequalities

$$\alpha^2 < 1\,,\ \overline{C} > \frac{1}{2} \tag{7.61}$$

$$\overline{C} - \alpha^2 + \gamma > 0 \tag{7.62}$$

$$\overline{C} - \overline{\beta}^2 - \gamma > 0 \tag{7.63}$$

$$\overline{\beta}^2 + \frac{1}{2} - \overline{C} > 0 \tag{7.64}$$

$$\sqrt{(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma)} > (1-\alpha)\overline{\beta} > 0 \tag{7.65}$$

$$\left(\overline{\beta}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - \alpha^2 + \gamma} > \overline{\beta}(1+\alpha)\sqrt{\overline{C} - \overline{\beta}^2 - \gamma} \tag{7.66}$$

then setting

$$\lambda := \sqrt{(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma)} - \eta\,,$$

the inequalities (7.56)-(7.60) are satisfied whenever η is a sufficiently small positive number.

Observe next that (7.64) is surely satisfied if the remaining inequalities are and hence we can drop it. Moreover, if $\overline{\beta}, \gamma$ and $\overline{C}$ satisfy

$$\overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{7.67}$$

$$\overline{C} - \overline{\beta}^2 - \gamma > 0 \tag{7.68}$$

$$\overline{C} - 1 + \gamma > 0 \tag{7.69}$$

$$\left(\overline{\beta}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - 1 + \gamma} > 2\overline{\beta}\sqrt{\overline{C} - \overline{\beta}^2 - \gamma} \tag{7.70}$$

then setting $\alpha = 1 - \vartheta$, the inequalities (7.61)-(7.66) hold provided $\vartheta > 0$ is chosen small enough.

Finally, choosing $\overline{C} = \frac{4}{5}\overline{\beta}^2$, $\gamma = -\frac{2}{5}\overline{\beta}^2$ and imposing $\overline{\beta} > \sqrt{\frac{5}{2}}$ we see that (7.67), (7.68) and (7.69) are automatically satisfied. Whereas (7.70) is equivalent to

$$\left(\frac{\overline{\beta}^2}{5} + \frac{1}{2}\right)\sqrt{\frac{2\overline{\beta}^2}{5} - 1} > \frac{2\overline{\beta}^2}{\sqrt{5}}\,.$$

However the latter inequality is surely satisfied for $\overline{\beta}$ large enough.

8. Classical solutions of the Riemann problem

We show here that, if we restrict our attention to BV selfsimilar solutions of (1.1)-(1.3) which do not depend on the variable x_1, then for the initial data of Theorem 1.1 and Corollary 1.2 the solutions of the Cauchy problem are unique. We mostly exploit classical results about the 1-dimensional Riemann problem for hyperbolic system of conservation laws. We however complement them with some recent results in the theory of transport equations: the resulting argument is then shorter and moreover yields uniqueness under milder assumptions (see Remark 8.2 below).

Proposition 8.1. *Consider $p(\rho) = \rho^2$ and any initial data of type* (1.3) *as in Lemma 6.1 (which therefore include the data of the proof of Theorem 1.1 and Corollary 1.2). Then there exists a unique admissible self-similar bounded BV_{loc} solution (i.e. of the form $(\rho, v)(x,t) = (r,w)(\frac{x_2}{t})$) of* (1.1) *with ρ bounded away from* 0.

Remark 8.2. In fact our proof of Proposition 8.1 has a stronger outcome. In particular the same uniqueness conclusion holds under the following more general assumptions:

- p satisfies the usual "hyperbolicity assumption" $p' > 0$ and the "genuinely nonlinearity condition" $2p'(r) + rp''(r) > 0\ \forall r > 0$;
- (ρ, v) is a bounded admissible solution with density bounded away from zero, whereas the BV regularity and the self-similarity hypotheses are assumed *only* for ρ and the second component of the velocity v.

Remark 8.3. The arguments given below can be adapted to show the same uniqueness statement for the Cauchy problem corresponding to the data generated by Lemma 7.1.

This would only require some lengthier ad hoc analysis of the classical Riemann problem for the system (8.3), with ρ^2 replaced by the pressures of Lemma 7.1.

Proof. Observe that the initial data for the first component v_1 is the constant $-\frac{1}{\rho_+}$. On the other hand:

- ρ is a bounded function of locally bounded variation;
- The vector field $\bar{v} = (0, v_2)$ is bounded, has locally bounded variation and solves the continuity equation

$$\partial_t \rho + \mathrm{div}_x(\rho \bar{v}) = 0\,; \tag{8.1}$$

- v_1 is an L^∞ weak solution of the transport equation

$$\begin{cases} \partial_t(\rho v_1) + \mathrm{div}(\rho \bar{v} v_1) = 0 \\ \\ v_1(0,\cdot) = -\frac{1}{\rho_+}\,. \end{cases} \tag{8.2}$$

Therefore, the vector field $\bar{v}$ is nearly incompressible in the sense of [9, Definition 3.6]. By the BV regularity of ρ and $\bar{v}$ we can apply Ambrosio's renormalization theorem [9, Theorem 4.1] and hence use [9, Lemma 5.10] to infer from (8.1) that the pair $(\rho, \bar{v})$ has the renormalization property of [9, Definition 3.9]. Thus we can apply [9, Corollary 3.14] to infer that there is a unique bounded weak solution of (8.2). Since the constant function is a solution, we therefore conclude that v_1 is identically equal to $-\frac{1}{\rho_+}$.

Set now $m(x_2, t) := \rho(x_2, t) v_2(x_2, t)$. The pair ρ, m is then a self-similar BV_{loc} weak solution of the 2×2 one-dimensional system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_{x_2} m = 0 \\ \\ \partial_t m + \partial_{x_2}\left(\frac{m^2}{\rho} + \rho^2\right) = 0\,, \end{cases} \tag{8.3}$$

that is the standard system of isentropic Euler in Eulerian coordinates with a particular polytropic pressure. It is well-known that such system is genuinely nonlinear in the sense of [7, Definition 7.5.1] and therefore, following the discusssion of [7, Section 9.1] we conclude that the functions (ρ, m) result from "patching" rarefaction waves and shocks connecting constant states, i.e. they are classical solutions of the so-called Riemann problem in the sense of [7, Section 9.3]. It is well-known that in the special case of (8.3) the latter property and the admissibility condition determine uniquely the functions (ρ, m). For instance, one can apply [17, Theorem 3.2]. □

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