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Mathematical Institute of the Charles University
Sokolovská 83, 186 75 Praha 8
http://ncmm.karlin.mff.cuni.cz/

Quasistatic adhesive contact delaminating in mixed mode and its numerical treatment

Martin Kružík
Institute of Information Theory and Automation of the ASCR, Pod vodárenskou věží 4, CZ–182 08 Praha 8, Czech Republic and *Faculty of Civil Engineering, Czech Technical University, Thákurova 7, CZ–166 29 Praha 6, Czech Republic*

Christos G. Panagiotopoulos
Group of Elasticity and Strength of Materials, Dept. of Continuum Mech., School of Engineering, University of Seville, Camino de los Descubrimientos s/n, E-41092 Sevilla, Spain

Tomáš Roubíček
Mathematical Institute, Charles University, Sokolovská 83, CZ–186 75 Praha 8, and *Institute of Thermomechanics of the ASCR, Dolejškova 5, CZ–182 08 Praha 8, Czech Republic*

Abstract. An adhesive unilateral contact between visco-elastic bodies at small strains and in a Kelvin-Voigt rheology is scrutinized, neglecting inertia. The flow-rule for debonding the adhesive is considered rate independent, unidirectional, and non-associative due to dependence on the mixity of modes of delamination, namely Mode I (opening) needs (=dissipates) less energy than Mode II (shearing). Such mode-mixity dependence of delamination is a very pronounced (and experimentally confirmed) phenomenon typically considered in engineering models. An efficient semi-implicit-in-time FEM discretization leading to recursive quadratic mathematical programs is devised. Its convergence and thus the existence of weak solutions is proved. Computational experiments implemented by BEM illustrate the modeling aspects and the numerical efficiency of the discretization.

Key Words. Adhesive contact, rate-independence, non-associative model, weak solution, semi-implicit discretization, finite-elements, convergence, quadratic mathematical programming.

AMS Subject Classification: 35K85, 65M60, 65Z05, 74M15, 74R20.

1 Introduction

In this article, we focus on adhesive contacts, which is a part of *nonlinear contact mechanics* with numerous practical applications. In particular, we focus on the modeling, analysis, and computations of an inelastic process called delamination (or debonding) of elastic bodies glued together along a prescribed delamination interfaces. On a microscopic level, some macromolecules of the adhesive may break upon loading and we assume that they can never be glued back, i.e., no "healing" is possible. This makes the process *unidirectional* and irreversible. On the glued interface, we consider the *delamination process* as *rate-independent* and, in the bulk, we also take into account rate-dependent *viscous effects*. An important feature appearing in engineering modeling (and so far mostly omitted in the mathematical literature), is the dependence of this process on the modes under which it proceeds. Indeed, Mode I (=opening) usually dissipates much less energy than Mode II (=shearing). The difference may be up to hundreds of percents, cf. [1, 14, 15, 35]. Moreover, the delamination process seldom occurs in such pure modes and, in reality, the *mixed mode* appears more frequently. The substantial difference in the dissipation in various modes is explained either by some roughness of the glued interface (to be overcome in Mode II but not in Mode I, cf. [6]) or by some plastification caused by shear in Mode II (but not by mere tension in Mode I) before the delamination itself happens, cf. [14, 42].

In this article, we focus on a standard engineering model which was rigorously analyzed already in [24] even in a full thermodynamical context but exploiting the concept of non-simple materials (see e.g. [40]) which whould be much more demanding to be implemented computationally. Although computational simulations are routinely performed in engineering and

successfully used in applications even in simple materials, where the strain energy depends only of the first gradient of the deformation, cf. e.g. [1, 3, 10, 36, 37] and references therein, the rigorous convergence/existence analysis is not at disposal. Such computations and the models themselves are thus completely unjustified. Here we concentrate on an *isothermal* situation in *simple visco-elastic materials*, and emphasize numerical aspects including the *efficient computational feasibility* of the model with a *guaranteed numerical stability and convergence*.

Let us just highlight main ingredients of the model formulated in detail in Section 2, in particular focusing on the mixity of delamination modes. We confine ourselves to *quasistatic* problems (i.e. intertial neglected) at *small strains* and, just for the sake of notational simplicity, we restrict the analysis to the case of two (instead of several) visco-elastic bodies Ω_+ and Ω_- glued together along the *contact interface* $\Gamma_{\rm C}$. We assume an elastic response of the adhesive, and then one speaks about *adhesive contact*. The elastic response in the adhesive is assumed linear, being determined by the (positive-definite) matrix of elastic moduli $\mathbb{A}$, and the adhesive layer itself is assumed infinitesimally thin. At a current time instant, the "volume fraction" of debonded molecular links will be "macroscopically" described by the *scalar delamination parameter* $z : \Gamma_{\rm C} \to [0,1]$, which can be referred to the modeling approach by M. Frémond, see [8, 9]. The state $z(x) = 1$ means that the adhesive at $x \in \Gamma_{\rm C}$ is still 100% undestroyed and thus fully effective, while the intermediate state $0 < z(x) < 1$ means that the fraction $1 - z(x)$ of molecular links have already been broken but the remaining portion $z(x)$ is still effective, and eventually $z(x) = 0$ means that the interface is already completely debonded at $x \in \Gamma_{\rm C}$. As already used e.g. in [11], in some simplification, it is assumed that a specific phenomenologically prescribed energy a (in J/m^2, in 3-dimensional situations) is needed to break the macromolecular structure of the adhesive, independently of the rate of this process. Thus, delamination is a *rate-independent* and activated phenomenon, ruled by the maximum dissipation principle, and we shall therefore consider a rate-independent flow rule for z. We will consider the whole energy spent for the delamination as dissipated; for a more general model cf. Remark 2.1 below.

Let us now emphasize main new features of the model, i.e. its mixity-sensitivity. A standard engineering approach as e.g. in [10,36,37] is to make the activation energy $a = a(\psi_{\rm G})$ depend on the so-called *mode-mixity angle* $\psi_{\rm G}$. For instance, if $\vec{n}_{\rm C} = (0,0,1)$ at some $x \in \Gamma_{\rm C}$ (with $\vec{n}_{\rm C}$ the unit normal to $\Gamma_{\rm C}$, oriented from Ω_+ to Ω_-), and $\mathbb{A} = {\rm diag}(\kappa_{\rm n}, \kappa_{\rm t}, \kappa_{\rm t})$, the mode-mixity angle is defined as $\psi_{\rm G} = \psi_{\rm G}(\llbracket u \rrbracket) := \arctan(k^{1/2}|\llbracket u \rrbracket_{\rm t}|/|\llbracket u \rrbracket_{\rm n}|)$ with $k = \kappa_{\rm t}/\kappa_{\rm n}$ where $\llbracket u \rrbracket_{\rm t}$ and $\llbracket u \rrbracket_{\rm n}$ stand for the tangential and the normal traction; i.e. the jump of displacement across the boundary $\Gamma_{\rm C}$ decomposes as $\llbracket u \rrbracket = \llbracket u \rrbracket_{\rm n}\vec{n}_{\rm C} + \llbracket u \rrbracket_{\rm t}$, with $\llbracket u \rrbracket_{\rm n} = \llbracket u \rrbracket \cdot \vec{n}_{\rm C}$. In fact, to avoid discontinuity of such formula at 0, rather a suitable regularization of this mode-mixity angle should be taken, e.g.

$$\psi_{\rm G}(\llbracket u \rrbracket) = \arctan\sqrt{\frac{\kappa_{\rm t}|\llbracket u \rrbracket_{\rm t}|^2}{\kappa_{\rm n}|\llbracket u \rrbracket_{\rm n}|^2 + \epsilon}} \quad \text{with a small } \epsilon > 0. \tag{1.1}$$

The coefficient $\kappa_{\rm t}$ is often smaller than $\kappa_{\rm n}$, and a typical phenomenological form of a used in engineering [10] is, e.g.,

$$a(\psi_{\rm G}) := a_{\rm I}\big(1 + \tan^2((1-\lambda)\psi_{\rm G})\big); \tag{1.2}$$

cf. also [1] for a similar formula. In (1.2), $a_{\rm I} = a(0)$ is the activation threshold for the delamination mode I and λ is the so-called *delamination-mode-sensitivity* parameter. Note that a moderately strong delamination-mode sensitivity occurs when the ratio $a_{\rm II}/a_{\rm I}$ is about 5-10 where $a_{\rm II} = a(90^\circ)$ is the activation threshold for the pure delamination mode II. Then, one has λ about 0.2-0.3; cf. [37].

Mathematical troubles arising in the analysis of the system of partial differential inequalities for mixity-sensitive delamination model are caused by an inelastic rate-independent process on the boundary $\Gamma_{\rm C}$ along which delamination performs. Mixity-dependence of the dissipation makes the model *non-associative*, in contrast to the mixity-insensitive case and to another model involving a interfacial plasticity as an additional internal variable, recently

devised and analyzed in an isothermal case in [29, 30, 32]. In this work, quadratic viscosity dissipation energy coupled with a stored energy density, asymptotically growing faster than the spatial dimension and exhibiting separate convexity in strain and the delamination variable allow us to prove the existence of a suitable weak solution. The viscosity has thus less growth than the stored energy, which, unfortunately, does not seem to allow for the mechanical energy conservation, cf. Remark 4.1 below.

The plan of the paper is as follows: in Section 2 we formulate the initial-boundary-value problem in its classical formulation, following essentially [23, 24]. After a suitable weak formulation based on the concept of local solutions due to [17, 41], in Sec. 3 we device a semi-implicit discretization in time combined with a finite-element discretization in space and prove its numerical stability, i.e. suitable a-priori estimates. The efficient numerical implementation (based on recursive quadratic programming combined with elimination of bulk nodes by a boundary-element method) as well as illustrative 2-dimensional computational simulations are performed in Sec. 4. Eventually, the convergence of the approximate solutions towards weak solutions is outlined in Sect. 5 by using the already derived a-priori estimates. In particular, we prove unconditional convergence of discrete solutions to a weak solution to the model whenever the mesh size and the time step tend to zero.

Let us emphasize that, to our best knowledge, this article represents a first attempt to pose the standard engineering model for the mixity-sensitive delamination of simple materials (together with a physically relevant concept of its solution) and to devise an efficiently implementable algorithm in a way which simultaneously allows for a rigorous mathematical support as far as numerical stability and guaranteed convergence.

2 The model in its classical formulation

Hereafter, we suppose that the visco-elastic/inelatic-adhesive structure occupies a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ composed from (for notational simplicity only) two visco-elastic bodies, denoted by Ω_+ and Ω_-, glued together on a common conctact boundary, denoted by $\Gamma_{\rm C}$, which represents a prescribed interface where delamination may occur. This means we consider

$$\Omega = \Omega_+ \cup \Gamma_{\rm C} \cup \Omega_- \,,$$

with Ω_+ and Ω_- disjoint Lipschitz subdomains. We denote by $\vec{n}$ the outward unit normal to $\partial\Omega$, and by $\vec{n}_{\rm C}$ the unit normal to $\Gamma_{\rm C}$, which we consider oriented from Ω_+ to Ω_-. Moreover, given v defined on $\Omega\backslash\Gamma_{\rm C}$, v^+ (respectively, v^-) signifies the restriction of v to Ω_+ (to Ω_-, resp.). We further suppose that the boundary of Ω splits as

$$\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N} \,,$$

with $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ open subsets in the relative topology of $\partial\Omega$, disjoint one from each other and each of them with a smooth (one-dimensional) boundary. Considering $T > 0$ a fixed time horizon, we set

$$Q := (0,T)\times\Omega, \quad \Sigma := (0,T)\times\partial\Omega, \quad \Sigma_{\rm C} := (0,T)\times\Gamma_{\rm C}, \quad \Sigma_{\rm D} := (0,T)\times\Gamma_{\rm D}, \quad \Sigma_{\rm N} := (0,T)\times\Gamma_{\rm N}.$$

For readers' convenience, let us summarize the basic notation used in what follows:

$d = 2, 3$ dimension of the problem,	$\mathbb{C} : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$ nonlinear elastic Hook law,
$u : \Omega\backslash\Gamma_{\rm C} \to \mathbb{R}^d$ displacement,	$\mathbb{D} \in \mathbb{R}^{d^4}$ viscosity constants,
$z : \Gamma_{\rm C} \to [0,1]$ delamination variable,	$\mathbb{A} \in \mathbb{R}^{d\times d}$ elastic coefficients of the adhesive,
$e = e(u) = \frac{1}{2}\nabla u^\top + \frac{1}{2}\nabla u$ small-strain tensor,	$\alpha = \alpha([\![u]\!])$ energy (per area) dissipated on $\Gamma_{\rm C}$,
$[\![u]\!] = u^+\vert_{\Gamma_{\rm C}} - u^-\vert_{\Gamma_{\rm C}}$ jump of u across $\Gamma_{\rm C}$,	$F : Q \to \mathbb{R}^d$ applied bulk force,
σ stress tensor,	$w_{\rm D}$ prescribed boundary displacement,
$\psi_{\rm G}$ mode-mixity angle,	$f : \Sigma_{\rm N} \to \mathbb{R}^d$ applied traction.

Table 1. Summary of the basic notation used thorough the paper.

The *state* is formed by the couple (u,z). We use *Kelvin-Voigt's rheology* and, rather for mathematical reasons to facilitate analysis in multidimensional cases, a (possibly only slightly) nonlinear static response. Hence we assume the *stress* $\sigma:(0,T)\times\Omega\to\mathbb{R}^{d\times d}$ in the form:

$$\sigma=\sigma(u,\dot u):=\underbrace{\mathbb{D}e(\dot u)}_{\substack{\text{viscous}\\ \text{stress}}}+\underbrace{\mathbb{C}(e(u))}_{\substack{\text{elastic}\\ \text{stress}}}, \tag{2.1}$$

Furthermore, we shall denote by $T=T(u,v)$ the traction stress on some two-dimensional surface Γ (later, we shall take either $\Gamma=\Gamma_{\rm C}$ or $\Gamma=\Gamma_{\rm N}$), i.e.

$$T(u,\dot u):=\sigma(u,\dot u)\big|_\Gamma\vec n\,, \tag{2.2}$$

where of course we take as $\vec n$ the unit normal $\vec n_{\rm C}$ to $\Gamma_{\rm C}$, if $\Gamma=\Gamma_{\rm C}$. Its normal and tangential components are defined on $\Gamma_{\rm C}\cup\Gamma_{\rm N}$ respectively by the formulas

$$T_{\rm n}(u,\dot u)=\vec n\cdot\sigma(u,\dot u)\big|_\Gamma\vec n\qquad\text{and}\qquad T_{\rm t}(u,\dot u)=T(u,\dot u)-T_{\rm n}(u,\dot u)\vec n. \tag{2.3}$$

We address the standard frictionless Signorini conditions on $\Gamma_{\rm C}$ for the displacement u.

Classical formulation of the adhesive contact problem. Beside the force equilibrium coupled with the heat equation inside $Q\backslash\Sigma_{\rm C}$ and supplemented with standard boundary conditions, we have two complementarity problems on $\Sigma_{\rm C}$. Altogether, we have the boundary-value problem

$$\begin{aligned}
&\operatorname{div}\big(\mathbb{D}e(\dot u)+\mathbb{C}(e(u))\big)+F=0, && \text{in } Q\backslash\Sigma_{\rm C}, && (2.4a)\\
&u=0 && \text{on } \Sigma_{\rm D}, && (2.4b)\\
&T(u,\dot u)=f && \text{on } \Sigma_{\rm N}, && (2.4c)\\
&\big[\!\big[\mathbb{D}e(\dot u)+\mathbb{C}(e(u))\big]\!\big]\vec n_{\rm C}=0 && \text{on } \Sigma_{\rm C}, && (2.4d)\\
&T_{\rm t}(u,\dot u)+z\big(\mathbb{A}u-\big((\mathbb{A}u)\cdot\vec n_{\rm C}\big)\vec n_{\rm C}\big)=0 && \text{on } \Sigma_{\rm C}, && (2.4e)\\
&\big[\!\big[u\big]\!\big]\cdot\vec n_{\rm C}\ge0\quad\text{and}\quad T_{\rm n}(u,\dot u)+z(\mathbb{A}\big[\!\big[u\big]\!\big])\cdot\vec n_{\rm C}\ge0 && \text{on } \Sigma_{\rm C}, && (2.4f)\\
&\big(T_{\rm n}(u,\dot u)+z(\mathbb{A}\big[\!\big[u\big]\!\big])\vec n_{\rm C}\big)\big(\big[\!\big[u\big]\!\big]\cdot\vec n_{\rm C}\big)=0 && \text{on } \Sigma_{\rm C}, && (2.4g)\\
&\dot z\le0\quad\text{and}\quad d\le\alpha(\big[\!\big[u\big]\!\big])\quad\text{and}\quad\dot z\,\big(d-\alpha(\big[\!\big[u\big]\!\big])\big)=0 && \text{on } \Sigma_{\rm C}, && (2.4h)\\
&d\in\partial I_{[0,1]}(z)+\tfrac12\mathbb{A}\big[\!\big[u\big]\!\big]\cdot\big[\!\big[u\big]\!\big] && \text{on } \Sigma_{\rm C}. && (2.4i)
\end{aligned}$$

As to the involved symbols, we assume that

$$\varphi:\mathbb{R}^{d\times d}_{\rm sym}\to\mathbb{R}\text{ convex smooth; }\exists\,\varepsilon_0,\varepsilon_1>0,p>d:\ \varepsilon_1(1+|e|^p)\ge\varphi(e)\ge\varepsilon_0(|e|^p-1), \tag{2.5a}$$

$$\mathbb{C}(e):=\varphi'(e):=\frac{\partial\varphi(e)}{\partial e}\quad\text{elastic stress tensor at strain } e, \tag{2.5b}$$

$$\mathbb{D}:\mathbb{R}^{d\times d}_{\rm sym}\to\mathbb{R}^{d\times d}_{\rm sym}\quad\text{linear positive definite}, \tag{2.5c}$$

$$\mathbb{A}:\mathbb{R}^d\to\mathbb{R}^d\quad\text{linear positive semidefinite}. \tag{2.5d}$$

The complementarity problem (2.4f)–(2.4g) describes the Signorini unilateral contact. The complementarity problem (2.4h)–(2.4i) corresponds to the *flow rule* governing the evolution of z:

$$\partial I_{(-\infty,0]}(\dot z)+\partial I_{[0,1]}(z)+\tfrac12\mathbb{A}\big[\!\big[u\big]\!\big]\cdot\big[\!\big[u\big]\!\big]\ni\alpha(\big[\!\big[u\big]\!\big])\qquad\text{in }\Sigma_{\rm C}, \tag{2.6}$$

with the indicator functions $I_{(-\infty,0]}$, $I_{[0,1]}:\mathbb{R}\to[0,+\infty]$ and their (convex analysis) subdif-

ferentials $\partial I_{(-\infty,0]}$, $\partial I_{[0,1]} : \mathbb{R} \rightrightarrows \mathbb{R}$. The energetics of the model is formally:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\underbrace{\int_{\Omega\setminus\Gamma_{\mathrm{C}}} \varphi(e(u))\,\mathrm{d}x}_{\substack{\text{elastic energy}\\ \text{in the bulk}}} + \underbrace{\int_{\Gamma_{\mathrm{C}}} \frac{z}{2}\mathbb{A}[\![u]\!]\cdot[\![u]\!]\,\mathrm{d}S}_{\substack{\text{elastic energy}\\ \text{in the adhesive}}} \Big) + \underbrace{\int_{\Omega\setminus\Gamma_{\mathrm{C}}} \frac{1}{2}\mathbb{D}e(\dot u){:}e(\dot u)\,\mathrm{d}x}_{\substack{\text{rate of viscous}\\ \text{dissipation in the bulk}}}$$
$$+ \underbrace{\int_{\Gamma_{\mathrm{C}}} \alpha([\![u]\!])\dot z\,\mathrm{d}S}_{\substack{\text{rate of dissipation by dela-}\\ \text{mination of the adhesive}}} = \underbrace{\int_{\Omega} F\cdot\dot u\,\mathrm{d}x}_{\substack{\text{power of bulk}\\ \text{mechanical load}}} + \underbrace{\int_{\Gamma_{\mathrm{N}}} f\cdot\dot u\,\mathrm{d}S}_{\substack{\text{power of surface}\\ \text{mechanical load}}}. \quad (2.7)$$

For more details about derivation of the model we refer to [23, 24]. We will consider the initial-value problem for (2.4) by prescribing the initial condition

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \qquad z(0) = z_0 \quad \text{a.e. in } \Gamma_{\mathrm{C}}. \quad (2.8)$$

Remark 2.1 (*Stored energy increased by delamination).* The energy needed for the delamination can be alternatively understood as contributing to the stored energy. This reflects the fact that any new surface represents some additional stored energy. In the isothermal unidirectional delamination, this alternative concept is mechanically equivalent. Yet, if temperature variations are considered, then it makes a difference because the stored energy variation does not contribute to the heat production. In fact, rather both parts (i.e. dissipative and stored) of the energy spent for delamination should more realistically be considered, cf. [24]. Also, if a bi-directional evolution of delamination (involving healing) would be considered, then the contribution to the stored energy becomes especially important because it just facilitates the driving force for possible healing, cf. [34].

Remark 2.2 (*Dynamical problems).* In some applications/regimes inertial forces cannot be neglected and then (2.4a) takes the form

$$\varrho\ddot u - \mathrm{div}\big(\mathbb{D}e(\dot u) + \mathbb{C}e(u) - \mathrm{div}\,\mathfrak{h}\big) = F \quad (2.9)$$

with $\varrho > 0$ mass density. Implicit discretization of this term is relatively easy to be incorporated in the analysis if a generalized concept of solution without energy preservation is accepted, cf. [24]. Here e.g. (3.12a) augments by the term $\tau^{-2}\int_\Omega \frac{1}{2}\varrho|u{-}2u_\tau^{k-1}{+}u_\tau^{k-2}|^2\mathrm{d}x$. Yet, it is well known that the implicit discretization of the inertial term is unsuitable for computational simulations due to spurious numerical attenuation and efficient calculations of wave propagation needs more sophisticated formulas. On the other hand, leaving the energy preservation out, we can also afford $\mathbb{D} = 0$ because the inertial term controls $[\![u]\!]$ "compactly" in $C(\overline{\Sigma}_{\mathrm{C}})$ via Aubin's-Lions' theorem, thus we get hyperbolic inviscid delamination problem.

Remark 2.3 (*Cohesive contacts).* We can also consider $z\mathbb{A}{+}z^2\mathbb{B}$ instead of $z\mathbb{A}$ in (2.4g-i) and $(\frac{1}{2}\mathbb{A}{+}z\mathbb{B})[\![u]\!]\cdot[\![u]\!] - \kappa\Delta z$ instead of $\frac{1}{2}\mathbb{A}[\![u]\!]\cdot[\![u]\!]$ in (2.4i), which would be based on the stored energy with the boundary term of the type

$$\int_{\Gamma_{\mathrm{C}}} \frac{1}{2}(z\mathbb{A}{+}z^2\mathbb{B})[\![u]\!]\cdot[\![u]\!] + \kappa|\nabla_{\mathrm{S}} z|^2\,\mathrm{d}S$$

with $\mathbb{B}$ positive semidefinite and $\kappa > 0$; for a more detailed discussion about this quadratic cohesion model cf. [29, Sect.6.1] and for analysis cf. [2]. Here, it leads to two quadratic mathematical programs after the semi-implicit discretization and numerical analysis works for P1-discretization of z and 2-dimensional problems simply by a mutual recovery sequence $\widetilde z_k := (\widetilde z - \|z_k{-}z\|_{C(\overline{\Gamma}_{\mathrm{C}})})^+$ suitably adjusted to spatial discretization, cf. the proof of stability of the limit below (while for 3-dimensional problems more sophisticated damage-type construction by M.Thomas et al. [38, 39] would be needed).

3 Weak formulation and semi-implicit discretization

We will use the standard notation $W^{1,p}(\Omega)$ for the Sobolev space of functions having the derivatives in the Lebesgue space $L^p(\Omega)$. If valued in $\mathbb{R}^n$ with $n \geq 2$, we will write $W^{1,p}(\Omega;\mathbb{R}^n)$, and furthermore, if $p = 2$, we use the shorthand notation $H^1(\Omega;\mathbb{R}^n) = W^{1,2}(\Omega;\mathbb{R}^n)$. Moreover, we will adopt the notation

$$W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) := \left\{v \in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) : \quad v = 0 \ \text{ on } \Gamma_{\rm D}\right\}.$$

For X a (separable) Banach space, we denote by $C_{\rm w}([0,T];X)$ and $BV([0,T];X)$ the Banach spaces of weakly continuous functions $[0,T] \to X$ and of the functions that have bounded variation on $[0,T]$, respectively. Notice that these functions are defined everywhere on $[0,T]$.

Hereafter, the external mechanical loading F and f will be qualified

$$F \in \begin{cases} L^2(0,T;L^{6/5}(\Omega;\mathbb{R}^d)) & \text{if } d = 3,\\ L^2(0,T;L^q(\Omega;\mathbb{R}^d)),\ q>1 & \text{if } d = 2; \end{cases} \tag{3.1a}$$

$$f \in \begin{cases} L^2(0,T;L^{4/3}(\Gamma_{\rm N};\mathbb{R}^d)) & \text{if } d = 3,\\ L^2(0,T;L^q(\Gamma_{\rm N};\mathbb{R}^d)),\ q>1 & \text{if } d = 2; \end{cases} \tag{3.1b}$$

cf. also Remark 3.4 below. As for the initial data, we impose the following

$$u_0 \in W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d), \quad \big[\!\big[u_0\big]\!\big]\cdot\vec{n}_{\rm C} \geq 0 \ \text{ on } \Gamma_{\rm C}, \tag{3.2a}$$

$$z_0 \in L^\infty(\Gamma_{\rm C}), \qquad 0 \leq z_0 \leq 1 \ \text{ a.e. on } \Gamma_{\rm C}. \tag{3.2b}$$

We will use the abbreviation for the stored energy Φ and the dissipation rate $\mathcal{R}$:

$$\Phi(u,z) := \begin{cases} \displaystyle\int_{\Omega\backslash\Gamma_{\rm C}} \varphi(e(u))\,{\rm d}x + \int_{\Gamma_{\rm C}} \frac{1}{2} z\mathbb{A}\big[\!\big[u\big]\!\big]\cdot\big[\!\big[u\big]\!\big]\,{\rm d}S & \text{if } [\![u]\!]\cdot\vec{n}_{\rm C} \geq 0 \text{ and } 0\leq z\leq 1 \text{ on } \Gamma_{\rm C},\\ +\infty & \text{otherwise, and} \end{cases} \tag{3.3}$$

$$\mathcal{R}\big(u;\dot u,\dot z\big) := \begin{cases} \displaystyle\int_\Omega \mathbb{D}e(\dot u){:}e(\dot u)\,{\rm d}x + \int_{\overline{\Gamma}_{\rm C}} \alpha(\big[\!\big[u\big]\!\big])|\dot z|\,{\rm d}S & \text{if } \dot z \leq 0 \text{ a.e. in } \Gamma_{\rm C},\\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Definition 3.1 (*Weak solution*)**.** Given an initial data (u_0,z_0) satisfying (3.2), we call a couple (u,z) a weak solution to the Cauchy problem for system (2.4) if

$$u \in C_{\rm w}([0,T];W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)) \cap H^1(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)), \tag{3.5a}$$

$$z \in L^\infty(\Sigma_{\rm C}) \cap BV([0,T];L^1(\Gamma_{\rm C})), \quad z(\cdot,x) \text{ nonincreasing on } [0,T] \text{ for a.a. } x\in\Gamma_{\rm C}, \tag{3.5b}$$

and the couple (u,z) complies, besides the initial condition (2.8), with:

(i) (weak formulation of the) momentum inclusion, i.e.:

$$\big[\!\big[u\big]\!\big]\cdot\vec{n}_{\rm C} \geq 0 \ \text{ on } \Sigma_{\rm C}, \quad \text{and} \tag{3.6a}$$

$$\int_{Q\backslash\Sigma_{\rm C}} \big(\mathbb{D}e(\dot u)+\mathbb{C}(e(u))\big){:}e(v-u)\,{\rm d}x{\rm d}t + \int_{\Sigma_{\rm C}} z\mathbb{A}\big[\!\big[u\big]\!\big]\cdot\big[\!\big[v-u\big]\!\big]{\rm d}S{\rm d}t \geq \int_Q F\cdot(v-u)\,{\rm d}x{\rm d}t + \int_{\Sigma_{\rm N}} f\cdot(v-u)\,{\rm d}S{\rm d}t \tag{3.6b}$$

for all v in $L^2(0,T;W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$ with $[\![v]\!]\cdot\vec{n}_{\rm C} \geq 0$ on $\Sigma_{\rm C}$,

(ii) energy inequality for almost all time instant $t_1 < t_2$, $[t_1,t_2] \subset [0,T]$:

$$\Phi\big(u(t_2),z(t_2)\big) + \int_{t_1}^{t_2} \mathcal{R}(u;\dot u,\dot z)\,{\rm d}t \leq \Phi\big(u(t_1),z(t_1)\big) + \int_{t_1}^{t_2}\int_\Omega F\cdot\dot u\,{\rm d}x{\rm d}t + \int_{t_1}^{t_2}\int_{\Gamma_{\rm N}} f\cdot\dot u\,{\rm d}S{\rm d}t, \tag{3.6c}$$

(iii) semistability for a.a. $t \in (0,T)$

$$\forall \tilde z \in L^\infty(\Gamma_{\rm C}): \qquad \Phi\big(u(t),z(t)\big) \le \Phi\big(u(t),\tilde z\big) + \mathcal{R}\big(u(t);0,\tilde z - z(t)\big). \tag{3.6d}$$

Remark 3.2. Due to cancellation of bulk terms, (3.6d) means just $\int_{\Gamma_{\rm C}}(z(t)-\tilde z)(\mathbb{A}[\![u(t)]\!]\cdot[\![u(t)]\!] - 2\alpha([\![u(t)]\!]))\,{\rm d}S \le 0$ for all $\tilde z \in L^\infty(\Gamma_{\rm C})$ such that $0 \le \tilde z \le z(t)$, which can be disintegrated so that (3.6d) is equivalent to

$$z(t,x)\mathbb{A}[\![u(t,x)]\!]\cdot[\![u(t,x)]\!] \le 2\alpha([\![u(t,x)]\!]) \quad \text{or} \quad z(t,x)=0 \qquad \text{for a.a. } x\in\Gamma_{\rm C}. \tag{3.7}$$

In [16, 19], a global stability condition combined with energy conservation was shown to provide the correct "weak" formulation of rate-independent flow rules. Due to the viscosity in the bulk, here the semistability (3.6d) plays the role of the global stability condition of [16, 19]. Moreover, here we do not require the energy conservation (2.7) but we only assume energy inequality (3.6c) between varying time instances $t_1 < t_2$, which is the general concept of so-called *local solutions* invented for purely rate-independent systems for a special crack problem in [41] and further generally investigated in [17], and proved to coincide with the concept of conventional weak solutions in [28, Prop.2.3]. Although this concept is very wide in general, here the viscosity together with the convexity of the stored energy in terms of z ensures good selectivity of this concept, cf. [26, Prop.5.2]. This viscosity/convexity attribute also excludes the undesired effect of too-early delamination unphysically sliding to less dissipative Mode I, which may occur in purely elastic model if energy conservation would be forced, cf. [30] in contrast to [32]. Also we point out that, disregarding the only one-sided inequality, (3.6c) is the integrated version of the total energy balance (2.7).

To solve the the initial-boundary value problem (2.4) and (2.8) numerically, we must make some discretization both in time and in space. Rather as an example, let us consider P1-elements for u and P0-elements for z. Assuming polygonal domains Ω_+ and Ω_-, we use a spatial discretization by considering a triangulation $\mathscr{T}_h$ of $\Omega\backslash\Gamma_{\rm C}$ with a mesh size $h>0$ and define the finite-dimensional subspaces

$$V_h := \big\{v\in W^{1,\infty}(\Omega;\mathbb{R}^d);\ \ \forall S\in\mathscr{T}_h:\ \ v|_S \text{ affine}\big\}, \tag{3.8a}$$

$$Z_h := \big\{z\in L^\infty(\Gamma_{\rm C});\ \ \forall S\in\mathscr{T}_h:\ \ z|_{\bar S\cap\Gamma_{\rm C}} \text{ constant}\big\}. \tag{3.8b}$$

Moreover, we make the *time-discretization* by using a suitable *semi-implicit scheme* using a popular fractional-step-like strategy, cf. also [25, Remark 8.25]. In contrast to anisothermal situation in [24], here this leads to alternating variational problems which are even convex, which allows for a constructive solution. Using an equidistant partition of the time interval $[0,T]$ with a time step $\tau>0$, we seek $u_{\tau h}^k \in V_h$ and $z_{\tau h}^k \in Z_h$ such that $[\![u_{\tau h}^k]\!]\cdot\vec n_{\rm C} \ge 0$ on $\Sigma_{\rm C}$ and

$$\begin{aligned}\int_{\Omega\backslash\Gamma_{\rm C}}\Big(\mathbb{D}e\big(\tfrac{u_{\tau h}^k-u_{\tau h}^{k-1}}{\tau}\big)+\mathbb{C}(e(u_{\tau h}^k))\Big):e(v-u_{\tau h}^k)\,{\rm d}x + \int_{\Gamma_{\rm C}} z_{\tau h}^{k-1}\mathbb{A}[\![u_{\tau h}^k]\!]\cdot[\![v-u_{\tau h}^k]\!]\,{\rm d}S \\ \ge \int_\Omega F_\tau^k\cdot(v-u_{\tau h}^k)\,{\rm d}x + \int_{\Gamma_{\rm N}} f_\tau^k\cdot(v-u_{\tau h}^k)\,{\rm d}S\end{aligned} \tag{3.9a}$$

$$\begin{aligned}\Phi\big(u_{\tau h}^k,z_{\tau h}^k\big) + \mathcal{R}\Big(u_{\tau h}^k;\tfrac{u_{\tau h}^k-u_{\tau h}^{k-1}}{\tau},\tfrac{z_{\tau h}^k-z_{\tau h}^{k-1}}{\tau}\Big) \le \Phi\big(u_{\tau h}^{k-1},z_{\tau h}^{k-1}\big) \\ + \int_\Omega F_\tau^k\cdot\tfrac{u_{\tau h}^k-u_{\tau h}^{k-1}}{\tau}\,{\rm d}x + \int_{\Gamma_{\rm N}} f_\tau^k\cdot\tfrac{u_{\tau h}^k-u_{\tau h}^{k-1}}{\tau}\,{\rm d}S,\end{aligned} \tag{3.9b}$$

$$\forall\tilde z\in Z_h: \qquad \Phi\big(u_{\tau h}^k,z_{\tau h}^k\big) \le \Phi\big(u_{\tau h}^k,\tilde z\big) + \mathcal{R}\big(u_{\tau h}^k;0,\tilde z - z_{\tau h}^k\big). \tag{3.9c}$$

with $F_\tau^k = \tau^{-1}\int_{(k-1)\tau}^{k\tau} F(s)\,{\rm d}s$, $f_\tau^k = \tau^{-1}\int_{(k-1)\tau}^{k\tau} f(s)\,{\rm d}s$, and proceeding recursively for $k = 1,\dots T/\tau \in \mathbb{N}$ with starting for $k=1$ from

$$u_{\tau h}^0 = u_0 \qquad \text{and} \qquad z_{\tau h}^0 = z_0. \tag{3.10}$$

The adjective "semi-implicit" is related with usage of $z_{\tau h}^{k-1}$ in (3.9a) instead of $z_{\tau h}^{k}$ which would lead to a fully implicit formula. Such usage of $z_{\tau h}^{k-1}$ leads to the decoupling of the problem: first we can solve (3.9a) for $u_{\tau h}^k$ and, only after it, the rest of (3.9b,c) for $z_{\tau h}^k$. Note that, in a simple way, we discretized rather the weak formulation (3.6) than the classical formulation (2.4) where we would have faced technical problems e.g. with the interaction of a piece-quadratic $\mathbb{A}[\![u_{\tau h}^k]\!]\cdot[\![u_{\tau h}^k]\!]$ with piecewise constant $z_{\tau h}^k - z_{\tau h}^{k-1}$ and a general nonlinear $\alpha([\![u_{\tau h}^k]\!])$ in (2.4h-i).

On top of it, we can employ the variational structure of both decoupled problems, cf. also [25, Remark 8.25]. To cope with the constraints more explicitly, we introduce the smooth stored energy $\mathcal{E}$ and the dissipation (pseudo)potentials $\mathcal{R}_1$ and $\mathcal{R}_2$ defined here by

$$\mathcal{E}(t,u,z) = \int_\Omega \varphi(e(u)) - F(t)\cdot u \,\mathrm{d}x + \int_{\Gamma_{\rm C}} \frac{1}{2} z\mathbb{A}[\![u]\!]\cdot[\![u]\!]\,\mathrm{d}S - \int_{\Gamma_{\rm N}} f(t)\cdot u\,\mathrm{d}S, \tag{3.11a}$$

$$\mathcal{R}_1(u;\dot z) = -\int_{\Gamma_{\rm C}} \alpha([\![u]\!])\dot z\,\mathrm{d}S\,, \qquad \mathcal{R}_2(\dot u) = \int_\Omega \frac{1}{2}\mathbb{D}e(\dot u){:}e(\dot u)\,\mathrm{d}x. \tag{3.11b}$$

Note that the constraints $[\![u]\!]\cdot\vec n_{\rm C} \ge 0$, $0 \le z \le 1$, and $\dot z \le 0$, originally contained in the stored energy Φ and the dissipation rate $\mathcal{R}$ in (3.3) and (3.4), are now included in (3.12) below so that we can equivalently use the smooth functionals $\mathcal{E}(t,\cdot,\cdot)$ and $\mathcal{R}_1(u;\cdot)$ and $\mathcal{R}_2$ in (3.11). Also note that $\mathcal{R}(u;\dot u,\dot z) = \mathcal{R}_1(u;\dot z) + 2\mathcal{R}_2(\dot u)$ and $\mathcal{R}_1(u;\cdot)$ is degree-1 homogeneous so that the factor τ does not show up in the functional in (3.12b), in contrast to the degree-2 homogeneous functional $\mathcal{R}_2(\cdot)$ in (3.12a). We thus obtain two convex minimization problems: first, we are to solve

$$\left.\begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau,u,z_{\tau h}^{k-1}) + \tau\mathcal{R}_2\Big(\dfrac{u-u_{\tau h}^{k-1}}{\tau}\Big) \\ \text{subject to} & u\in V_h,\quad [\![u]\!]\cdot\vec n_{\rm C}\ge 0 \end{array}\right\} \tag{3.12a}$$

and, denoting its unique solution by $u_{\tau h}^k$, then we solve

$$\left.\begin{array}{ll} \text{minimize} & \mathcal{E}\big(k\tau,u_{\tau h}^k,z\big) + \mathcal{R}_1\big(u_{\tau h}^k;z-z_{\tau h}^{k-1}\big) \\ \text{subject to} & z\in Z_h,\quad 0\le z\le z_{\tau h}^{k-1}. \end{array}\right\} \tag{3.12b}$$

For $\tau>0$ fixed, we denote the left-continuous and the right-continuous *piecewise constants*, and the *piecewise linear* interpolants of the discrete solutions $\{u_\tau^k\}_{k=1}^{T/\tau}$ by $\overline{u}_{\tau h} : (0,T)\to W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$, $\underline{u}_{\tau h} : (0,T)\to W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$, and $u_{\tau h} : (0,T)\to W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ defined by

$$\overline{u}_{\tau h}(t) = u_\tau^k,\quad \underline{u}_{\tau h}(t) = u_\tau^{k-1},\quad u_{\tau h}(t) = \frac{t-t_\tau^{k-1}}{\tau}u_\tau^k + \frac{t_\tau^k - t}{\tau}u_\tau^{k-1} \quad \text{for } t\in(t_\tau^{k-1},t_\tau^k]. \tag{3.13}$$

In the same way, we shall denote the interpolants of $\{z_\tau^k\}_{k=1}^{T/\tau}$, and of F_k^τ, and f_k^τ.

Both for supporting convergence analysis (cf. Sect. 5 below) and for implementation, the important attribute of the above devised discrete scheme is its numerical stability, i.e. the numerical results do not exhibit spurious mesh dependency:

Proposition 3.3 (Numerical stability of the discretization)**.** *Let us assume* (2.5), (3.1), (3.2), $\inf\alpha(\cdot)>0$, $\mathrm{meas}_{d-1}(\partial\Omega_+\cap\Gamma_{\rm D})>0$, *and* $\mathrm{meas}_{d-1}(\partial\Omega_-\cap\Gamma_{\rm D})>0$, *where* meas_{d-1} *denotes the* $(d-1)$*-dimensional measure on* Γ. *Then, for all* $\tau>0$ *and* $h>0$ *and for some constant* $S_0>0$ *independent of* τ *and* $h>0$, *the approximate solutions* $(\overline{u}_{\tau h},\overline{z}_{\tau h},u_{\tau h},z_{\tau h})$ *satisfy*

$$\big\|\overline{u}_{\tau h}\big\|_{L^\infty(0,T;W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))}\le S_0\,, \tag{3.14a}$$

$$\big\|u_{\tau h}\big\|_{H^1(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))}\le S_0\,, \tag{3.14b}$$

$$\big\|\overline{z}_{\tau h}\big\|_{L^\infty(\Sigma_{\rm C})}\le S_0\,, \tag{3.14c}$$

$$\big\|\overline{z}_{\tau h}\big\|_{BV([0,T];L^1(\Gamma_{\rm C}))}\le S_0. \tag{3.14d}$$

Sketch of the proof. We only sketch the calculations for proving (3.14), since the argument closely follows the proof of [23, Lemma 7.7] or also [24, Lemma 5.6].

A discrete analog of (3.6c) can be obtained by testing the optimality conditions for (3.12a) and (3.12b) respectively by $u^k_{\tau h}-u^{k-1}_{\tau h}$ and $z^k_{\tau h}-z^{k-1}_{\tau h}$ (which, in fact, means plugging $v=u^{k-1}_{\tau h}$ into a discrete version of (3.6b) for the former test), and by adding it, benefiting from the cancellation of the terms $\pm\Phi(u^k_{\tau h},z^{k-1}_{\tau h})$ and by the separate convexity of $\Phi(\cdot,\cdot)$, i.e. both $\Phi(u,\cdot)$ and $\Phi(\cdot,z)$ are convex. This gives the estimate

$$\begin{aligned}
&\Phi(u^k_{\tau h},z^k_{\tau h})+\tau\sum_{l=1}^{k}\mathcal{R}\Big(u^l_{\tau h};\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau},\frac{z^l_{\tau h}-z^{l-1}_{\tau h}}{\tau}\Big)\\
&\le\Phi(u_0,z_0)+\tau\sum_{l=1}^{k}\int_\Omega F^l_\tau\cdot\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau}\,\mathrm{d}x+\int_{\Gamma_{\rm N}}f^l_\tau\cdot\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau}\,\mathrm{d}S\\
&\le\Phi(u_0,z_0)+\|F^l_\tau\|_{L^{6/5}(\Omega;\mathbb{R}^d)}\Big\|\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau}\Big\|_{L^6(\Omega;\mathbb{R}^d)}+\|f^l_\tau\|_{L^{4/3}(\Gamma_{\rm N};\mathbb{R}^d)}\Big\|\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau}\Big\|_{L^4(\Gamma_{\rm N};\mathbb{R}^d)},\\
&\le\Phi(u_0,z_0)+C_\delta\|F^l_\tau\|^2_{L^{6/5}(\Omega;\mathbb{R}^d)}+C_\delta\|f^l_\tau\|^2_{L^{4/3}(\Gamma_{\rm N};\mathbb{R}^d)}+\delta\Big\|\frac{u^l_{\tau h}-u^{l-1}_{\tau h}}{\tau}\Big\|^2_{H^1(\Gamma_{\rm N};\mathbb{R}^d)},
\end{aligned}\tag{3.15}$$

where $\delta>0$ and C_δ depends, beside δ, also on the norms of the embedding $H^1(\Omega\backslash\Gamma_{\rm C})\subset L^6(\Omega)$ and of the trace operator $H^1(\Omega\backslash\Gamma_{\rm C})\to L^4(\Gamma_{\rm N})$. Then we choose $\delta>0$ so small that the last term can be absorbed in the $\mathcal{R}$-term by using the assumption (2.5c). Then all the a-priori estimates (3.14) easily follow. □

Remark 3.4. The integrability in (3.1) designed rather for $d=3$ can be improved for $d=2$. One can also consider the alternative qualification, e.g. for $p>d$, one can consider $F\in W^{1,1}(I;L^1(\Omega;\mathbb{R}^d))$ and $f\in W^{1,1}(I;L^1(\Gamma_{\rm N};\mathbb{R}^d))$ and then to perform the a-priori estimate (3.15) by using the discrete by-part integration (=summation) and the discrete Gronwall inequality, and the coercivity (2.5a) instead of (2.5c). For the purpose of a-priori estimates only, one can also weaken (2.5c) to positive semi-definiteness (and in particular the rate-independent, inviscid problem with $\mathbb{D}=0$), although the convergence seems not guaranteed, cf. also Remark 5.4 below.

Remark 3.5. If we assume time-dependent boundary conditions such that $u(t)=u_{\rm D}(t)$ a.e. on $\Gamma_{\rm D}$ for every $t\in[0,T]$ for some $u_{\rm D}(t)\in W^{1,p}(\Omega;\mathbb{R}^d)$ Then the shift $u\mapsto u+u_{\rm D}(t)$ transform the problem to zero boundary conditions for u, i.e., $u_0=0$ on $\Gamma_{\rm D}$.

4 Computer implementation and illustrative simulations

We demonstrate varying mode-mixity of delamination on a relatively simple two-dimensional example motivated by the pull-push shear experimental test used in engineering practice [4]. Intentionally, we use the same geometry, shown in Fig. 1, as in [30, 32] in order to have a comparison of our weak solution of the engineering non-associative visco-elastic model with a maximally-dissipative local solution and the energetic solution of the associative inviscid model presented respectively in [32] and in [30]. In contrast to Sections 2–3, only one bulk domain is considered and $\Gamma_{\rm C}$ is a part of its boundary but this modification is straightforward; alternatively, one may also think about Ω_- as a completely rigid body in the previous setting. Here Ω_+ is a two-dimensional rectangular domain glued on the most of its bottom side $\Gamma_{\rm C}$ with the Dirichlet loading acting on the right-hand side $\Gamma_{\rm D}$ in the direction $(1,0.6)$, cf. Fig. 1, increasing linearly in time with velocity 0.3 mm/s.

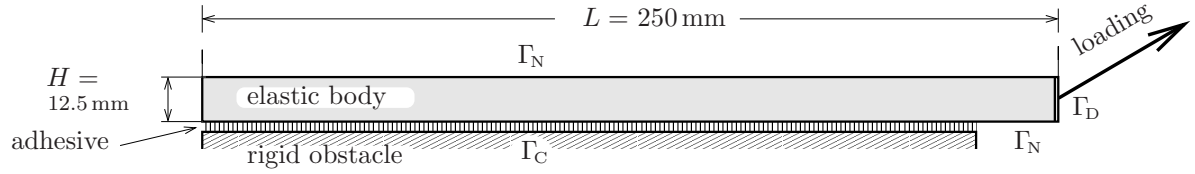

Fig. 1. Geometry and boundary conditions of the problem considered. The length of the initially glued part $\Gamma_{\rm C}$ is $0.9L=225\,$mm, the adhesive layer has zero thickness.

The bulk material is considered linear, homogeneous, and isotropic with the Young modulus $E = 70$ GPa and Poisson's ratio $\nu = 0.35$ (which corresponds to aluminum); thus $\mathbb{C}_{ijkl} = \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with δ_{ij} standing for the Kronecker symbol. For the viscosity tensor we consider $\mathbb{D} = \chi\mathbb{C}$ with a relaxation time $\chi = 0.001\,\mathrm{s}$, which is very small with the relation to the loading speed we have considered; actually, we did not see any essential difference for just merely elastic material $\chi = 0$ although the model has only a limited validity for this case, cf. Remark 5.4 below.

For the adhesive, we took a normal stiffness $\kappa_{\rm n}$ =150 GPa/m, a tangential stiffness with $\kappa_{\rm t} = \kappa_{\rm n}/2$, and the Mode-I fracture toughness $a_{\rm I} = 187.5\ \mathrm{J/m^2}$. Furthermore, the engineering model (1.2) was used with $\lambda = 0.333$ (and with $\epsilon = 0$) which corresponds to a rather moderate mode-sensitivity $a(90^\circ)/a_{\rm I} = a_{\rm II}/a_{\rm I} = 4$.

The numerical stability, i.e. the a-priori estimates (3.14), fully applies. The (rather negligible) shortcuts is that we take $p = 2$ (instead of $p > 2$) so that the convergence in Appendix below applies only up to this (small) discrepancy.

For the computer implementation, it is important that (3.12b) represents a *linear-quadratic program*, which allows for algorithmically a very efficient solution. Furthermore, (3.12b) is even a *linear program*. On top of it, as we used P0-elements, cf. (3.8b), it localizes on particular elements on $\Gamma_{\rm C}$, so that its solution is trivial and fast; for this effect, it is also important that we do not need to consider any gradient of z, in contrast to the associative model [29, 30, 32]. For the Dirichlet loading, we have used Remark 3.5.

A noteworthy attribute of our problem is that the inelastic process of delamination occurs on the boundary $\Gamma_{\rm C}$ while in the bulk domain(s) it is linear. This allows for elimination of nodal values arising by using P1-elements (3.8a) inside Ω and considerable reduction of degrees of freedom by considering only nodal or element values on $\Gamma_{\rm C}$. In fact, this idea has been systematically exploited even on the continuous level when implementing the boundary-element method, cf. [21,22,30–32,37], although it is still not fully supported by a convergence analysis like Proposition 5.1 below due to general substantial theoretical difficulties related to this method.

Anyhow, for the computational experiments presented here partially also with the goal to document the usually not investigated modelling issues (in particular the energetics), we use a shortcut in implementing the spacial discretization (3.12) with (3.8) and, instead of an algebraic elimination of the interior nodal point, we made this elimination by the collocation boundary-element method. Here also our choice $\mathbb{D} = \chi\mathbb{C}$ makes the implementation of this method simpler, cf. [27, Remark 6.2]. For the results presented on Figures 2-3, we have used 81 elements on $\Gamma_{\rm C}$, i.e. $h = 2.7\bar{7}\,\mathrm{mm}$ (=the size of a boundary segment in equidistant discretization), and the time step $\tau = 2.2\bar{2}\,\mathrm{ms}$.

This example exhibits remarkably varying mode of delamination. At the beginning the delamination is performed by a mixed mode close to Mode I given essentially by the direction of the Dirichlet loading, cf. Figure 1, while later it turns rather to nearly pure Mode II. Yet, at the very end of the process, due to elastic bending the delamination starts performing also from the left-hand side of the bar opposite to the loading side, and thus again a mixed mode occurs. This relatively complicated mixed-mode behaviour is depicted in Figure 2(right), showing essential qualitative difference from the energetic solution which exhibits a non-physical tendency to slide to less-dissipative Mode I, cf. [30, Fig. 7]. We have here the energy (im)balance (3.6c) as an important ingredient and thus, in contrast to usual engineering calculations, we trace also the energetics of the model, depicted on Figure 2(left).

The evolution of the deformation of the visco-elastic domain (through the displacement u) and spacial distribution of the delamination parameter z are depicted in Figure 3 at eight snapshots selected not uniformly to visualize interesting effects when delamination starts to be completed. In particular, the delamination propagating from both sides at the very end (mentioned already above) is seen there.

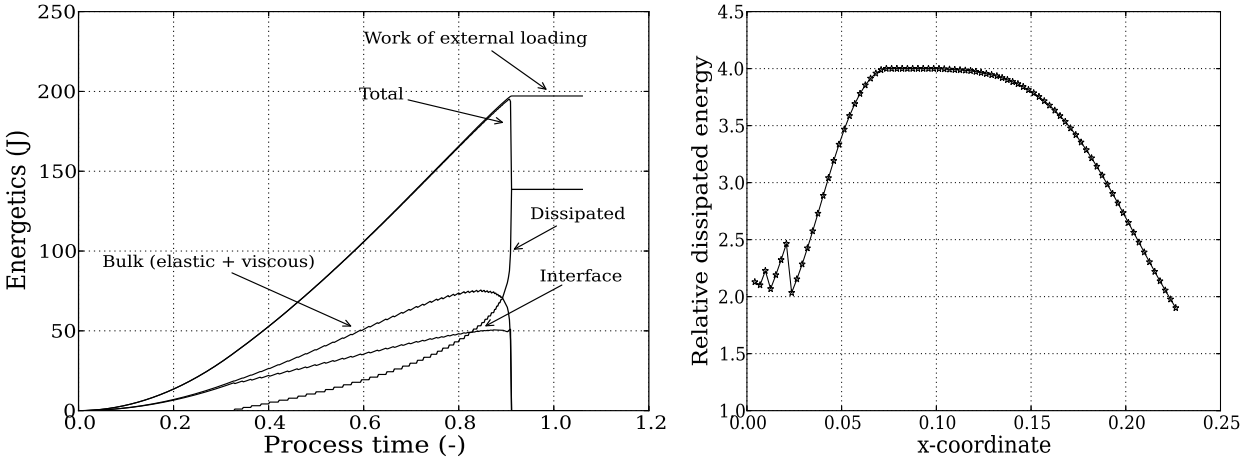

Fig. 2. Left: Time evolution of the energies: the bulk and the interfacial parts of $\mathcal{E}(t,\bar{u}_\tau(t),\bar{z}_\tau(t)) + \int_0^t \mathcal{R}_2(\dot{u}_\tau)\mathrm{d}t$, the interfacial dissipated energy $\mathcal{R}_1(\bar{z}_\tau(t)-z_0)$, their sum=total energy (=the left-hand side of (3.6c)), and the work of external loading (=the right-hand side of (3.6c)).
Right: Mixity-mode distribution along $\Gamma_{\rm C}$ evaluated according the overall dissipated energy related to $a_{\rm I}$ after the delamination has been completed: value $=1\sim$ Mode I, value $=4=a_{\rm II}/a_{\rm I}\sim$ Mode II.

Eventually, the joint convergence from Proposition 5.1 below for time- and FEM-spatial discretization (although here implemented by BEM) is demonstrated in Figures 4 and 5 for two different gradually refining space/time discretizations. We choose the scenario keeping the ratio τ/h constant, although Proposition 5.1 itself does not give any particular suggestion in this respect. Anyhow, the tendency of convergences is clearly seen, although we naturally do not know the exact solution so that we cannot evaluate any actual error. On top of it, the exact solution does not need to be unique so we even do not have guaranteed the convergence of the whole sequence of the approximate solutions and, moreover, the simplified implementation by collocation BEM does not have guaranteed convergence, in contrast to FEM stated in Proposition 5.1.

Remark 4.1 (*Non-conservation of energy)*. It is interesting to check the energy (im)balance (3.6c). In Figure 2(left), we can see it depicted for $t_1 = 0$ as a function of time t_2: the upper line is the right-hand side of (3.6c) while the line below is the left-hand side of (3.6c). We can clearly see that the difference is not zero and is increasing in time, which is in accord with (3.6c) because otherwise, if the difference would decrease on some time interval $[t_1, t_2]$, (3.6c) could not be valid on this interval. This non-vanishing difference between the left- and the right-hand sides of (3.6c) is likely not because of a possible numerical possible error (as Fig. 4 shows a nice convergence) but seems to have a physical meaning that some part of energy is lost (dissipated) due to some neglected dissipation mechanisms which would guarantee energy conservation. This would certainly happen if the bulk viscosity would be completely neglected, as now well documented in [27, 31]. Here, although we have bulk viscosity, this viscosity is of a slower growth than the stored energy and thus might not be strong enough for fast processes, as observed at the fast completion of the delamination process here. Nevertheless, superquadratic growth of the viscosity would recast our model to the so-called doubly-nonlinear problem and would bring additional mathematical difficulties.

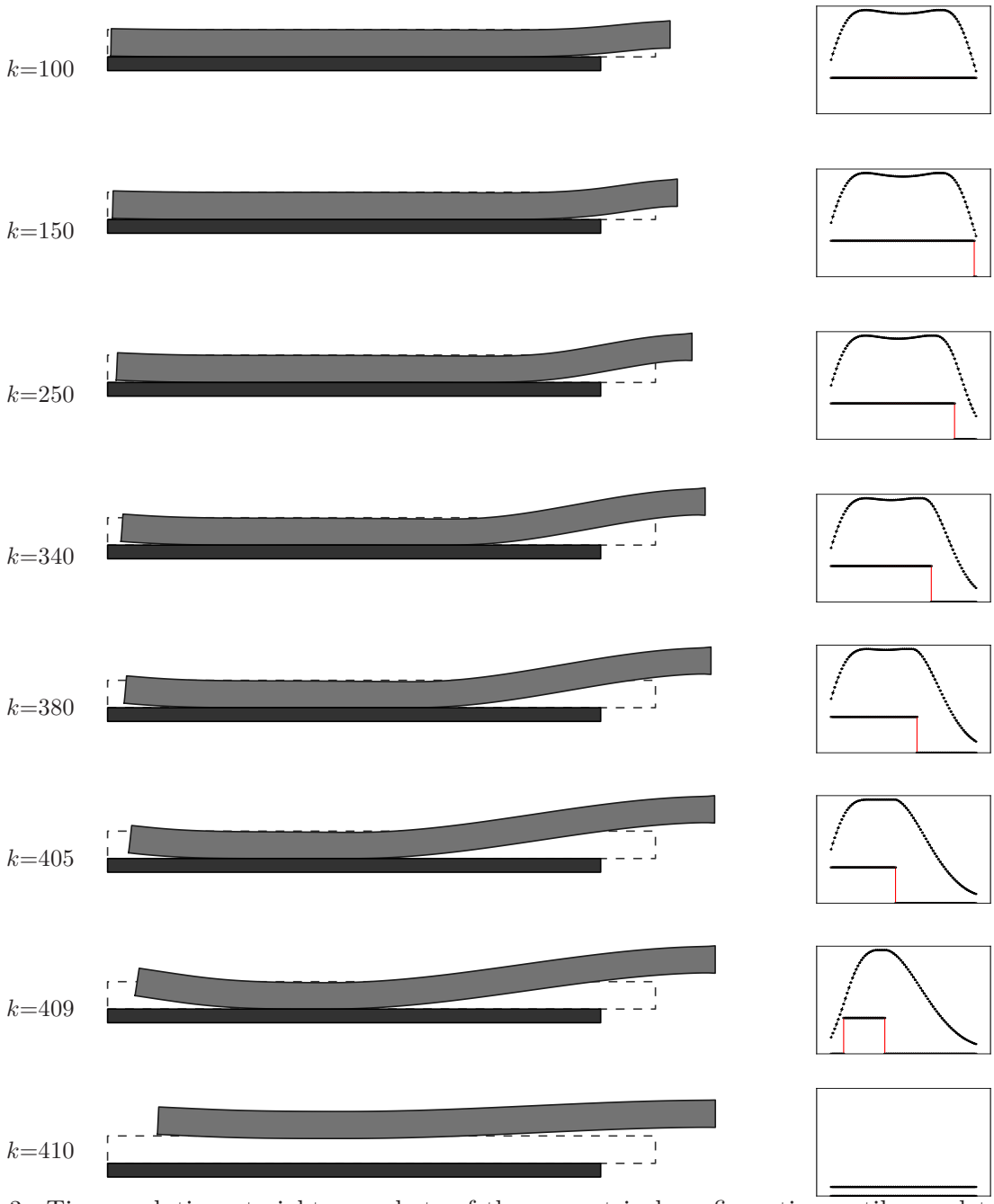

Fig. 3. Time evolution at eight snapshots of the geometrical configuration until complete delamination (displacement depicted magnified 100 ×) and the spacial distribution of the mode-mixity angle $\psi_{\rm G}$ and the delamination parameter z along the interface $\Gamma_{\rm C}$.

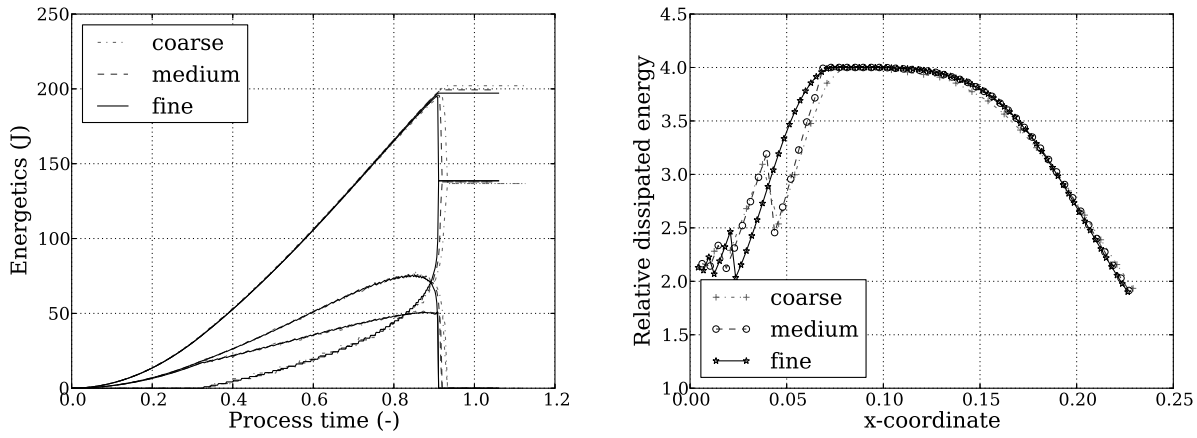

Fig. 4. Convergence test of the quantities from Figure 2. Coarse = 27 elements on $\Gamma_{\rm C}$, medium = 54 elements, and fine = 81 elements. The ratio τ/h is constant.

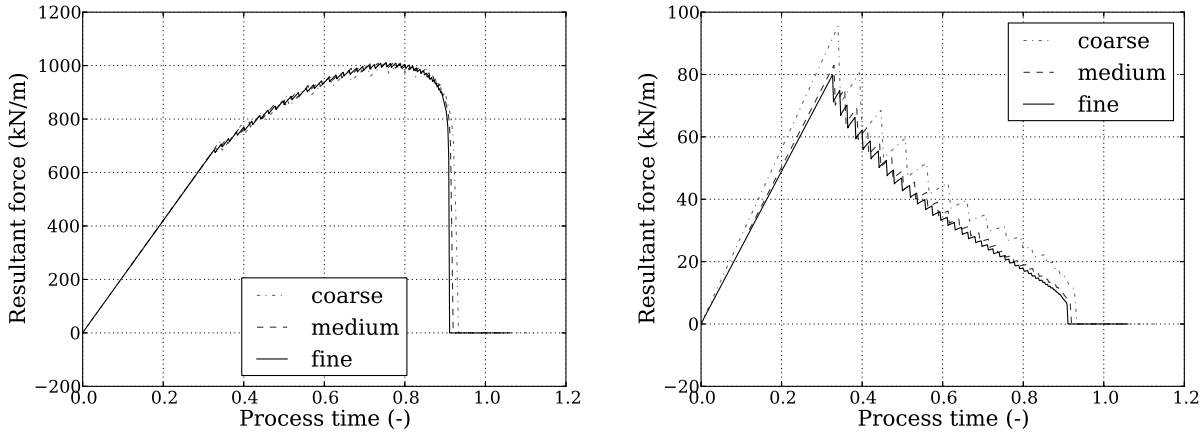

Fig. 5. Convergence test of the total force response evolving in time: the horizontal (left) and the vertical (right) components.

Remark 4.2 (*Comparison with other models/solutions).* As the viscosity has been chosen low, this example bears a good qualitative comparison (including the energy gap discussed in Remark 4.1) with the associative inviscid model with interfacial plasticity from [29,30], provided maximally-dissipative local solutions are considered [32]. The non-associative model allows for a general monotone dependence $a(\cdot)$ not necessarily just the special ansatz (1.2) which enables a possible fitting to that associative model which involves only few parameters. Also, the non-associative model allows do not involve any gradients of z which allows simpler computational implementation, as mentioned already above. On the other hand, the "vanishing-viscosity" asymptotics of the non-associative model towards the inviscid, rate-independent model is not entirely clear, cf. Remark 5.4 below.

5 Appendix: convergence

In terms of the piecewise constant/affine interpolants introduced in Section 3, cf. (3.13), the discrete scheme (3.9) summed for $k=1,...,T/\tau$ can be written "more compactly" as:

$$\forall v\in L^1(I;V_h):\quad \int_{Q\setminus\Sigma_{\rm C}}\Big(\mathbb{D}e\big(\dot{\bar u}_{\tau h}\big)+\mathbb{C}\big(e(\bar u_{\tau h})\big)\Big):e(v-\bar u_{\tau h})\,{\rm d}x{\rm d}t + \int_{\Sigma_C}\underline{z}_{\tau h}\mathbb{A}\big[\!\big[\bar u_{\tau h}\big]\!\big]\cdot\big[\!\big[v-\bar u_{\tau h}\big]\!\big]\,{\rm d}S{\rm d}t \geq \int_Q \bar F_\tau\cdot(v-\bar u_{\tau h})\,{\rm d}x{\rm d}t+\int_{\Sigma_N}\bar f_\tau\cdot(v-\bar u_{\tau h})\,{\rm d}S{\rm d}t \tag{5.1a}$$

$$\forall t_1<t_2\,,\ t_1,t_2\in\{k\tau\}_{k=1}^{T/\tau}:\quad \Phi\big(u_{\tau h}(t_2),z_{\tau h}(t_2)\big)+\int_{t_1}^{t_2}\mathcal{R}_1\big(\bar u_{\tau h};\dot z_{\tau h}\big)+\mathcal{R}_2\big(\dot u_{\tau h}\big)\,{\rm d}t \leq \Phi\big(u_{\tau h}(t_1),z_{\tau h}(t_1)\big)+\int_{t_1}^{t_2}\!\!\int_\Omega \bar F_\tau\cdot\dot u_{\tau h}\,{\rm d}x{\rm d}t+\int_{t_1}^{t_2}\!\!\int_{\Gamma_{\rm N}}\bar f_\tau\cdot\dot u_{\tau h}\,{\rm d}S{\rm d}t, \tag{5.1b}$$

$$\forall\tilde z\in Z_h:\qquad \Phi\big(\bar u_{\tau h}(t),\bar z_{\tau h}(t)\big)\leq\Phi\big(\bar u_{\tau h}(t),\tilde z\big)+\mathcal{R}\big(\bar u_{\tau h}(t);0,\tilde z-\bar z_{\tau h}(t)\big). \tag{5.1c}$$

We now enlist further assumptions needed for convergence:

$$\exists\eta>0\ \forall e_1,e_2\in\mathbb{R}^{d\times d}_{\rm sym}:\qquad (\varphi'(e_1)-\varphi'(e_2)):(e_1-e_2)\geq\eta|e_1-e_2|^p, \tag{5.2a}$$

$$\alpha\in C(\mathbb{R}^d;\mathbb{R}), \tag{5.2b}$$

$$\bigcup_{h>0}V_h \text{ dense in } W^{1,p}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d)\ \&\ V_{h_0}\subset V_{h_1}\text{ if }h_1<h_0\ . \tag{5.2c}$$

Proposition 5.1 (Unconditional convergence)**.** *Let the assumptions of Proposition 3.3 with $p>d$ as well as* (5.2) *be fulfilled. Then, for $\tau\to0$ and $h\to0$, the approximate solutions*

$(u_{\tau h}, z_{\tau h})$ converge in terms of subsequences in the sense that

$$u_{\tau h} \to u \qquad \text{in } L^q(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)), \tag{5.3a}$$

$$\bar z_{\tau h}(t) \overset{*}{\rightharpoonup} z(t) \qquad \text{in } L^\infty(\Gamma_{\rm C}) \qquad \text{for all } t\in\bar I \tag{5.3b}$$

for any $1\le q<\infty$, and any (u,z) obtained by such converging subsequences is a weak solution to the adhesive contact problem due to Definition 3.1. In particular, such weak solutions do exist.

Sketch of the proof. By the a-priori estimates (3.14) and by Banach's principle, we can select a weakly* converging subsequence in the spaces indicated in (3.14). Moreover, by Helly's selection principle, we can consider this subsequence so that (5.3b) holds.

Now we improve the weak* convergence $u_{\tau h} \overset{*}{\rightharpoonup} u$ in the space $L^\infty(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\cap H^1(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$ by proving also the strong convergence in $L^p(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$. To this goal, let us exploit the uniform monotonicity of the operator $-\mathrm{div}\,\mathbb{C}(e(\cdot))$, cf. (5.2a), and use (5.1a) for $v=2\bar u_{\tau h}-v_h$ with some $v_h\in L^\infty(I;V_h)$:

$$\begin{aligned}
\eta\|e(\bar u_{\tau h}-v_h)\|^p_{L^p(Q;\mathbb{R}^{d\times d})} &\le \int_{Q\backslash\Sigma_{\rm C}} \big(\mathbb{C}(e(\bar u_{\tau h}))-\mathbb{C}(e(v_h))\big):e(\bar u_{\tau h}-v_h)\,{\rm d}x{\rm d}t\\
&\le \int_{Q\backslash\Sigma_{\rm C}} \bar F_\tau\cdot(\bar u_{\tau h}-v_h) - \big(\mathbb{D}e(\dot u_{\tau h})+\mathbb{C}(e(v_h))\big):e(\bar u_{\tau h}-v_h)\,{\rm d}x{\rm d}t\\
&+\int_{\Sigma_{\rm C}} \underline{z}_{\tau h}\mathbb{A}\big[\![\bar u_{\tau h}]\!]\cdot\big[\![\bar u_{\tau h}-v_h]\!]\,{\rm d}S{\rm d}t + \int_{\Sigma_{\rm N}}\bar f_\tau\cdot(\bar u_{\tau h}-v_h)\,{\rm d}S{\rm d}t\\
&\le \int_{Q\backslash\Sigma_{\rm C}} \bar F_\tau\cdot(\bar u_{\tau h}-v_h) - \mathbb{C}(e(v_h)):e(\bar u_{\tau h}-v_h) - \mathbb{D}e(\dot u_{\tau h}):e(\bar u_{\tau h}-v_h)\,{\rm d}x{\rm d}t\\
&+\int_{\Sigma_{\rm C}} \underline{z}_{\tau h}\mathbb{A}\big[\![\bar u_{\tau h}]\!]\cdot\big[\![\bar u_{\tau h}-v_h]\!]\,{\rm d}S{\rm d}t + \int_{\Sigma_{\rm N}}\bar f_\tau\cdot(\bar u_{\tau h}-v_h)\,{\rm d}S{\rm d}t - \int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}e(\dot u_{\tau h}):e(\bar u_{\tau h}-v_h)\,{\rm d}x{\rm d}t\,.
\end{aligned} \tag{5.4}$$

The last term can be estimated as follows, which shows its nonpositivity

$$\begin{aligned}
-\int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}e(\dot u_{\tau h}):e(\bar u_{\tau h}-v_h)\,{\rm d}x{\rm d}t &= \sum_{k=1}^{T/\tau}\int_{\Omega\backslash\Gamma_{\rm C}}\mathbb{D}e(u^{k-1}_{\tau h}-u^k_{\tau h}):e(u^k_{\tau h}) + \int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}e(\dot u_{\tau h}):e(v_h)\,{\rm d}x{\rm d}t\\
&\le \sum_{k=1}^{T/\tau}\int_{\Omega\backslash\Gamma_{\rm C}}\frac12\mathbb{D}e(u^{k-1}_{\tau h}):e(u^{k-1}_{\tau h}) - \frac12\mathbb{D}e(u^k_{\tau h}):e(u^k_{\tau h})\,{\rm d}x + \int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}e(\dot u_{\tau h}):e(v_h)\,{\rm d}x{\rm d}t\\
&= \frac12\int_{\Omega\backslash\Sigma_{\rm C}}\mathbb{D}e(u_0):e(u_0) - \mathbb{D}e(u^{T/\tau}_{\tau h}):e(u^{T/\tau}_{\tau h})\,{\rm d}x + \int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}e(\dot u_{\tau h}):e(v_h)\,{\rm d}x{\rm d}t.
\end{aligned} \tag{5.5}$$

Now, using (5.2c), we take $v_h\to u$ in $L^p(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$ and notice that the right-hand side in (5.5) tends to zero which forces the left-hand side of the expression in (5.4) to vanish, as well. This yields strong convergence $\bar u_{\tau h}\to u$ in $L^p(I;W^{1,p}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))$. By interpolation of $L^\infty(I;W^{1,p}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))$ and $L^p(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$, we eventually obtain (5.3a).

The limit passage in (5.1a) is simple with the kinds of convergence proved above.

For the limit passage in (5.1b), we can always consider $\tau>0$ so small, that chosen time instants $t_1<t_2$ are elements of $\{k\tau\}_k$ and we write

$$\begin{aligned}
\int_{t_1}^{t_2}\mathcal{R}_1\big(\bar u_{\tau h};\dot z_{\tau h}\big)\,{\rm d}t &= \int_{t_1}^{t_2}\int_{\Gamma_C}\alpha(\big[\![\overline{u}_{\tau h}]\!]\big)|\dot z_{\tau h}|\,{\rm d}S{\rm d}t\\
&= \int_{t_1}^{t_2}\int_{\Gamma_C}\alpha\big(\big[\![u_{\tau h}]\!]\big)|\dot z_{\tau h}|\,{\rm d}S{\rm d}t + \int_{t_1}^{t_2}\int_{\Gamma_C}\big(\alpha\big(\big[\![\overline{u}_{\tau h}]\!]\big)-\alpha\big(\big[\![u_{\tau h}]\!]\big)\big)|\dot z_{\tau h}|\,{\rm d}S{\rm d}t.
\end{aligned} \tag{5.6}$$

We can see that we need the convergence of $\alpha([\![u_{\tau h}]\!]) \to \alpha([\![u]\!])$ in $C(\overline{\Sigma}_{\rm C})$. We observe that the sequence $\{u_{\tau h} : I \to H^1(\Omega\backslash\Gamma_{\rm C})\}_{\tau>0,h>0}$ is equicontinuous because

$$\begin{aligned}\big\|u_{\tau h}(t_1)-u_{\tau h}(t_2)\big\|_{H^1(\Omega\backslash\Gamma_{\rm C})} &\le \Big\| \int_{t_1}^{t_2} \dot u_{\tau h}\,{\rm d}t\Big\|_{H^1(\Omega\backslash\Gamma_{\rm C})} \le \int_{t_1}^{t_2} \big\|\dot u_{\tau h}\big\|_{H^1(\Omega\backslash\Gamma_{\rm C})}{\rm d}t\\ &\le \big\|\dot u_{\tau h}\big\|_{L^2(I;H^1(\Omega\backslash\Gamma_{\rm C}))}\|1\|_{L^2([t_1,t_2])} = |t_1-t_2|^{1/2}\big\|\dot u_{\tau h}\big\|_{L^2(I;H^1(\Omega\backslash\Gamma_{\rm C}))}\end{aligned} \tag{5.7}$$

for any $0 \le t_1 < t_2 \le T$. By compactness of the embedding $W^{1,p}(\Omega\backslash\Gamma_{\rm C}) \subset W^{1-\epsilon,p}(\Omega\backslash\Gamma_{\rm C})$ and by Arzelá-Ascoli-type arguments based on (5.7), cf. [25, Lemma 7.10], we have the strong convergence $u_{\tau h} \to u$ in $C(\bar I; W^{1-\epsilon,p}(\Omega\backslash\Gamma_{\rm C}))$ with a small $\epsilon > 0$. By the trace operator $W^{1-\epsilon,p}(\Omega\backslash\Gamma_{\rm C}) \to C(\overline{\Gamma}_{\rm C})$, we then have also the convergence $[\![u_{\tau h}]\!] \to [\![u]\!]$ in $C(\overline{\Sigma}_{\rm C})$. Using (5.2b), we eventually have $\alpha([\![u_{\tau h}]\!]) \to \alpha([\![u]\!])$ in $C(\overline{\Sigma}_{\rm C})$. This allows us to pass to the limit in the first term of (5.6). Since $|\dot z_{\tau h}| \to |\dot z|$ weakly* in the sense of measures on $\overline{\Sigma}_{\rm C}$, we conclude that

$$\alpha\big([\![u_{\tau h}]\!]\big)|\dot z_{\tau h}| \to \alpha\big([\![u]\!]\big)|\dot z| \qquad \text{weakly* in the sense of measures on } \overline{\Sigma}_{\rm C}, \tag{5.8}$$

where $|\dot z|$ denotes the variation of the measure $\dot z$; in fact, here simply $|\dot z| = -\dot z$, since $\dot z \le 0$. Using again (5.7), we have

$$\big\|\bar u_{\tau h}-u_{\tau h}\big\|_{L^\infty(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))} \le \tau^{1/2}\big\|\dot u_{\tau h}\big\|_{L^2(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))} \tag{5.9}$$

and, by interpolation with the $L^\infty(I;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$-estimate, we also obtain that for $\epsilon > 0$ arbitrarily small $\big\|\bar u_{\tau h}-u_{\tau h}\big\|_{L^\infty(I;W^{1,p-\epsilon}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))} \to 0$. For $\epsilon < p-d$, we have still the continuous trace operator $W^{1,p-\epsilon}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \to C(\overline{\Gamma}_{\rm C})$. Using (5.2b), we also have $\alpha([\![\bar u_{\tau h}]\!]) - \alpha([\![u_{\tau h}]\!]) \to 0$ in $L^\infty(I;C(\overline{\Gamma}_{\rm C}))$. Since $\{\dot z_{\tau h}\}_{\tau>0,h>0}$ is bounded in $L^1(\Sigma_{\rm C})$, we then conclude that the second term on the right-hand side of (5.6) tends to zero as $\tau \to 0$. Thus, altogether $\int_{t_1}^{t_2}\mathcal{R}_1(\bar u_{\tau h};\dot z_{\tau h})\,{\rm d}t \to \int_{t_1}^{t_2}\mathcal{R}_1(u;\dot z)\,{\rm d}t$. Furthermore, by weak lower semicontinuity, we have $\liminf_{\tau\to0,\,h\to0}\int_{t_1}^{t_2}\mathcal{R}_2(\dot u_{\tau h})\,{\rm d}t \ge \int_{t_1}^{t_2}\mathcal{R}_2(\dot u)\,{\rm d}t$. If $t_1 \in [0,T)$ is such that $u_{\tau h}(t_1) \to u(t_1)$ strongly in $W^{1,p}(\Omega\setminus\Gamma_C;\mathbb{R}^d)$ then the remaining terms in (5.1b) can be handled even by semicontinuity and continuity to obtain (notice that φ is p-Lipschitz continuous)

$$\Phi\big(u(t_2),z(t_2)\big) + \int_{t_1}^{t_2}\mathcal{R}(u;\dot u,\dot z)\,{\rm d}t \le \Phi\big(u(t_1),z(t_1)\big) + \int_{t_1}^{t_2}\int_\Omega F\cdot\dot u\,{\rm d}x{\rm d}t + \int_{t_1}^{t_2}\int_{\Gamma_{\rm N}} f\cdot\dot u\,{\rm d}S{\rm d}t\,. \tag{5.10}$$

The limit passage in (5.1c) needs an explicit construction of a so-called mutual recovery sequence. Take $t \in [0,T]$ for which (5.1c) holds. We want to prove (3.6d). Take arbitrary $\tilde z \in L^\infty(\Gamma_{\rm C})$ such that $\tilde z \le z(t,\cdot)$ because otherwise the inequality holds trivially. Here we can take

$$\tilde z_{\tau h}(t,x) = \begin{cases} \bar z_{\tau h}(t,x)\Pi_h\Big(\frac{\tilde z(x)}{z(t,x)}\Big) & \text{if } z(t,x) \ne 0\\ 0 & \text{otherwise,}\end{cases} \tag{5.11}$$

where Π_h is the projector $L^\infty(\Gamma_{\rm C}) \to Z_h$ making piecewise constant averages, cf. [18, Sect.4.6]. We see that $\tilde z_{\tau h} \le z_{\tau h}$. Notice that $\Pi_h(\tilde z/z) \to \tilde z/z$ as $h \to 0$ in any $L^p(\Gamma_{\rm C})$ with $+\infty > p \ge 1$. Hence, $\tilde z_{\tau h} \to \tilde z$ weakly* in $L^\infty(\Gamma_{\rm C})$. We get from (5.1c) that

$$\begin{aligned}0 &\le \lim_{\tau\to0}\lim_{h\to0}\big(\Phi(\bar u_{\tau h}(t),\tilde z_{\tau h}) + \mathcal{R}_1(\bar u_{\tau h}(t),\tilde z_{\tau h}-\bar z_{\tau h}(t)) - \Phi(\bar u_{\tau h}(t),\bar z_{\tau h})\big)\\ &= \lim_{\tau\to0}\lim_{h\to0}\int_{\Gamma_{\rm C}}\Big(\frac12\mathbb{A}[\![\bar u_{\tau h}]\!]\cdot[\![\bar u_{\tau h}]\!] + \alpha[\![\bar u_{\tau h}]\!]\Big)\,(\tilde z_{\tau h}-\bar z_{\tau h}(t))\,{\rm d}S\\ &= \int_{\Gamma_{\rm C}}\Big(\frac12\mathbb{A}[\![u]\!]\cdot[\![u]\!] + \alpha[\![u]\!]\Big)(\tilde z-z(t))\,{\rm d}S = \Phi(u(t),\tilde z) + \mathcal{R}_1(u(t),\tilde z-z(t)) - \Phi(u(t),z(t))\,,\end{aligned}$$

which proves (3.6d).

As the considered finite-element discretization always exists as well as solutions to the recursive problem (3.12), the existence of an energetic solution is proved by this constructive method. □

Remark 5.2 (*Energy conservation).* The equality in (3.6c) is still unclear. Let us outline the difficulties. Within this model, the proof would need (2.5a) to hold for $p = 2$, which however contradicts our assumption $p > d$ in the physically interesting multidimensional cases. Assume further also that (5.2a) hold with $p = 2$, so that we also have

$$\exists C > 0\ \forall e \in \mathbb{R}^{d\times d}_{\rm sym}: \qquad |\varphi'(e)| \le C(1 + |e|)\ . \tag{5.12}$$

Then we can prove the energy inequality opposite to (3.6c). Consider for simplicity $t_1 = 0$ and $t_2 = T$. Following [24] we notice that

$$\Phi(u(T), z(T)) - \Phi(u_0, z_0) + \int_{\overline{\Sigma}_{\rm C}} \alpha(\llbracket u \rrbracket)|\dot z|\,{\rm d}S{\rm d}t \ge \int_0^T \big\langle \lambda, \dot u\big\rangle\,{\rm d}t\ , \tag{5.13}$$

where $\lambda \in L^2(0,T; W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)$ with $\lambda(t) \in \partial_u\Phi(u(t), z(t))$ for a.a. $t \in (0,T)$ and where "$|\dot z|\,{\rm d}S{\rm d}t$" denotes the integration with respect to the Borel measure $|\dot z|$ on $\overline{\Sigma}_{\rm C}$. We have that $\lambda \in \partial_u\Phi(u,z)$ if and only if there is $\ell \in \partial\mathcal{J}(u)$ and for all $v \in W^{1,p}(\Omega \setminus \Gamma_{\rm C};\mathbb{R}^d)$

$$\big\langle \lambda, v\big\rangle = \int_\Omega \mathbb{C}(e(u)){:}e(v)\,{\rm d}x + \int_{\Gamma_{\rm C}} z\mathbb{A}\llbracket u\rrbracket\cdot\llbracket v\rrbracket\,{\rm d}S + \big\langle \ell, v\big\rangle\ ,$$

where $\langle\cdot,\cdot\rangle$ denotes the appropriate duality pairing. Here the convex function $\mathcal{J} : H^1(\Omega \setminus \Gamma_{\rm C};\mathbb{R}^d) \to [0;+\infty]$ is defined as follows

$$\mathcal{J}(u) = \begin{cases} 0 & \text{if } \llbracket u\rrbracket\cdot\vec{n}_{\rm C} \ge 0\ , \\ +\infty & \text{otherwise.}\end{cases}$$

In order to prove (5.13) for a chosen selection $\lambda(t) \in \partial_u\Phi(u(t), z(t))$ we use (3.6d) and approximations of Stieltjes integrals by Riemann sums; [5]. Take $n \in \mathbb{N}$ and a partition $0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = T$ such that $\max_i(t_{i+1}^n - t_i^n) < 1/n$ and such that the function $\mathcal{A}_n : [0,T] \to L^\infty(\Gamma_{\rm C})$ defined as $\mathcal{A}_n(t) = \alpha(\llbracket u(t_{i-1}^n)\rrbracket)$ if $t_{i-1}^n < t \le t_i^n$. We can assume, without loss of generality, that (3.6d) holds for all timesteps $\{t_i^n\}_{i=0}^{N_n-1}$. The sequence $\{\mathcal{A}_n\}$ satisfies

$$\mathcal{A}_n \to \alpha(\llbracket u\rrbracket) \text{ in } L^\infty(\Gamma_{\rm C}) \text{ as } n \to \infty\ . \tag{5.14}$$

This limit passage follows from the fact that $u \in H^1([0,T]; H^1(\Omega \setminus \Gamma_{\rm C};\mathbb{R}^d))$, so that $u : [0,T] \to H^1(\Omega \setminus \Gamma_{\rm C};\mathbb{R}^d)$ is Hölder continuous on $[0,T]$, which implies uniform continuity of $\llbracket u\rrbracket : [0,T] \to L^\infty(\Gamma_{\rm C})$. Finally, (5.14) follows by approximations of uniformly continuous functions by piecewise constant interpolants. Further, we test (3.6d) at t_{i-1}^n against $\tilde z = z(t_i^n)$ to get

$$\begin{aligned}\Phi(u(t_{i-1}^n), z(t_{i-1}^n)) &\le \Phi((u(t_{i-1}^n), z(t_i^n)) + \mathcal{R}_1(u(t_{i-1}^n), z(t_i^n) - z(t_{i-1}^n))\\ &= \Phi((u(t_i^n), z(t_i^n)) - \int_{\Gamma_{\rm C}} \mathcal{A}_n(t)(z(t_i^n) - z(t_{i-1}^n))\,{\rm d}S - \int_{t_{i-1}^n}^{t_i^n} \big\langle \lambda_n(s), \dot u(s)\big\rangle\,{\rm d}s\end{aligned} \tag{5.15}$$

whenever $\lambda_n(s) \in \partial_u\Phi(u(s), z(t_i^n))$ for any $s \in (t_{i-1}^n, t_i^n]$. The last equality is based on the chain rule for $\frac{\rm d}{{\rm d}t}\Phi(u(t), z(t_i^n))$; cf. [43, Prop.IX.4.11]. Notice that $\langle\lambda_n, \dot u\rangle$ is well-defined even for $\dot u \in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ due to (5.12). Summing up (5.15) for $i = 1,\ldots,N_n$ and passing to the

limit inferior for $n\to\infty$ (in fact, the for the dissipative term on the left hand side even the limit exists), we have in view of (5.14) and [24] that

$$\Phi(u(T),z(T))-\Phi(u_0,z_0)-\int_{\overline{\Sigma}_{\rm C}}\alpha(\llbracket u\rrbracket)\dot z\,{\rm d}S{\rm d}t\ge\int_0^T\big\langle\lambda(s),\dot u(s)\big\rangle\,{\rm d}s\ .$$

Further, notice that $\int_{t_1^n}^{t_2^n}\langle\ell,\dot u\rangle\,{\rm d}s=\mathcal{J}(u(t_2^n))-\mathcal{J}(u(t_1^n))=0$, hence we have

$$\Phi(u(T),z(T))-\Phi(u_0,z_0)-\int_{\overline{\Sigma}_{\rm C}}\alpha(\llbracket u\rrbracket)\dot z\,{\rm d}S{\rm d}t\ge\int_Q\mathbb{C}(e(u)){:}e(\dot u)\,{\rm d}x{\rm d}t+\int_{\Sigma_{\rm C}}z\mathbb{A}\llbracket u\rrbracket\cdot\llbracket\dot u\rrbracket\,{\rm d}S\,{\rm d}t\,.\tag{5.16}$$

Taking into account (3.6b) we see that $\mathfrak{J}$ defined by

$$\big\langle\mathfrak{J},v\big\rangle:=\int_{\Omega\setminus\Gamma_{\rm C}}\big(\mathbb{D}e(\dot u)+\mathbb{C}(e(u))\big){:}e(v)\,{\rm d}x+\int_{\Gamma_{\rm C}}z\mathbb{A}\llbracket u\rrbracket\cdot\llbracket v\rrbracket{\rm d}S-\int_\Omega F\cdot v\,{\rm d}x-\int_{\Gamma_{\rm N}}f\cdot v\,{\rm d}S\tag{5.17}$$

is such that

$$\int_0^T\mathcal{J}(v)\,{\rm d}t-\int_0^T\mathcal{J}(u)\,{\rm d}t\ge\int_0^T\big\langle\mathfrak{J},v{-}u\big\rangle\,{\rm d}t\ ,$$

i.e. $\mathfrak{J}(t)\in\partial\mathcal{J}(u(t))$. Therefore, $\int_0^T\langle\mathfrak{J},\dot u\rangle\ {\rm d}t=0$ and together with (5.16) we get that

$$\Phi\big(u(t),z(t)\big)+\int_0^t\mathcal{R}(u;\dot u,\dot z)\,{\rm d}t\ge\Phi\big(u_0,z_0\big)+\int_0^t\int_\Omega F\cdot\dot u\,{\rm d}x{\rm d}t+\int_0^t\int_{\Gamma_{\rm N}}f\cdot\dot u\,{\rm d}S{\rm d}t\ .$$

Remark 5.3 (*Linear but nonlocal elasticity).* We can also consider linear but nonlocal stress

$$\sigma=\mathbb{D}e(\dot u)+\mathbb{C}e(u)+\int_\Omega\mathbb{H}(x,\xi)\big(e(u(x){-}u(\xi))\big)\,{\rm d}\xi,\tag{5.18}$$

with some (presumably only small) 4th-order tensor $\mathbb{H}$. This yields a contribution

$$\frac12\sum_{i,j,k,l=1}^d\int_{\Omega\times\Omega}\mathbb{H}_{ijkl}(x,\xi)e_{ij}(u(x){-}u(\xi))e_{kl}(u(x){-}u(\xi))\,{\rm d}\xi{\rm d}x$$

to the stored energy. Assuming that, for some $\epsilon>0$ and $\zeta>0$ and for all $x,\xi\in\Omega$, it holds $\mathbb{H}_{ijkl}(x,\xi)\ge\zeta\delta_{ik}\delta_{jl}/|x{-}\xi|^{d+2\epsilon}$, we obtain estimates in $W^{1+\epsilon,2}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d)$ instead of $W^{1,p}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d)$ we used above. For $d=2$, any $0<\epsilon<1$ is sufficient for the convergence and, for $\epsilon<1/2$, we can still use P1-elements for the space discretization. For $d=3$, $\epsilon>1/2$ is needed for the compact embedding $W^{1+\epsilon,2}(\Omega\setminus\Gamma_{\rm C})\subset C(\bar\Omega_+\cup\bar\Omega_-)$ and thus for the convergence, which unfortunately excludes discontinuities of gradients and thus P1-element so that P2-elements would have to be used. In the latter case, one could also consider the concept of nonsimple materials. A definite benefit would be an energy conservation [24].

Remark 5.4 (*Towards the rate-independent evolution).* We can replace $\mathbb{D}$ with $\varepsilon\mathbb{D}$ in the model and, like in the mixity-insensitive case [27, 31], study the vanishing-viscosity limit for $\varepsilon\searrow0$ towards the rate-independent limit. Let us denote $(u_\varepsilon,z_\varepsilon)$ a weak solution to the problem (2.4), i.e., it satisfies Definition 3.1 with our modified viscosity. Then we get the following a-priori estimate $\sqrt{\varepsilon}\|e(\dot u_\varepsilon)\|_{L^2(Q\setminus\Sigma_{\rm C})}\le S_0$. We consider a weak$*$ limit $(u,z)\in L^\infty(0,T;W^{1,p}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))\times(L^\infty(\Sigma_{\rm C})\cap{\rm BV}([0,T];L^1(\Gamma_{\rm C})))$ of a (sub)sequence of solutions for

$\varepsilon\searrow 0$. Further, using (5.2a), we estimate for $\varepsilon\searrow 0$ that:

$$\begin{aligned}\eta\big\|e(u_\varepsilon{-}u)\big\|^p_{L^p(Q;\mathbb{R}^{d\times d})} &\le \int_{Q\setminus\Sigma_{\rm C}} \big(\mathbb{C}(e(u_\varepsilon)) - \mathbb{C}(e(u))\big){:}e(u_\varepsilon{-}u)\,{\rm d}x{\rm d}t\\ &\le \int_{Q\setminus\Sigma_{\rm C}} (\mathbb{C}(e(u_\varepsilon)) - \mathbb{C}(e(u))){:}e(u_\varepsilon{-}u)\,{\rm d}x{\rm d}t\\ &\qquad + \int_{\Sigma_{\rm C}} z_\varepsilon \mathbb{A}\big[\!\big[u_\varepsilon{-}u\big]\!\big]\cdot\big[\!\big[u_\varepsilon{-}u\big]\!\big]\,{\rm d}S{\rm d}t\\ &\le -\int_{Q\setminus\Sigma_{\rm C}} \varepsilon\mathbb{D}e(\dot u_\varepsilon){:}e(u_\varepsilon{-}u) + \mathbb{C}(e(u)){:}e(u_\varepsilon{-}u)\,{\rm d}x{\rm d}t\\ &\qquad - \int_{\Sigma_{\rm C}} z_\varepsilon\mathbb{A}\big[\!\big[u\big]\!\big]\cdot\big[\!\big[u_\varepsilon{-}u\big]\!\big] + (z_\varepsilon{-}z)\mathbb{A}\big[\!\big[u_\varepsilon\big]\!\big]\cdot\big[\!\big[u_\varepsilon\big]\!\big]\,{\rm d}x{\rm d}t\\ &\qquad + \int_Q F\cdot(u_\varepsilon{-}u)\,{\rm d}x{\rm d}t + \int_{\Sigma_{\rm N}} f\cdot(u_\varepsilon{-}u)\,{\rm d}S{\rm d}t \to 0\ .\end{aligned}$$

Hence, we have for almost every $t\in[0,T]$ that $e(u_\varepsilon(t))\to e(u(t))$ in $L^p(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^{d\times d})$. We thus obtain an approximable solution in the sense of [7,12,41] to the rate-independent mixity-sensitive model. Moreover, also (3.6d) holds for a.a. $t\in[0,T]$. It is not obvious, however, if it holds that $\liminf_{\varepsilon\to 0}\int_{t_1}^{t_2}\int_{\Gamma_{\rm C}}\alpha([\![u_\varepsilon]\!])\dot z_\varepsilon\,{\rm d}S{\rm d}t \ge \int_{t_1}^{t_2}\int_{\Gamma_{\rm C}}\alpha([\![u]\!])\dot z\,{\rm d}S{\rm d}t$, which is needed to pass to the limit in the energy (im)balance. In this sense, the associative model from [29–32] remains still more justified for the merely rate-independent mixity-sensitive evolution.

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