ANALYSIS AND APPROXIMATION OF A STRAIN-LIMITING NONLINEAR ELASTIC MODEL

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Abstract. Elastic solids with strain-limiting response to external loading represent an interesting class of material models, capable of describing stress concentration at strains with small magnitude. A theoretical justification of this class of models comes naturally from implicit constitutive theory. We investigate mathematical properties of static deformations for such strain-limiting nonlinear models. Focusing on the spatially periodic setting, we obtain results concerning existence, uniqueness and regularity of weak solutions, and existence of renormalized solutions for the full range of the positive scalar parameter featuring in the model. These solutions are constructed via a Fourier spectral method. We formulate a sufficient condition for ensuring that a renormalized solution is in fact a weak solution, and we comment on the extension of the analysis to non-periodic boundary-value problems.

1. Introduction

The recently developed *implicit constitutive theory* (see [10], [11]) expands quite considerably the possibilities for describing nonlinear responses of materials, even though the quantities involved in implicit constitutive models are the same as in classical linear models, which bear the names of Hooke, Lamé, Navier, Stokes, Darcy, and others. Another significant feature of the implicit constitutive theory is that it provides a firm theoretical foundation for various models in fluid and solid mechanics that were proposed by engineers, physicists and chemists in an ad hoc manner.

Concerning solids, that are the subject of this study, one of the main achievements of implicit constitutive theory is in providing a theoretical background and justification for nonlinear models involving the linearized strain. In particular, it is thus possible to have models in which the linearized strain is in all circumstances a bounded function, even when the stress is very large. This class of implicit constitutive models, developed by Rajagopal in [11], and which are referred to as *limiting strain models*, has the potential to be useful in describing the behavior of brittle materials near crack tips or notches, or concentrated loads inside the body or on its boundary. Both of these effects lead to stress concentration even though the gradient of the displacement is small. It is these limiting linearized strain models[1] that are the subject of the present study, which is focused on the mathematical analysis of the existence and uniqueness of solutions to boundary-value problems in these models.

Limiting strain models have been thus far studied in several situations. Firstly, in the case of special deformations such as shearing, compressions, torsion, etc., Rajagopal himself, and Bustamante and Rajagopal aimed to assess whether the models exhibit the expected responses (cf. [2], [14], [13]). Secondly, in the case of anti-plane strain (stress) problems, considered in domains with nonconvex cross-sections (including thus the domains with V-notches or cracks), the resulting scalar problem in two space dimensions has been analyzed by methods of asymptotic analysis in

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[1]Models with limiting *finite* strain are found to be useful in modeling the response of various soft tissues that exhibit the phenomenon of finite extensibility. Also, Rajagopal's elastic models stemming from implicit constitutive theory seem to provide good description of Fung's experimental data concerning the passive response of biological tissues that indicate that the stress/strain response of the tissue is, to a good approximation, exponential. Referring to Fung's experimental results, Freed in his book [4] states: "Hooke's law is applicable for infinitesimal strains. Fung's law is applicable for moderate strains. The implicit theory of elasticity, or Rajagopal's elasticity, is applicable at finite strains." We refer also to [5] for another important application of elastic models based on implicit constitutive theory.

[17], by performing systematic computational tests in [8], and by analytical methods from the theory of nonlinear partial differential equations in [1]; the last result establishes the existence of weak solutions in nonconvex domains for values of the model parameter r in the range $r \in (0,2)$, see equation (4) below, and in convex domains for the range $r \in (0,\infty)$. Thirdly, a detailed computational study of the complete problem in planar domains was performed in Ortiz et al. [9].

The present work is the first one with focus on the mathematical analysis of general boundary-value problems (which include systems of $\frac{1}{2}d(d+3)$ time-independent nonlinear partial differential equations of first order), featuring in limiting strain models, in bounded subsets of $\mathbb{R}^d$, $d \geq 2$.

2. Formulation of the problem, simplifications and main results

Let $\Omega \subset \mathbb{R}^d$ denote the set occupied by an elastic body being at a static state achieved as a result of the action of body forces $f : \Omega \to \mathbb{R}^d$ and traction forces $g : \Gamma_N \to \mathbb{R}^d$ whereas we assume that the boundary $\partial\Omega$ of the set Ω consists of two parts, Γ_N and Γ_D, and the displacement $u : \Omega \to \mathbb{R}^d$ is given on Γ_D. Assuming that the response of the body is described by a constitutive equation relating the Cauchy stress tensor $T : \Omega \to \mathbb{R}^{d\times d}_{\rm sym}$ and the deformation gradient[2] through the Galilean invariant Cauchy–Green deformation tensor B implicitly (see [10], [11], [12]), then we arrive at the following problem: find u and T such that

$$
\begin{aligned}
-\operatorname{div} T = f, \quad G(T,B) &= 0 && \text{in } \Omega,\\
Tn &= g && \text{on } \Gamma_N,\\
u &= 0 && \text{on } \Gamma_D,
\end{aligned}
\tag{1}
$$

where $G : \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$ is given and n stands for the outer unit normal vector to the boundary of Ω. One should consider only those relations $G(T,B) = 0$ that are thermodynamically consistent, which means that the relations $G(T,B) = 0$ should automatically guarantee that there is no dissipation of energy associated with the class of materials considered and the responses considered are elastic. This compatibility of the model with the laws of thermodynamics is addressed in [15] and [16].

Problem (1) includes as a special case models described by explicit relations of the form

$$B = H(T).$$

If the material is isotropic, then a representation theorem leads to an expression of the form

$$B = \alpha_0 I + \alpha_1 T + \alpha_2 T^2,$$

where α_i, $i = 0,1,2$, are functions of the invariants of T, i.e., $\alpha_i = \alpha_i(\operatorname{tr} T, \operatorname{tr} T^2, \operatorname{tr} T^3)$.

If we assume that the displacement gradient is small in the sense that $\sup_{x\in\Omega} |\nabla u(x)| \ll 1$, and if we set $\alpha_2 = 0$, we obtain

$$\varepsilon = \alpha_0 I + \alpha_1 T \qquad \text{with} \qquad \varepsilon = \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

Within this framework Rajagopal proposed (see [14], [13]) several limiting strain models that can be described by the relation

$$\varepsilon = \alpha_0(\operatorname{tr} T, \operatorname{tr} T^2) I + \frac{T}{\mu_0(1 + (\operatorname{tr} T^2)^{r/2})^{1/r}} = \alpha_0(\operatorname{tr} T, \operatorname{tr} T^2) I + \frac{T}{\mu_0(1 + |T|^r)^{1/r}}, \tag{2}$$

which we aim to analyze. The parameter μ_0 is a positive constant.

In this study, we make several simplifications, which we shall now specify. We shall also explain our reasons for doing so.

Firstly, we neglect the spherical part of the Cauchy stress and we deal only with one operator that encodes the key mathematical difficulties. Neglecting the spherical part of the Cauchy stress

[2] Since we are studying a simplified problem throughout the whole paper, we restrict ourselves to introducing only those concepts that we need. Let $\mathcal{X} \mapsto \chi(\mathcal{X}) =: x$ represent the motion of the body assigned to a typical point $\mathcal{X}$ in the reference configuration whose current position is x. We then define the displacement vector $u := x - \mathcal{X}$, the deformation gradient $F := \frac{\partial \chi}{\partial \mathcal{X}}$, the (left and right) Cauchy–Green deformation tensors $B = FF^T$ and $C = F^T F$, and the linearized strain tensor $\varepsilon = \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$.

helps us to simplify the presentation. We believe that retaining the neglected spherical term will not alter the mathematical analysis in an essential way.

Secondly, we consider a domain of a special form: namely an axiparallel parallelepiped, with spatially periodic boundary conditions in the various co-ordinate directions. This essential simplification helps us to introduce not only the concept of weak solution to the problem under consideration, but also the concept of a renormalized solution. The spatially periodic setting also helps us to *construct* the solution via a specific numerical method, namely the Fourier spectral method. Thus our proof of existence of weak and renormalized solutions to the model is at the same time a proof of the convergence of the sequence of numerical approximations to the unknown analytical solution.

Thirdly, in the periodic setting the various bounds that are obtained on the sequence of approximating solutions are, in a sense, optimal, as there are no boundary effects, and the analysis therefore highlights the ideal objective that one should aim for in the case of other (mixed) boundary-value problems in general domains. We shall state some of the relevant open problems and conjectures concerning the extension of the present analysis to such nonperiodic boundary-value problems at the end of the paper.

These simplifications also allow us to provide a fairly complete picture regarding the existence and uniqueness of solutions for a nontrivial example of a strain-limiting nonlinear elastic model.

Since standard notations in continuum mechanics and in the theory of partial differential equations differ and since we are neglecting the spherical part of the Cauchy stress T, we make the following two changes to our notation: henceforth we shall write

$$S \text{ instead of } T \quad \text{ and } \quad D(u) \text{ instead of } \varepsilon = \varepsilon(u).$$

We also set $\mu_0 = 1$.

The problem under consideration here is thus the following: suppose that $\Omega = (0, 2\pi)^d$, with $d \geq 2$, $r > 0$ is a parameter in the model, and f is a given d-component vector-function (the load-vector), which is 2π-periodic in each of the d co-ordinate directions. The objective is to show the existence of a unique pair (S, u), where S is the stress tensor and u is the displacement, which belong to suitable function spaces consisting of $d \times d$ matrix functions and d-component vector functions, respectively, that are 2π-periodic in each co-ordinate direction, such that

$$-\operatorname{div} S = f \tag{3}$$

and

$$D(u) = S(1 + |S|^r)^{-\frac{1}{r}}. \tag{4}$$

In terms of the parameter r featuring in the model the main results of the paper are the following:

(a) for $r \in \left(0, \frac{2}{d}\right)$, we prove the existence of a unique weak solution to the problem;

(b) for $r \in \left[\frac{2}{d}, \infty\right)$, we prove the existence of a renormalized solution to the problem and specify the conditions on the regularity of the stress tensor S that suffice in order to deduce that the renormalized solution is in fact a weak solution.

We note that despite its geometrical simplicity, one can still use the framework presented here to study the effects of concentrated loads that are active in the neighborhood of the center point of the periodic cell, assuming that the side-lengths of the cell are large enough so that the effects of concentrations are not effective in the neighbourhood of the boundary of the cell.

The paper is structured as follows. In Section 3 we define the sequence $\{(S_N, u_N)\}_{N\geq 1}$ of approximating solutions, which our subsequent weak compactness argument will be based upon, and we show the existence and uniqueness of the approximating solution for given f and $r > 0$. In Section 4 we prove appropriate bounds on S_N and u_N, which then permit us to extract a subsequence from the sequence $\{(S_N, u_N)\}_{N\geq 1}$ that converges to a limiting object (S, u), which is then identified as a weak solution to the problem under consideration. Having done so, we show that the weak solution is unique. The analysis in Section 4 relies on certain uniform bounds (with respect to N) on S_N and u_N, which are only valid for $r \in \left(0, \frac{2}{d}\right)$. In order to show the existence of solutions for $r \in \big[\frac{2}{d}, \infty\big)$, we shall introduce in Section 5 the notion of renormalized solution to the problem and will prove the existence of a renormalized solution for all $r \in \big[\frac{2}{d}, \infty\big)$. As was already noted above, we shall also clarify the conditions on the regularity of the stress tensor S that suffice to ensure that the renormalized solution is in fact a weak solution. In the Conclusion, we relate the results achieved in the spatially periodic setting to possible results and open problems regarding the analysis of the more general problem (1). The paper closes with an Appendix, which contains the proofs of various Korn-type inequalities in periodic Lebesgue spaces, that we were unable to find in the literature and have therefore decided to include for the sake of completeness.

3. Definition of the approximation: existence and uniqueness of solutions

Consider the domain $\Omega := (0, 2\pi)^d$ in $\mathbb{R}^d$, $d \geq 2$. All function spaces consisting of real-valued 2π-periodic functions (by which we mean 2π-periodic in each of the d co-ordinate directions) will be labelled with the subscript #; subspaces of these, consisting of 2π-periodic functions whose integral over Ω is equal to 0, will be labelled with the subscript $*$; in order to avoid notational clutter we shall not use the symbols # and $*$ in the various norm signs. It will be clear from the argument of the norm which of the symbols # or $*$ is intended. For example, $L^p_{\#}(\Omega)$ will denote the Lebesgue space of all real-valued 2π-periodic functions v such that $|v|^p$ is integrable of Ω, equipped with the norm $\|\cdot\|_{L^p(\Omega)}$. It is understood that the usual modification is made when $p = \infty$. Spaces of d-component vector functions, where each component belongs to a certain function space X, will be denoted by $[X]^d$, while spaces of $d \times d$ component matrix functions each of whose components is an element of X will be signified by $[X]^{d\times d}$. Letting $C^\infty_{\#}(\overline{\Omega})$ denote the linear space consisting of the restriction to $\overline{\Omega}$ of all real-valued 2π-periodic C^∞ functions defined on $\mathbb{R}^d$, we note that $C^\infty_{\#}(\overline{\Omega})$ is dense in $L^p_{\#}(\Omega)$ for all $p \in [1, \infty)$; analogously, $C^\infty_*(\overline{\Omega})$ is dense in $L^p_*(\Omega)$ for $1 \leq p < \infty$. The Sobolev space $W^{1,p}_{\#}(\Omega)$, $1 \leq p < \infty$, will be defined as the closure of $C^\infty_*(\overline{\Omega})$ in the Sobolev norm $\|\cdot\|_{W^{1,p}(\Omega)}$, where

$$\|v\|_{W^{1,p}(\Omega)} := \Big(\|v\|^p_{L^p(\Omega)} + \|\nabla v\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}};$$

here, $\|\nabla v\|_{L^p(\Omega)} := \||\nabla v|\|_{L^p(\Omega)}$, where $|\nabla v|$ denotes the Euclidean norm of ∇v. Analogously, $W^{1,p}_*(\Omega)$, $1 \leq p < \infty$, will be defined as the closure of $C^\infty_*(\overline{\Omega})$ in the Sobolev norm $\|\cdot\|_{W^{1,p}(\Omega)}$. In the case of a d-component vector-valued function v, the definition of the norm $\|v\|_{W^{1,p}(\Omega)}$ is the same as above, except that $\|v\|_{L^p(\Omega)} := \||v|\|_{L^p(\Omega)}$, with $|\cdot|$ again signifying the Euclidean norm, while $\|\nabla v\|_{L^p(\Omega)} := \||\nabla v|\|_{L^p(\Omega)}$, where now $|\nabla v|$ denotes the Frobenius norm of the $d \times d$ matrix ∇v. We recall that the Frobenius norm on $\mathbb{R}^{d\times d}$ is defined by $|X|^2 := X : X = \mathrm{tr}(X^{\mathrm{T}}X)$.

We further define

$$H_{\#}(\mathrm{div};\Omega) := \{v \in [L^2_{\#}(\Omega)]^d \,:\, \text{such that } \mathrm{div}\, v \in L^2_{\#}(\Omega)\},$$

equipped with the norm

$$\|v\|_{H(\mathrm{div};\Omega)} := \Big(\|v\|^2_{L^2(\Omega)} + \|\mathrm{div}\, v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

Let

$$\Sigma_N \subset H_{\#,\mathrm{symm}}(\mathrm{div};\Omega) := \{S \in [L^2_{\#}(\Omega)]^{d\times d} \,:\, S = S^{\mathrm{T}},\ \mathrm{div}\, S \in [L^2_{\#}(\Omega)]^d\},$$

equipped with norm

$$\|S\|_{H(\mathrm{div};\Omega)} := \Big(\|S\|^2_{L^2(\Omega)} + \|\mathrm{div} S\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}},$$

and

$$V_N \subset [W^{1,2}_*(\Omega)]^d := \left\{v \in [W^{1,2}_{\#}(\Omega)]^d \,:\, \int_\Omega v(x)\,\mathrm{d}x = 0\right\}$$

be a pair of finite-dimensional spaces consisting of, respectively, $\mathbb{R}^{d\times d}$-valued and $\mathbb{R}^d$-valued functions, whose components are 2π-periodic real-valued trigonometric polynomials of degree N,

$N \geq 1$, in each of the d-coordinate directions. The pair of spaces (Σ_N, V_N) satisfies the following inf-sup condition: let $b(v,T) := -(v, \operatorname{div} T)$; then, there exists a positive constant $c_{\text{inf-sup}}$, independent of N, such that

$$\inf_{v_N \in V_N \setminus \{0\}} \sup_{T_N \in \Sigma_N \setminus \{0\}} \frac{b(v_N, T_N)}{\|v_N\|_{L^2(\Omega)} \|T_N\|_{H(\operatorname{div};\Omega)}} \geq c_{\text{inf-sup}}. \tag{5}$$

For a short proof of (5) we refer to the Appendix at the end of the paper, where it is also shown that $c_{\text{inf-sup}} \geq 1/3$.

Assume further that $r > 0$ and define

$$F(X) := X(1 + |X|^r)^{-1/r}, \qquad X \in \mathbb{R}^{d\times d},$$

where, again, $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{d\times d}$.

Suppose that $f \in [W^{1,t}_*(\Omega)]^d := \{g \in [W^{1,t}_\#(\Omega)]^d : \int_\Omega g(x)\,\mathrm{d}x = 0\}$, for some $t > 1$. We consider the following discrete problem: find $(S_N, u_N) \in \Sigma_N \times V_N$ such that

$$-(\operatorname{div} S_N, v_N) = (f, v_N) \qquad \forall\, v_N \in V_N, \tag{6}$$

$$\hat{D}_N := F(S_N), \tag{7}$$

$$(D(u_N), T_N) = (\hat{D}_N, T_N) \qquad \forall\, T_N \in \Sigma_N. \tag{8}$$

Lemma 1. *For any $y \geq 0$ and $r > 0$, we have that*

$$\min(1, 2^{-1+1/r})\,(1+y) \leq (1+y^r)^{1/r} \leq \max(1, 2^{-1+1/r})\,(1+y).$$

Proof. Consider the function $y \in [0,\infty) \mapsto g(y) := (1+y^r)^{1/r}/(1+y) \in (0,\infty)$. Note that $g \in C([0,\infty)) \cap C^\infty((0,\infty))$, $g(0) = 1$, $\lim_{y\to\infty} g(y) = 1$, and

$$g'(y) = \frac{(1+y^r)^{-1+1/r}}{(1+y)^2}(y^{r-1} - 1).$$

Thus g has a unique stationary point in the interval $(0,\infty)$, at $y = 1$. As

$$g(1) = 2^{-1+1/r} \begin{cases} \geq 1 & \text{if } 0 < r \leq 1, \\ \leq 1 & \text{if } r \geq 1, \end{cases}$$

we deduce that $\max_{y\in[0,\infty)} g(y) = g(1)$ if $0 < r \leq 1$ and $\max_{y\in[0,\infty)} g(y) = g(0)$ if $r \geq 1$. Hence the desired upper bound. Similarly, $\min_{y\in[0,\infty)} g(y) = \min\{g(0), g(1)\}$. □

Lemma 2. *Let $r > 0$, and consider the mapping*

$$X \in \mathbb{R}^{d\times d} \mapsto F(X) := X(1+|X|^r)^{-1/r} \in \mathbb{R}^{d\times d}.$$

Then, for each $A, B \in \mathbb{R}^{d\times d}$, we have that

$$|F(A) - F(B)| \leq 2|A - B|,$$

and

$$(F(A) - F(B)) : (A - B) \geq \min(1, 2^{r-1/r})\,|A-B|^2\,(1 + |A| + |B|)^{-r-1}.$$

Proof. We begin by observing that

$$\begin{aligned} F(A) - F(B) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} F(\theta A + (1-\theta)B)\,\mathrm{d}\theta \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \Big[\big(\theta A + (1-\theta)B\big)(1 + |\theta A + (1-\theta)B|^r)^{-1/r}\Big]\,\mathrm{d}\theta. \end{aligned}$$

Thanks to the definition of the matrix norm $|\cdot|$, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\Big[\big(\theta A + (1-\theta)B\big)(1 + |\theta A + (1-\theta)B|^r)^{-1/r}\Big] &= \frac{A - B}{(1 + |\theta A + (1-\theta)B|^r)^{1/r}} \\ - (\theta A + (1-\theta)B)\frac{|\theta A + (1-\theta)B|^{r-1}}{(1 + |\theta A + (1-\theta)B|^r)^{1+1/r}}\, &\frac{(\theta A + (1-\theta)B) : (A - B)}{|\theta A + (1-\theta)B|}. \end{aligned}$$

Hence, for any $A, B, C \in \mathbb{R}^{d\times d}$, we have that

$$(F(A) - F(B)) : C = \int_0^1 \frac{(A-B) : C}{(1 + |\theta A + (1-\theta)B|^r)^{1/r}}\,\mathrm{d}\theta$$
$$- \int_0^1 [(\theta A + (1-\theta)B) : C]\,[(\theta A + (1-\theta)B) : (A-B)]\,\frac{|\theta A + (1-\theta)B|^{r-2}}{(1 + |\theta A + (1-\theta)B|^r)^{1+1/r}}\,\mathrm{d}\theta,$$

and therefore, by applying the Cauchy–Schwarz inequality on the right-hand side,

$$|F(A) - F(B)| \le 2|A-B| \qquad \forall\, A, B \in \mathbb{R}^{d\times d}.$$

The proof of the second inequality in the statement of the lemma proceeds similarly:

$$(F(A) - F(B)) : (A-B) = \int_0^1 \frac{|A-B|^2}{(1 + |\theta A + (1-\theta)B|^r)^{1/r}}\,\mathrm{d}\theta$$
$$- \int_0^1 [(\theta A + (1-\theta)B) : (A-B)]^2\,\frac{|\theta A + (1-\theta)B|^{r-2}}{(1 + |\theta A + (1-\theta)B|^r)^{1+1/r}}\,\mathrm{d}\theta,$$

and by the Cauchy–Schwarz inequality

$$[(A-B) : (\theta A + (1-\theta)B)]^2 \le |A-B|^2\,|\theta A + (1-\theta)B|^2.$$

Thus,

$$(F(A) - F(B)) : (A-B) \ge \int_0^1 \frac{|A-B|^2}{(1 + |\theta A + (1-\theta)B|^r)^{1/r}}\,\mathrm{d}\theta$$
$$- \int_0^1 |A-B|^2|\theta A + (1-\theta)B)|^2\,\frac{|\theta A + (1-\theta)B|^{r-2}}{(1 + |\theta A + (1-\theta)B|^r)^{1+1/r}}\,\mathrm{d}\theta$$
$$= |A-B|^2 \int_0^1 (1 + |\theta A + (1-\theta)B|^r)^{-1-1/r}\,\mathrm{d}\theta.$$

It follows from Lemma 1 that, for any $\theta \in [0,1]$ and any $A, B \in \mathbb{R}^{d\times d}$,

$$(1 + |\theta A + (1-\theta)B|^r)^{1/r} \le \max(1, 2^{-1+1/r})\,(1 + |\theta A + (1-\theta)B|)$$
$$\le \max(1, 2^{-1+1/r})\,(1 + |A| + |B|).$$

Hence

$$(1 + |\theta A + (1-\theta)B|^r)^{(r+1)/r} \le \max(1, 2^{1/r-r})\,(1 + |A| + |B|)^{r+1}.$$

Applying this to the integrand, we deduce that

$$(F(A) - F(B)) : (A-B) \ge \min(1, 2^{r-1/r})\,|A-B|^2\,(1 + |A| + |B|)^{-r-1},$$

which is the second inequality in the statement of the lemma. □

With these preliminary results in place, we are now ready to embark on the proof of existence and uniqueness of solutions to the discrete problem (6)–(8).

Existence and uniqueness of solutions

Assuming for the moment the existence of a solution $(S_N, u_N) \in \Sigma_N \times V_N$ to (6)–(8), we shall show that the solution must be unique. Suppose otherwise, that there exist $(S_N^i, u_N^i) \in \Sigma_N \times V_N$ that solve (6)–(8) for $i = 1, 2$. Hence,

$$-(\operatorname{div}(S_N^1 - S_N^2), v_N) - (D(u_N^1 - u_N^2), T_N) + \big(F(S_N^1) - F(S_N^2), T_N\big) = 0$$

for all $(T_N, v_N) \in \Sigma_N \times V_N$. We take $T_N = S_N^1 - S_N^2$ and $v_N = u_N^1 - u_N^2$, and note that, after partial integration in the first term,

$$-(\operatorname{div}(S_N^1 - S_N^2), u_N^1 - u_N^2) - (D(u_N^1 - u_N^2), S_N^1 - S_N^2)$$
$$= (S_N^1 - S_N^2, \nabla(u_N^1 - u_N^2)) - (D(u_N^1 - u_N^2), S_N^1 - S_N^2)$$
$$= (S_N^1 - S_N^2, D(u_N^1 - u_N^2)) - (D(u_N^1 - u_N^2), S_N^1 - S_N^2) = 0.$$

Consequently,

$$\big(F(S_N^1) - F(S_N^2), S_N^1 - S_N^2\big) = 0.$$

Lemma 2 then implies that $S_N^1 \equiv S_N^2$ on Ω, and hence $\hat{D}_N^1 \equiv \hat{D}_N^2$ on Ω, which yields that $D(u_N^1 - u_N^2) \equiv 0$ on Ω. By Korn's inequality (cf. Lemma A.2), we then have that $u_N^1 - u_N^2 \equiv 0$ on Ω, thus completing the proof of uniqueness of the solution to discrete problem (6)–(8).

Next we prove the existence of a solution to (6)–(8). First we choose any $\hat{S}_N \in \Sigma_N$ such that $-(\operatorname{div} \hat{S}_N, v_N) = (f, v_N)$ for all $v_N \in V_N$, and let $S_{N,0} := S_N - \hat{S}_N$. The existence of such an $\hat{S}_N$ will be shown below; for the time being, we shall proceed by taking the existence of such an $\hat{S}_N$ for granted. Clearly, $-(\operatorname{div} S_{N,0}, v_N) = 0$ for all $v_N \in V_N$, which then motivates us to define

$$\Sigma_{N,0} := \{T_N \in \Sigma_N \,:\, -(\operatorname{div} T_N, v_N) = 0 \;\text{ for all } v_N \in V_N\}.$$

As $0 \in \Sigma_{N,0}$, the set $\Sigma_{N,0}$ is nonempty. Problem (6)–(8) can be therefore restated in the following equivalent form: find $(S_{N,0}, u_N) \in \Sigma_{N,0} \times V_N$ such that

$$(D(u_N), T_N) = \Big(F(S_{N,0} + \hat{S}_N), T_N\Big) \qquad \forall\, T_N \in \Sigma_N. \tag{9}$$

Now, for $T_N \in \Sigma_{N,0}$, $(D(v_N), T_N) = (\nabla v_N, T_N) = -(v_N, \operatorname{div} T_N) = -(\operatorname{div} T_N, v_N) = 0$ for all $v_N \in V_N$. Hence, (9) indicates that we should seek $S_{N,0} \in \Sigma_{N,0}$ such that

$$\Big(F(S_{N,0} + \hat{S}_N), T_N\Big) = 0 \qquad \forall\, T_N \in \Sigma_{N,0}. \tag{10}$$

Let us consider the nonlinear operator $\mathfrak{F} : \Sigma_{N,0} \to \Sigma_{N,0}$, defined on the finite-dimensional Hilbert space $\Sigma_{N,0}$, equipped with the inner product and norm of $[L^2_{\#}(\Omega)]^{d\times d}$, by

$$\mathfrak{F}(U_N) := P_N F(U_N + \hat{S}_N), \qquad U_N \in \Sigma_{N,0},$$

where P_N denotes the orthogonal projector in $[L^2_{\#}(\Omega)]^{d\times d}$ onto $\Sigma_{N,0}$.

Thanks to the first inequality in the statement of Lemma 2, we then have that

$$\|\mathfrak{F}(U_N^1) - \mathfrak{F}(U_N^2)\|_{L^2(\Omega)} \leq 2\|U_N^1 - U_N^2\|_{L^2(\Omega)} \qquad \forall\, U_N^1, U_N^2 \in \Sigma_{N,0},$$

and therefore $\mathfrak{F} : \Sigma_{N,0} \to \Sigma_{N,0}$ is (globally) Lipschitz continuous on $\Sigma_{N,0}$.

Note further that

$$(\mathfrak{F}(U_N), U_N) = \left(\frac{U_N + \hat{S}_N}{(1 + |U_N + \hat{S}_N|^r)^{1/r}}, U_N\right) \geq \frac{1}{2}\int_\Omega \frac{|U_N|^2 - |\hat{S}_N|^2}{(1 + |U_N + \hat{S}_N|^r)^{1/r}}\,\mathrm{d}x.$$

Thus, thanks to Lemma 1 and noting that $(1 + |U_N + \hat{S}_N|^r)^{1/r} \geq 1$, we have that

$$(\mathfrak{F}(U_N), U_N) \geq \frac{1}{2}\min(1, 2^{1-1/r})\int_\Omega \frac{|U_N|^2}{1 + |U_N| + |\hat{S}_N|}\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\hat{S}_N|^2\,\mathrm{d}x.$$

Now,

$$\frac{|U_N|^2}{1 + |U_N| + |\hat{S}_N|} \geq \frac{|U_N|^2}{(1 + \|\hat{S}_N\|_{L^\infty(\Omega)})(1 + \|U_N\|_{L^\infty(\Omega)})},$$

and therefore, by the Nikol'skiĭ inequality $\|U_N\|_{L^\infty(\Omega)} \leq C_{\mathrm{inv}} N^{d/2}\|U_N\|_{L^2(\Omega)}$, we deduce that

$$(\mathfrak{F}(U_N), U_N) \geq \frac{1}{2}\,\frac{\min(1, 2^{1-1/r})}{1 + \|\hat{S}_N\|_{L^\infty(\Omega)}}\,\frac{1}{1 + C_{\mathrm{inv}} N^{d/2}\|U_N\|_{L^2(\Omega)}}\|U_N\|^2_{L^2(\Omega)} - \frac{1}{2}\|\hat{S}_N\|^2_{L^2(\Omega)}.$$

Thus, for any $U_N \in \Sigma_{N,0}$ such that $\|U_N\|_{L^2(\Omega)} = \mu > 0$, we have that

$$(\mathfrak{F}(U_N), U_N) \geq \frac{1}{2}\,\frac{\min(1, 2^{1-1/r})}{1 + \|\hat{S}_N\|_{L^\infty(\Omega)}}\,\frac{\mu^2}{1 + C_{\mathrm{inv}} N^{d/2}\mu} - \frac{1}{2}\|\hat{S}_N\|^2_{L^2(\Omega)}.$$

For $N \geq 1$ fixed (and therefore $\|\hat{S}_N\|_{L^\infty(\Omega)}$ and $\|\hat{S}_N\|_{L^2(\Omega)}$ also fixed), the expression on the right-hand side of the last displayed inequality is a continuous function of $\mu \in (0, \infty)$, which converges to $+\infty$ as $\mu \to +\infty$; thus, there exists a $\mu_0 = \mu_0(d, r, N, \|\hat{S}_N\|_{L^\infty(\Omega)}, \|\hat{S}_N\|_{L^2(\Omega)})$, such that $(\mathfrak{F}(U_N), U_N) > 0$ for all $U_N \in \Sigma_{N,0}$ satisfying $\|U_N\|_{L^2(\Omega)} = \mu$, for $\mu > \mu_0$.

We shall invoke the following corollary of Brouwer's fixed point theorem (cf. Girault & Raviart [7], Corollary 1.1, p.279).

Lemma 3. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space whose inner product is denoted by $(\cdot,\cdot)_{\mathcal{H}}$ and the corresponding norm by $\|\cdot\|_{\mathcal{H}}$. Let $\mathfrak{F}$ be a continuous mapping from $\mathcal{H}$ into $\mathcal{H}$ with the following property: there exists a $\mu>0$ such that $(\mathfrak{F}(v),v)_{\mathcal{H}}>0$ for all $v\in\mathcal{H}$ with $\|v\|_{\mathcal{H}}=\mu$. Then, there exists an element $u\in\mathcal{H}$ such that $\|u\|_{\mathcal{H}}\le\mu$ and $\mathfrak{F}(u)=0$.*

By taking $\mathcal{H}=\Sigma_{N,0}$, equipped with the inner product and norm of $[L^2_\#(\Omega)]^{d\times d}$, we deduce from Lemma 3 the existence of an $S_{N,0}\in\Sigma_{N,0}$ that solves (10), and thus, recalling that $S_N=S_{N,0}+\hat{S}_N$, we have also shown the existence of an $S_N\in\Sigma_N$ such that $-(\mathrm{div} S_N,v_N)=(f,v_N)$ for all $v_N\in V_N$.

Having shown the existence of $S_N\in\Sigma_N$, we return to (9) in order to show the existence of a $u_N\in V_N$ such that

$$(D(u_N),T_N)=(F(S_N),T_N)\qquad\forall\,T_N\in\Sigma_N.$$

Equivalently, we wish to show the existence of a $u_N\in V_N$ such that

$$b(u_N,T_N)=\ell(T_N)\qquad\forall\,T_N\in\Sigma_N, \tag{11}$$

where

$$b(v_N,T_N):=-(v_N,\mathrm{div}\,T_N)\quad\text{and}\quad\ell(T_N):=(F(S_N),T_N)\,.$$

We note that $\ell(T_N)=0$ for all $T_N\in\Sigma_{N,0}$, i.e., $\ell\in(\Sigma_{N,0})^0$ (the annihilator of $\Sigma_{N,0}$).

The existence of a unique $u_N\in V_N$ satisfying (11) then follows, thanks to the inf-sup condition (5), from the fundamental theorem of the theory of mixed variational problems stated in Lemma 4.1(ii) on p.40 of Girault & Raviart [6].

At the very beginning of our proof of existence of solutions we postulated the existence of an $\hat{S}_N\in\Sigma_N$ such that $-(\mathrm{div}\,\hat{S}_N,v_N)=(f,v_N)$ for all $v_N\in V_N$. Part (iii) of Lemma 4.1 on p.40 of Girault & Raviart [6] implies, again thanks to the inf-sup condition (5), the existence of an $\hat{S}_N\in\Sigma_N$ such that $b(v_N,\hat{S}_N)=(f,v_N)$ for all $v_N\in V_N$; i.e., $-(\mathrm{div}\,\hat{S}_N,v_N)=(f,v_N)$ for all $v_N\in V_N$. Thus we have proved both the existence and the uniqueness of solutions to the discrete problem (6)–(8).

4. Convergence of the sequence of approximate solutions

It remains to show that the sequence of approximate solutions generated by (6)–(8) converges in a suitable sense, and that the limiting function is a (weak) solution to the problem under consideration. We define the function space

$$D^{1,\infty}_*(\Omega):=\left\{w\in[L^1_\#(\Omega)]^d\;:\;D(w)\in[L^\infty_\#(\Omega)]^{d\times d},\;\int_\Omega w(x)\,\mathrm{d}x=0\right\}.$$

Trivially, $V_N\subset D^{1,\infty}_*(\Omega)$ for each $N\ge1$. As, by Hölder's inequality, $\|D(w)\|_{L^p(\Omega)}<\infty$ for any $w\in D^{1,\infty}_*(\Omega)$ and any $p\in[1,\infty)$, Korn's inequality (cf. Lemma A.1) implies that the seminorm $w\in D^{1,\infty}_*(\Omega)\mapsto\|D(w)\|_{L^\infty(\Omega)}$ is in fact a norm on $D^{1,\infty}_*(\Omega)$.

Lemma 4. *$[C^\infty_*(\overline{\Omega})]^d$ is weak-$*$ dense in $D^{1,\infty}_*(\Omega)$ against $[L^1_\#(\Omega)]^{d\times d}$, in the sense that for each $v\in D^{1,\infty}_*(\Omega)$ there exists a sequence $\{v_n\}_{n\ge1}\subset[C^\infty_*(\overline{\Omega})]^d$ such that*

$$\int_\Omega T(x):D(v_n(x))\,\mathrm{d}x\overset{n\to\infty}{\to}\int_\Omega T(x):D(v(x))\,\mathrm{d}x\qquad\forall\,T\in[L^1_\#(\Omega)]^{d\times d}.$$

Proof. Let $v\in[L^1_*(\Omega)]^d$. The function v then has the Fourier series expansion

$$v(x)=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{v}(k)\,\mathrm{e}^{ik\cdot x},\quad\text{where}\quad\hat{v}(k):=\frac{1}{(2\pi)^d}\int_\Omega v(x)\,\mathrm{e}^{-ik\cdot x}\,\mathrm{d}x.$$

Let $r:=(r_1,\dots,r_d)$ with $0<r_j<1$ for $j=1,\dots,d$, and, for a multi-index $m\in\mathbb{Z}^d$, define $r^{|m|}:=r_1^{m_1}\cdots r_d^{m_d}$. We consider the *Poisson kernel*:

$$P_r(x):=\sum_{m\in\mathbb{Z}^d}r^{|m|}\,\mathrm{e}^{\imath\,m\cdot x}=\prod_{j=1}^d\frac{1-r_j^2}{1-2r_j\cos x_j+r_j^2},\qquad x\in\Omega,$$

and, given any $v \in D^{1,\infty}_*(\Omega)$, we define the *Poisson integral of* v as $P_r * v$. We then take

$$v_n(x) := P_{r_n} * v, \qquad \text{with } r_n := (1 - \tfrac{1}{n}, \dots, 1 - \tfrac{1}{n}) \text{ and } x \in \Omega,$$

(so that $r_n \to (1, \dots, 1)$ as $n \to \infty$), with the convolution $*$ understood in the sense that each of the d components of v is individually convolved with P_{r_n}. For each fixed $n \geq 1$, $v_n \in [C^\infty_\#(\overline{\Omega})]^d$. By the convolution theorem $\hat{v}_n(k) = (2\pi)^d\, \widehat{P_{r_n}}(k)\, \hat{v}(k)$ for all $k \in \mathbb{Z}^d$, and therefore, as $0 = \int_\Omega v(x)\,\mathrm{d}x = (2\pi)^d\, \hat{v}(0)$, also $\hat{v}_n(0) = 0$, meaning that $\int_\Omega v_n(x)\,\mathrm{d}x = 0$; i.e., $\{v_n\}_{n\geq 1} \subset [C^\infty_*(\overline{\Omega})]^d$. Furthermore, by properties of the convolution, $D(v_n) = D(P_{r_n} * v) = P_{r_n} * D(v)$. Note also that $P_{r_n}(x) = P_{r_n}(-x)$ for all $x \in \Omega$ and each $n = 1, 2 \dots$. Thus we have that

$$\begin{aligned}
&\left| \int_\Omega T(x) : [P_{r_n}(x) * D(v(x))]\,\mathrm{d}x - \int_\Omega T(x) : D(v(x))\,\mathrm{d}x \right| \\
&\quad = \left| \int_\Omega [P_{r_n}(x) * T(x) - T(x)] : D(v(x))\,\mathrm{d}x \right| \leq \|P_{r_n} * T - T\|_{L^1(\Omega)}\, \|D(v)\|_{L^\infty(\Omega)}.
\end{aligned}$$

By Corollary 2.15 to Theorem 2.11 pp. 256 and 253, respectively, of Stein & Weiss [19], we have that $\|P_{r_n} * T - T\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}
\int_\Omega T(x) : D(v_n(x))\,\mathrm{d}x &= \int_\Omega T(x) : [P_{r_n}(x) * D(v(x))]\,\mathrm{d}x \\
&\overset{n\to\infty}{\to} \int_\Omega T(x) : D(v(x))\,\mathrm{d}x \qquad \forall\, T \in [L^1_\#(\Omega)]^{d\times d}.
\end{aligned}$$

That completes the proof. □

Theorem 1. *Suppose that* $f \in [W^{1,t}_*(\Omega)]^d$ *for some* $t > 1$*; then, there exists a unique pair* $(S, u) \in [L^1_\#(\Omega)]^{d\times d} \times D^{1,\infty}_*(\Omega)$*, such that*

$$(S, D(v)) = (f, v) \qquad \forall\, v \in D^{1,\infty}_*(\Omega),$$

and

$$D(u) = F(S) = S(1 + |S|^r)^{-\frac{1}{r}} \quad \textit{with} \begin{cases} r \in (0, 1] & \textit{if } d = 2, \\ r \in (0, 2/d) & \textit{if } d > 2. \end{cases}$$

Furthermore, the sequence of (uniquely defined) solution pairs $(S_N, u_N) \in \Sigma_N \times V_N$*,* $N \geq 1$*, generated by* (6)–(8)*, converges to* (S, u) *in the following sense:*

(a) *The sequence* $\{u_N\}_{N\geq 1}$ *converges to* u *strongly in* $[L^p_\#(\Omega)]^d$ *and weakly in* $[W^{1,p}_\#(\Omega)]^d$ *for all* $p \in [1, \infty)$*;*
(b) *The sequence* $\{D(u_N)\}_{N\geq 1}$ *converges to* $D(u)$ *weakly in* $[L^p_\#(\Omega)]^{d\times d}$ *for all* $p \in [1, \infty)$*;*
(c) *The sequence* $\{S_N\}_{N\geq 1}$ *converges to* S *strongly in* $[L^s_\#(\Omega)]^{d\times d}$ *for all values of* s *in the range* $[1, d(1 - r)/(d - 2))$ *for* $r \in (0, 2/d)$ *when* $d > 2$*, and for all values of* $s \in [1, \infty)$ *if* $r \in (0, 1]$ *and* $d = 2$*;*
(d) *The sequence* $\{D(u_N)\}_{N\geq 1}$ *converges to* $D(u)$ *weakly in* $[W^{1,2}_\#(\Omega)]^{d\times d}$*, and therefore also strongly in* $[L^p(\Omega)]^{d\times d}$ *for all* $p \in [1, 2d/(d - 2))$*,* $d \geq 2$*;*
(e) *If* $r \in (0, 1/(d - 1))$*,* $d \geq 2$*, then the sequence* $\{S_N\}_{N\geq 1}$ *converges to* S *weakly in* $[W^{1,\theta}_\#(\Omega)]^{d\times d}$ *for all* $\theta \in [1, d(1 - r)/(d - r - 1))$*.*

Proof. The proof consists of two parts. First we shall prove existence of solutions; having done so, we shall proceed to prove uniqueness of the solution.

Existence of solutions. We begin by noting that, thanks to the definition of $\hat{D}_N$, we immediately have that, for all $N \geq 1$,

$$\|\hat{D}_N\|_{L^\infty(\Omega)} \leq 1. \tag{12}$$

Consequently, denoting by P_N the $[L^2_\#(\Omega)]^{d\times d}$ orthogonal projector onto Σ_N, we also have that, for all $N \geq 1$,

$$\|D(u_N)\|_{L^p(\Omega)} = \|P_N \hat{D}_N\|_{L^p(\Omega)} \leq C_0, \qquad p \in (1, \infty),$$

thanks to the stability of the $[L^2_\#(\Omega)]^{d\times d}$ orthogonal projector in the $[L^p_\#(\Omega)]^{d\times d}$ norm (cf. Canuto, Hussaini, Quarteroni, Zang [3], ineq. (5.1.14) on p.271), where $C_0 = C_0(p)$ signifies a generic positive constant, independent of N, whose value may change from one occurrence to another. Thus, by Lemma A.1, also $\|u_N\|_{L^p(\Omega)} \le C_0$ for all $N \ge 1$, and any $p \in (1,\infty)$; as Ω is bounded, also, $\|u_N\|_{L^1(\Omega)} \le C_0$ for all $N \ge 1$.

¿From these bounds, by the compact embedding of $W^{1,p}_*(\Omega)$ into $C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 1 - d/p$, for $p \in (d,\infty)$, we deduce the existence of a function $u \in [W^{1,p}_*(\Omega)]^d$ such that (upon extraction of a subsequence, which we shall not denote here and henceforth), as $N \to \infty$ we have that

$$u_N \to u \qquad \text{weakly in } [W^{1,p}_*(\Omega)]^d \text{ for all } p \in [1,\infty), \tag{13}$$

$$u_N \to u \qquad \text{strongly in } [C^{0,\alpha}(\overline{\Omega})]^d \text{ for all } \alpha \in (0,1), \tag{14}$$

$$D(u_N) \to D(u) \qquad \text{weakly in } [L^p_\#(\Omega)]^{d\times d} \text{ for all } p \in [1,\infty), \tag{15}$$

$$\hat{D}_N \to D(u) \qquad \text{weakly in } [L^p_\#(\Omega)]^{d\times d} \text{ for all } p \in [1,\infty). \tag{16}$$

To see that $\{D(u_N)\}_{N\ge1}$ and $\{\hat{D}_N\}_{N\ge1}$ converge weakly in $[L^p_\#(\Omega)]^{d\times d}$ to the *same* limit, denote their respective weak limits by $D(u)$ (as in (15)) and $\hat{D}$. We will show that $D(u) = \hat{D}$, which will then fully justify the statement (16). Indeed,

$$(D(u) - \hat{D}, T) = (D(u) - D(u_N), T) + (D(u_N) - \hat{D}_N, T) + (\hat{D}_N - \hat{D}, T) \quad \forall\, T \in [W^{1,p'}_\#(\Omega)]^{d\times d}.$$

The first and third term on the right-hand side tend to 0 as $N \to \infty$ by the definitions of the respective weak limits, so it remains to show that the second term also tends to 0. Thanks to (8), for any fixed $T \in [W^{1,p'}_\#(\Omega)]^{d\times d}$, we have that

$$(D(u_N) - \hat{D}_N, T) = (D(u_N) - \hat{D}_N, T - T_N) \le \|D(u_N) - \hat{D}_N\|_{L^p(\Omega)} \, \|T - T_N\|_{L^{p'}(\Omega)} \quad \forall\, T_N \in \Sigma_N,$$

and hence, by boundedness in $[L^p_\#(\Omega)]^{d\times d}$ of the weakly convergent sequences $\{D(u_N)\}_{N\ge1}$ and $\{\hat{D}_N\}_{N\ge1}$, we have that

$$|(D(u_N) - \hat{D}_N, T)| \le C_0 \inf_{T_N \in \Sigma_N} \|T - T_N\|_{L^{p'}(\Omega)} \qquad \forall\, T \in [W^{1,p'}_\#(\Omega)]^{d\times d}.$$

As $N \to \infty$ the right-hand side converges to zero; therefore the same is true of the left-hand side. This proves that $(D(u) - \hat{D}, T) = 0$ for all $T \in [W^{1,p'}_\#(\Omega)]^{d\times d}$, which by the density of $[W^{1,p'}_\#(\Omega)]^{d\times d}$ in $[L^{p'}_\#(\Omega)]^{d\times d}$ implies that $D(u) = \hat{D}$, i.e., that $D(u_N)$ and $\hat{D}_N$ converge weakly in $[L^p_\#(\Omega)]^{d\times d}$ to the same limit, $D(u) \in [L^p_\#(\Omega)]^{d\times d}$, $p \in (1,\infty)$; hence, also for $p = 1$. We note also that, by (12), $\{\hat{D}_N\}_{N\ge1}$ has a subsequence that converges weak-$*$ in $[L^\infty_\#(\Omega)]^{d\times d}$ to some $\chi \in [L^\infty_\#(\Omega)]^{d\times d}$. By uniqueness of the weak limit, however $\chi = D(u)$; hence, $D(u) \in [L^\infty_\#(\Omega)]^{d\times d}$; also, $\int_\Omega u(x)\,\mathrm{d}x = 0$. Thus, in view of (14)–(15), we have proved (a) and (b) in the statement of the theorem, as well as that $u \in D^{1,\infty}_*(\Omega)$.

In order to prove (c), we take $v_N = u_N$ in (6), integrate by parts and note (7); thus,

$$(f, u_N) = (S_N, D(u_N)) = (S_N, \hat{D}_N) = \int_\Omega \frac{|S_N(x)|^2}{(1 + |S_N(x)|^r)^{1/r}}\,\mathrm{d}x.$$

Hence, for all $N \ge 1$,

$$\int_\Omega \frac{|S_N(x)|^2}{(1 + |S_N(x)|^r)^{1/r}}\,\mathrm{d}x \le \|f\|_{L^{p'}(\Omega)} \|u_N\|_{L^p(\Omega)} \le C_0, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p' \le t,$$

whereby $\|S_N\|_{L^1(\Omega)} \le C_0$. This last statement follows by noting that, on the one hand,

$$\int_{\{x\in\Omega\,:\,|S_N(x)|\ge1\}} |S_N(x)|\,\mathrm{d}x \le 2^{1/r} \int_{\{x\in\Omega\,:\,|S_N(x)|\ge1\}} \frac{|S_N(x)|^2}{(1 + |S_N(x)|^r)^{1/r}}\,\mathrm{d}x \le C_0,$$

(where we have used that $y/(1+y^r)^{1/r} \geq 1/2^r$ for all $y \geq 1$ thanks to the fact that $y \in [0,\infty) \mapsto y/(1+y^r)^{1/r}$ is strictly monotonic increasing), and on the other, by the Cauchy–Schwarz inequality,

$$\int_{\{x\in\Omega\,:\,|S_N(x)|\leq 1\}} |S_N(x)|\,\mathrm{d}x \leq \left(\int_{\{x\in\Omega\,:\,|S_N(x)|\leq 1\}} (1+|S_N(x)|^r)^{1/r}\,\mathrm{d}x\right)^{1/2} \times \left(\int_{\{x\in\Omega\,:\,|S_N(x)|\leq 1\}} \frac{|S_N(x)|^2}{(1+|S_N(x)|^r)^{1/r}}\,\mathrm{d}x\right)^{1/2}$$
$$\leq 2^{1/(2r)}|\Omega|^{1/2}\left(\int_{\{x\in\Omega\,:\,|S_N(x)|\leq 1\}} \frac{|S_N(x)|^2}{(1+|S_N(x)|^r)^{1/r}}\,\mathrm{d}x\right)^{1/2} \leq C_0,$$

whereby, upon adding the bounds on the integrals over the sets $\{x \in \Omega : |S_N(x)| \geq 1\}$ and $\{x \in \Omega : |S_N(x)| \leq 1\}$, we have that, for all $N \geq 1$,

$$\|S_N\|_{L^1(\Omega)} \leq C_0\,. \tag{17}$$

We need bounds on S_N in stronger norms in order to be able to pass to the limit as $N \to \infty$. To this end, we take $v_N = -\mathrm{div}\, D(u_N)$ in (6); note that such a v_N belongs to V_N and is therefore a legitimate choice of test function in (6). After performing integrations by parts on both the left-hand side and the right-hand side we deduce that, for all $N \geq 1$,

$$(\nabla S_N, \nabla D(u_N)) = (\nabla f, D(u_N)).$$

Note further that

$$(\nabla S_N, \nabla D(u_N)) = -(\mathrm{div}\,(\nabla S_N), D(u_N)) = -(\mathrm{div}\,(\nabla S_N), \hat{D}_N) = (\nabla S_N, \nabla \hat{D}_N)$$

and

$$(\nabla f, D(u_N)) = (P_N(\nabla f), D(u_N)) = (P_N(\nabla f), \hat{D}_N).$$

We thus have that

$$(\nabla S_N, \nabla \hat{D}_N) = (P_N(\nabla f), \hat{D}_N).$$

Differentiating the definition of $\hat{D}_N$ in (7) and noting that

$$\nabla S_N : \nabla \hat{D}_N \geq \frac{|\nabla S_N|^2}{(1+|S_N|^r)^{1/r}} - \frac{|S_N|^r\,|\nabla S_N|^2}{(1+|S_N|^r)^{1+1/r}} = \frac{|\nabla S_N|^2}{(1+|S_N|^r)^{1+1/r}},$$

we thus have that, for all $N \geq 1$,

$$\int_\Omega \frac{|\nabla S_N|^2}{(1+|S_N|^r)^{1+1/r}}\,\mathrm{d}x \leq (P_N(\nabla f), \hat{D}_N) \leq \|P_N(\nabla f)\|_{L^p(\Omega)}\|\hat{D}_N\|_{L^{p'}(\Omega)}$$
$$\leq |\Omega|^{1/p'}\|P_N(\nabla f)\|_{L^p(\Omega)} \leq c_p|\Omega|^{1/p'}\|\nabla f\|_{L^p(\Omega)} \leq C_0, \tag{18}$$

where in the penultimate inequality we used the stability of the $[L^2_\#(\Omega)]^{d\times d}$ projector P_N in the $[L^p_\#(\Omega)]^{d\times d}$ norm, with $\frac{1}{p}+\frac{1}{p'}=1$, $1<p\leq t$.

It follows from Lemma 1 that

$$(1+y^r)^{1+1/r} \leq \max(1, 2^{1/r-r})(1+y)^{r+1}, \qquad y \geq 0,$$

and therefore, assuming that $r \neq 1$ (which then implies that $0<r<2/d$ for all $d \geq 2$),

$$\|\nabla(1+|S_N|)^{\frac{1-r}{2}}\|^2_{L^2(\Omega)} \leq \left(\frac{1-r}{2}\right)^2 \int_\Omega \frac{|\nabla S_N|^2}{(1+|S_N|)^{r+1}}\,\mathrm{d}x \leq C_0 \qquad \forall\, N \geq 1. \tag{19}$$

As $0<r<2/d$, we have by Sobolev embedding that, for all p such that $2<p<2d/(d-2)$ and $0<r<1-(2/p)<2/d$,

$$\int_\Omega |S_N|^{p(1-r)/2}\,\mathrm{d}x \leq \int_\Omega \left[(1+|S_N|)^{\frac{1-r}{2}}\right]^p \mathrm{d}x = \|(1+|S_N|)^{\frac{1-r}{2}}\|^p_{L^p(\Omega)} \leq C_0 \quad \forall\, N \geq 1, \tag{20}$$

and the sequence $\{S_N\}_{N\geq 1}$ is therefore bounded in $[L^{p(1-r)/2}_\#(\Omega)]^{d\times d}$, with $p(1-r)/2>1$. Thus we can extract a subsequence, which is weakly convergent in $[L^{p(1-r)/2}_\#(\Omega)]^{d\times d}$; we denote the corresponding weak limit by S; $S \in [L^{p(1-r)/2}_{\#}(\Omega)]^{d\times d}$. As p approaches $2d/(d-2)$ from below, $1-(2/p)$ approaches $2/d$ from below; thus, for any r such that $0<r<2/d$, p can be chosen to be arbitrarily close to $2d/(d-2)$; hence,

$$S_N \overset{N\to\infty}{\rightharpoonup} S \quad \text{weakly in } [L^s_{\#}(\Omega)]^{d\times d} \text{ for all } s \text{ such that } 1\le s<\tfrac{d(1-r)}{d-2}, \begin{cases} d\ge 2, \\ 0<r<2/d. \end{cases} \tag{21}$$

For $0<r<2/d$, we have that $1<d(1-r)/(d-2)$, and the range of such s is therefore a nonempty half-open interval.

We return to the case of $r=1$, which was excluded in the argument above; this particular case is only of interest when $d=2$ as it corresponds to the upper limit $r=2/d=1$ in the range of admissible values for r. To this end, we take $r=1$ in (18) to deduce that

$$\|\nabla \log(1+|S_N|)\|^2_{L^2(\Omega)} \le C_0 \qquad \forall\, N\ge 1.$$

By the continuous embedding of $W^{1,2}_{\#}(\Omega)$ (for $d=2$) into the Orlicz space $L^{\Psi}_{\#}(\Omega)$, with $\Psi(t)=\mathrm{e}^{t^2}-1$ (as expressed by the Trudinger–Donaldson inequality, for example,) we deduce that, for some constant $\gamma>0$ (independent of N),

$$\int_\Omega \mathrm{e}^{\gamma[\log(1+|S_N|)]^2}\,\mathrm{d}x \le C_0 \qquad \forall\, N\ge 1,$$

and therefore $\{S_N\}_{N\ge1}$ is bounded in $L^s_{\#}(\Omega)$ for any $s\in[1,\infty)$. Upon extraction of a subsequence, we have weak convergence of $\{S_N\}_{N\ge1}$ in $[L^s_{\#}(\Omega)]^{d\times d}$ for all $s\in(1,\infty)$, and therefore also for $s=1$. In other words, the following modification of (21) holds:

$$S_N \overset{N\to\infty}{\rightharpoonup} S \quad \text{weakly in } [L^s_{\#}(\Omega)]^{d\times d} \text{ for all } s \text{ such that } 1\le s<\infty,\ d=2,\ r=1. \tag{22}$$

We postpone the proof of strong convergence of the sequence $\{S_N\}_{N\ge1}$ until after we have shown that the pair of functions (S,u) thus identified by the limiting procedure is a weak solution of the problem under consideration; in particular, we now shall show that $(S,D(v))=(f,v)$ for all $v\in D^{1,\infty}_*(\Omega)$, that $F(S)=D(u)$, and that $\{F(S_N)\}_{N\ge1}$ converges to $F(S)$ weak-$*$ in $[L^\infty_{\#}(\Omega)]^{d\times d}$. The argument is based on Minty's method.

Let S denote the weak limit in $[L^1_{\#}(\Omega)]^{N\times N}$ of the sequence $\{S_N\}_{N\ge1}$. Hence, for any $v\in[C^\infty_*(\overline{\Omega})]^d$ and any $v_N\in V_N$, we have that

$$\begin{aligned}|(S_N,D(v))-(f,v)| &= |(S_N,D(v)-D(v_N))+(f,v_N-v)| \\ &\le \left(\|S_N\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)}\right)\|v-v_N\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

Thus, by (17),

$$|(S_N,D(v))-(f,v)| \le \left(C_0+\|f\|_{L^1(\Omega)}\right)\inf_{v_N\in V_N}\|v-v_N\|_{W^{1,\infty}(\Omega)} \qquad \forall\, v\in[C^\infty_*(\overline{\Omega})]^d.$$

By letting $N\to\infty$ and noting that the right-hand side converges to 0 we deduce, using the weak convergence of $\{S_N\}_{N\ge1}$ to S in $[L^1_{\#}(\Omega)]^{d\times d}$ that

$$(S,D(v))=(f,v) \qquad \forall\, v\in[C^\infty_*(\overline{\Omega})]^d.$$

Hence, by noting the definition of $D^{1,\infty}_*(\Omega)$ and Lemma 4, we get that

$$(S,D(v))=(f,v) \qquad \forall\, v\in D^{1,\infty}_*(\Omega). \tag{23}$$

It remains to show that $D(u)=F(S)$. We begin by observing that $\{\hat{D}_N\}_{N\ge1}=\{F(S_N)\}_{N\ge1}$ is a bounded sequence in $[L^\infty_{\#}(\Omega)]^{d\times d}$. It therefore has a weak-$*$ convergent subsequence, still denoted by $\{\hat{D}_N\}_{N\ge1}$, with limit $\chi\in[L^\infty_{\#}(\Omega)]^{d\times d}$, say. We have already shown in the discussion following equation (16) that $\{D(u_N)\}_{N\ge1}$ and $\{\hat{D}_N\}_{N\ge1}=\{F(S_N)\}_{N\ge1}$ possess the same weak limit, $D(u)$; therefore, $\chi=D(u)$; i.e.,

$$F(S_N) \overset{N\to\infty}{\rightharpoonup} D(u) \qquad \text{weak-}* \text{ in } [L^\infty_{\#}(\Omega)]^{d\times d}. \tag{24}$$

We now will show that $D(u) = F(S)$. Note that, thanks to Lemma 2, and equations (7), (8), (6), for any $T \in [L^1_\#(\Omega)]^{d\times d}$,

$$\begin{aligned} 0 &\le (F(S_N) - F(T), S_N - T) = (F(S_N), S_N) - (F(S_N), T) - (F(T), S_N - T) \\ &= (\hat{D}_N, S_N) - (F(S_N), T) - (F(T), S_N - T) \\ &= (D(u_N), S_N) - (F(S_N), T) - (F(T), S_N - T) \\ &= (-\mathrm{div}\, S_N, u_N) - (F(S_N), T) - (F(T), S_N - T) \\ &= (f, u_N) - (F(S_N), T) - (F(T), S_N - T) \\ &\to (f, u) - (\chi, T) - (F(T), S - T) \qquad \text{as } N \to \infty. \end{aligned} \tag{25}$$

Thus, by (23) and since $\chi = D(u)$, we have that

$$0 \le (S, D(u)) - (D(u), T) - (F(T), S - T) \qquad \forall\, T \in [L^1_\#(\Omega)]^{d\times d}.$$

Equivalently,

$$0 \le (D(u) - F(T), S - T) \qquad \forall\, T \in [L^1_\#(\Omega)]^{d\times d}. \tag{26}$$

Now consider any $W \in [L^1_\#(\Omega)]^{d\times d}$ and take $T = S - \lambda W$, with $\lambda > 0$ in (26). Upon division by λ,

$$0 \le (D(u) - F(S - \lambda W), W) \qquad \forall\, W \in [L^1_\#(\Omega)]^{d\times d},\ \forall\, \lambda > 0.$$

Passing to the limit $\lambda \to 0_+$ (note that $\lambda \in [0,\infty) \in \mathbb{R} \mapsto (F(S - \lambda W), W) \in \mathbb{R}$ is a continuous function for each fixed T and W), we have that

$$0 \le (D(u) - F(S), W) \qquad \forall\, W \in [L^1_\#(\Omega)]^{d\times d}.$$

Since $[L^1_\#(\Omega)]^{d\times d}$ is a linear space, and therefore the last inequality also holds with W replaced by $-W$, we have that

$$(D(u) - F(S), W) = 0 \qquad \forall\, W \in [L^1_\#(\Omega)]^{d\times d}.$$

Thus we have shown that $D(u) = F(S)$, as an equality in $[L^\infty_\#(\Omega)]^{d\times d}$, i.e., that

$$F(S_N) \overset{N\to\infty}{\rightharpoonup} F(S) = D(u) \qquad \text{weak-}* \text{ in } [L^\infty_\#(\Omega)]^{d\times d}. \tag{27}$$

It remains to prove strong convergence of the sequence $\{S_N\}_{N\ge1}$. Define

$$E_N(x) := (F(S_N(x)) - F(S(x))) : (S_N(x) - S_N(x)).$$

Then, thanks to Lemma 2 and noting that by the triangle inequality $|S_N| \le |S_N - S| + |S|$, we obtain the following lower bound on E_N:

$$E_N(x) \ge 2^{r-\frac{1}{r}} \frac{|S_N(x) - S(x)|^2}{(1 + |S_N(x) - S(x)| + 2|S(x)|)^{r+1}}, \qquad \text{a.e. } x \in \Omega. \tag{28}$$

On the other hand, by taking $T = S$ in (25) and noting the fifth line of (25), we have (because E_N is nonnegative) and by (23) with $v = u$, that

$$\begin{aligned} \int_\Omega |E_N(x)|\,\mathrm{d}x = \int_\Omega E_N(x)\,\mathrm{d}x &= (f, u_N) - (F(S_N), S) - (F(S), S_N - S) \\ &= (f, u_N - u) + (F(S), S - S_N) + (D(u) - F(S_N), S) \qquad \forall N \ge 1. \end{aligned}$$

According to (14), (21) (or (22) if $r = 1$ and $d = 2$) and (24) the right-hand side converges to 0 as $N \to \infty$; thus, $E_N \to 0$ strongly in $L^1_\#(\Omega)$. Hence we can extract a subsequence, still denoted by E_N, which converges to 0 almost everywhere on Ω. The right-hand side of (28) is a nonnegative, continuous, strictly monotonic increasing function of $|S_N(x) - S(x)|$, which vanishes if, and only if, $|S_N(x) - S(x)| = 0$. Therefore, $|S_N(x) - S(x)|$ must also converge to 0 almost everywhere on Ω. In other words, $\{S_N\}_{N\ge1}$ converges to S almost everywhere on Ω. By Vitali's theorem we deduce from this, together with (21) (or (22) if $r = 1$ and $d = 2$), that $\{S_N\}_{N\ge1}$ converges to S strongly in $[L^s(\Omega)]^{d\times d}$ for $1 \le s < \frac{d(1-r)}{d-2}$ when $d > 2$, and for $1 \le s < \infty$ when $r \in (0,1]$ and $d = 2$. That proves part (c).

Next we prove part (d). By (7) and (19) (cf. also (18) for the case of $r = 1$ and $d = 2$), we have that

$$\int_\Omega |\nabla \hat{D}_N|^2 \,\mathrm{d}x \le \int_\Omega \left[\frac{2|\nabla S_N|}{(1+|S_N|^r)^{\frac{1}{r}}}\right]^2 \mathrm{d}x = 4\int_\Omega \frac{|\nabla S_N|^2}{(1+|S_N|)^{r+1}}\,\frac{(1+|S_N|)^{r+1}}{(1+|S_N|^r)^{2/r}}\,\mathrm{d}x \le C_0$$

for all $N \ge 1$, since the fraction appearing in the integrand as second factor is bounded by 1 for all $r \in (0,1]$ by Lemma 1. As, also, $\|\hat{D}_N\|_{L^\infty(\Omega)} \le 1$, we deduce that $\{\hat{D}_N\}_{N\ge1}$ is bounded in $[W^{1,2}_\#(\Omega)]^{d\times d}$. Further, by (8), $D(u_N) = P_N \hat{D}_N$; note also that P_N commutes with differentiation and is stable in the $[L^2(\Omega)]^{d\times d\times d}$ norm. Thus we have that

$$\|\nabla D(u_N)\|_{L^2(\Omega)} = \|\nabla P_N \hat{D}_N\|_{L^2(\Omega)} = \|P_N \nabla \hat{D}_N\|_{L^2(\Omega)} \le \|\nabla \hat{D}_N\|_{L^2(\Omega)} \le C_0 \qquad \forall\, N \ge 1.$$

As $\{D(u_N)\}_{N\ge1}$ is already known to be bounded in $[L^p(\Omega)]^{d\times d}$ for all $p \in [1,\infty)$, (cf. part (a)), we then have that $\{D(u_N)\}_{N\ge1}$ is bounded in $[W^{1,2}_\#(\Omega)]^{d\times d}$, just as $\{\hat{D}_N\}_{N\ge1}$. The statement in part (d) regarding weak convergence of the sequence $\{D(u_N)\}_{N\ge1}$ in $[W^{1,2}_\#(\Omega)]^{d\times d}$ then directly follows.

Finally, we prove part (e), now under the more restrictive hypothesis that $0 < r < 1/(d-1)$. For any $\theta \in (1,2)$ (to be fixed later on in the argument) note that, thanks to Hölder's inequality (with conjugate exponents $2/\theta$ and $2/(2-\theta)$) and (19), we have, for any $N \ge 1$,

$$\begin{aligned}\int_\Omega |\nabla S_N|^\theta \,\mathrm{d}x &= \int_\Omega \left[\frac{|\nabla S_N|^2}{(1+|S_N|)^{r+1}}\right]^{\frac{\theta}{2}} (1+|S_N|)^{\frac{(r+1)\theta}{2}}\,\mathrm{d}x \\ &\le \left[\int_\Omega \frac{|\nabla S_N|^2}{(1+|S_N|)^{r+1}}\,\mathrm{d}x\right]^{\frac{\theta}{2}} \left[\int_\Omega (1+|S_N|)^{\frac{(r+1)\theta}{2-\theta}}\,\mathrm{d}x\right]^{1-\frac{\theta}{2}} \\ &\le C_0 \left[\int_\Omega (1+|S_N|)^{\frac{(r+1)\theta}{2-\theta}}\,\mathrm{d}x\right]^{1-\frac{\theta}{2}}.\end{aligned}$$

It remains to bound the term in the square brackets on the right-hand side. We have from the bound (20) that

$$\int_\Omega (1+|S_N|)^{\frac{(1-r)p}{2}}\,\mathrm{d}x \le C_0 \qquad \forall\, p \in [1, 2d/(d-2)), \quad \forall N \ge 1, \tag{29}$$

which motivates us to link θ to p by demanding that

$$\frac{(r+1)\theta}{2-\theta} = \frac{(1-r)p}{2}, \tag{30}$$

i.e., that

$$\theta = \frac{2p(1-r)}{2(r+1)+p(1-r)}.$$

Trivially, $\theta < 2$; in order to ensure that $\theta > 1$, we demand that $p > 2(r+1)/(1-r)$. Thanks to the assumption that $0 < r < 1/(d-1)$, we have that

$$\frac{2(r+1)}{1-r} < \frac{2d}{d-2},$$

and the set of p and θ that satisfy the requirements that $1 \le p < 2d/(d-2)$, $1 < \theta < 2$, such that the equality (30) holds, is therefore nonempty. With such θ and p, for $0 < r < 1/(d-1)$ and $d \ge 2$ fixed, we have that

$$\int_\Omega |\nabla S_N|^\theta \,\mathrm{d}x \le C_0 \qquad \forall\, N \ge 1. \tag{31}$$

By noting the definition of θ in terms of p and r and the restriction on the range of p, i.e., that $1 \le p < 2d/(d-2)$, we deduce that $\{S_N\}_{N\ge1}$ converges weakly in $[W^{1,\theta}(\Omega)]^{d\times d}$ for all $\theta \in [1, d(1-r)/(d-r-1))$. That completes the proof of (e).

Uniqueness of the solution. Suppose that $(S_i, u_i) \in [L^1(\Omega)]^{d\times d} \times D^{1,\infty}_*(\Omega)$, $i = 1, 2$, are such that

$$(S_i, D(v)) = (f, v) \qquad \forall\, v \in D^{1,\infty}_*(\Omega),$$

with $D(u_i) = F(S_i) = S_i(1 + |S_i|^r)^{1/r} \in [L^\infty_\#(\Omega)]^{d\times d}$, where $0 < r < 2/d$ when $d > 2$ or $0 < r \le 1$ when $d = 2$. Upon subtracting and taking $v = u_1 - u_2$ (note that this is an admissible choice, since $u_i \in D^{1,\infty}_*(\Omega)$, $i = 1, 2$), we have that

$$(S_1 - S_2, D(u_1) - D(u_2)) = 0.$$

On the other hand, $(S_1 - S_2, D(u_1) - D(u_2)) = (S_1 - S_2, F(S_1) - F(S_2)) > 0$ by Lemma 2; the resulting contradiction implies the uniqueness of the solution. □

5. Renormalized solutions

In this section we introduce an alternative notion of solution to problem (3), (4): that of a *renormalized solution*. The necessity for introducing this notion of solution is associated with the fact that for $r > 1$ (if $d = 2$) or $r \ge 2/d$ (if $d > 3$) we are only able to show the validity of (3), (4) up to a set $\mathcal{Z}$ of zero Lebesgue measure, and the potential loss of equality between the left-hand side and the right-hand side of (3) on $\mathcal{Z}$ can be caused by an a priori unknown singular measure concentrated on the set $\mathcal{Z}$. We shall show however that the renormalized solution will coincide with a weak solution provided that S has an improved integrability property. This statement is made more precise in the following theorem.

Theorem 2. *Suppose that $f \in [W^{1,t}_*(\Omega)]^d$ for some $t > 1$, and let $r > 0$ be arbitrary; then, there exists a pair $(S, u) \in [L^1_\#(\Omega)]^{d\times d} \times D^{1,\infty}_*(\Omega)$ such that*

$$D(u) = F(S) = S(1 + |S|^r)^{-\frac{1}{r}}$$

and

$$(S, D(v)) + \langle \chi, D(v)\rangle = (f, v) \qquad \forall\, v \in [C^1_*(\Omega)]^d, \tag{32}$$

where $\chi \in [\mathcal{M}_\#(\Omega)]^{d\times d}$ is a symmetric periodic Radon measure that is not absolutely continuous with respect to the Lebesgue measure and is supported on a subset of Ω of zero Lebesgue measure. Moreover, the following energy inequality holds:

$$(S, D(u)) \le (f, u). \tag{33}$$

In addition, for any $g \in \mathcal{D}(\mathbb{R})$ and any $v \in [C^\infty_*(\Omega)]^d$ the following renormalized equation holds:

$$(S, g(|S|)D(v)) + (S, \nabla g(|S|) \otimes v) = (f, g(|S|)v). \tag{34}$$

Furthermore, the sequence of (uniquely defined) solution pairs $(S_N, u_N) \in \Sigma_N \times V_N$, $N \ge 1$, generated by (6)–(8), converges to (S, u) in the following sense:

- (a) *The sequence $\{u_N\}_{N\ge1}$ converges to u strongly in $[L^p_*(\Omega)]^d$ and weakly in $[W^{1,p}_*(\Omega)]^d$ for all $p \in [1, \infty)$;*
- (b) *The sequence $\{D(u_N)\}_{N\ge1}$ converges to $D(u)$ weakly in $[L^p_\#(\Omega)]^{d\times d}$ for all $p \in [1, \infty)$;*
- (c) *The sequence $\{S_N\}_{N\ge1}$ converges to $S + \chi$ weak-$*$ in $[\mathcal{M}_\#(\Omega)]^{d\times d}$;*
- (d) *The sequence $\{D(u_N)\}_{N\ge1}$ converges to $D(u)$ strongly in $[L^p_\#(\Omega)]^{d\times d}$ for all $p \in [1, \infty)$;*
- (e) *The sequence $\{S_N\}_{N\ge1}$ converges to S a.e. in Ω.*

Proof. First, using the sequence generated by (6)–(8), we apply the same procedure as in the preceding section to extract a subsequence that we do not relabel such that (a) and (b) hold. Further, thanks to the boundedness of the sequence $\{S_N\}_{N\ge1}$ in $[L^1_\#(\Omega)]^{d\times d}$, by applying the Banach–Alaoglou theorem and Lebesgue's decomposition theorem, we deduce the existence of a symmetric periodic Radon measure $S \in [\mathcal{M}_\#(\Omega)]^{d\times d}$ that is absolutely continuous with respect to the Lebesgue measure, and of a symmetric periodic Radon measure $\chi \in [\mathcal{M}_\#(\Omega)]^{d\times d}$ that is not absolutely continuous with respect to the Lebesgue measure, such that (a subsequence of) $\{S_N\}_{N\ge1}$ converges weak-$*$ to $S + \chi$ in $[\mathcal{M}_\#(\Omega)]^{d\times d}$. On the other hand, Chacon's biting lemma implies the existence of a nondecreasing sequence $\{\Omega_k\}_{k=1}^\infty$ of Lebesgue-measurable sets,

$\Omega_k \subset \Omega_{k+1} \subset \cdots \subset \Omega$, such that $\lim_{k\to\infty} |\Omega \setminus \Omega_k| = 0$, and of a (Lebesgue measurable) function $\hat{S} \in [L^1_\#(\Omega)]^{d\times d}$, such that, for each fixed $k \geq 1$, we have

$$S_N \rightharpoonup \hat{S} \qquad \text{weakly in } [L^1(\Omega_k)]^{d\times d}. \tag{35}$$

In particular, $\hat{S}$ can be assigned a Radon measure, in $[\mathcal{M}(\Omega_k)]^{d\times d}$, whose Radon–Nikodým derivative is precisely $\hat{S}$; hence the assigned Radon measure is absolutely continuous with respect to the Lebesgue measure. As $\{S_N\}_{N\geq 1}$ converges weak-$*$ to $S + \chi$ in $[\mathcal{M}(\Omega_k)]^{d\times d}$, for each $k \geq 1$, we deduce by uniqueness of the weak limit that $S + \chi = \hat{S}$ in $[\mathcal{M}(\Omega_k)]^{d\times d}$, for each $k \geq 1$, with $\hat{S}$ on the right-hand side now understood as an element of $[\mathcal{M}(\Omega_k)]^{d\times d}$. However S is absolutely continuous with respect to the Lebesgue measure (as is $\hat{S}$), while χ is not absolutely continuous with respect to the Lebesgue measure. Hence, $\chi = 0$ on each Ω_k, and thus $S = \hat{S} \in L^1(\Omega_k)$ for each $k \geq 1$. As $\chi = 0$ on each Ω_k, and $|\Omega \setminus \Omega_k| \to 0$, it follows that χ is supported on subset of Ω of zero Lebesgue measure. Therefore, $S = \hat{S}$ almost everywhere in Ω and $S = \hat{S} \in [L^1_\#(\Omega)]^{d\times d}$.

Next we prove (e). To do so, we first recall (18), which implies that, for all $N \geq 1$,

$$\int_\Omega \frac{|\nabla S_N|^2}{(1 + |S_N|)^{1+r}}\,\mathrm{d}x \leq C_0.$$

Thus, defining

$$B_N := \frac{S_N}{(1 + |S_N|)^{r+1}},$$
$$a_N := \frac{1}{(1 + |S_N|)^{r}},$$

it follows that, for all $N \geq 1$,

$$\|B_N\|_{L^\infty(\Omega)} + \|a_N\|_{L^\infty(\Omega)} + \int_\Omega |\nabla B_N|^2 + |\nabla a_N|^2\,\mathrm{d}x \leq 2 + C\int_\Omega \frac{|\nabla S_N|^2}{(1 + |S_N|)^{1+r}}\,\mathrm{d}x \leq C\,,$$

where the last inequality follows from Hölder's inequality. Therefore, thanks to the compactness of the Sobolev embedding of $W^{1,2}_\#(\Omega)$ into $L^1_\#(\Omega)$, there exist subsequences (not indicated) such that

$$\begin{aligned} B_N &\to B \quad \text{strongly in } [L^1_\#(\Omega)]^{d\times d},\\ a_N &\to a \quad \text{strongly in } L^1_\#(\Omega),\\ B_N &\to B \quad \text{a.e. in } \Omega,\\ a_N &\to a \quad \text{a.e. in } \Omega. \end{aligned}$$

Moreover, since $\{S_N\}_{N\geq 1}$ is a bounded sequence in $[L^1_\#(\Omega)]^{d\times d}$, the nonnegativity of a_N implies that

$$a^{-\frac{1}{r}} \in L^1_\#(\Omega) \implies a > 0 \text{ a.e. in } \Omega.$$

Finally, since

$$S_N = B_N (a_N)^{-\frac{r+1}{r}},$$

the above pointwise convergence result implies that

$$S_N \to \overline{S} \text{ a.e. in } \Omega,$$

where

$$\overline{S} := B a^{-\frac{r+1}{r}},$$

which is a measurable function that is finite a.e. in Ω. On the other hand, from (35) we have weak convergence to S on Ω_k and due to the uniqueness of the limit we obtain that

$$S_N \to S \text{ a.e. in } \Omega_k.$$

Thanks to the properties of the sets Ω_k it then follows that

$$S_N \to S \text{ a.e. in } \Omega.$$

Moreover, using Fatou's lemma, we deduce that

$$\int_\Omega |S| \, \mathrm{d}x \le \liminf_{N\to\infty} \int_\Omega |S_N| \, \mathrm{d}x \le C,$$

which completes the proof of (e). Then, (d) easily follows from the definition of $D(u_N)$ in terms of S_N. Similarly, one can obtain the energy inequality (33).

The rest of the proof is devoted to establishing the validity of (34). Having shown pointwise convergence of the sequence $\{S_N\}_{N\ge 1}$, we deduce from the above a priori bounds that

$$\begin{aligned} \frac{S_N}{(1+|S_N|)^{r+1}} &\rightharpoonup \frac{S}{(1+|S|)^{r+1}} && \text{weakly in } [W^{1,2}_\#(\Omega)]^{d\times d}, \\ \frac{1}{(1+|S_N|)^{r}} &\rightharpoonup \frac{1}{(1+|S|)^{r}} && \text{weakly in } W^{1,2}_\#(\Omega). \end{aligned} \tag{36}$$

Next, let $g \in C_0^\infty(\mathbb{R})$ be arbitrary. Then, the pointwise convergence of S_N implies that

$$g(|S_N|) \to g(|S|) \qquad \text{strongly in } L^p_\#(\Omega) \text{ for all } p \in [1,\infty). \tag{37}$$

Moreover, it also follows from (36) that

$$g(|S_N|) \rightharpoonup g(|S|) \qquad \text{weakly in } W^{1,2}_\#(\Omega).$$

In addition, as g has compact support, Lebesgue's dominated convergence theorem implies that

$$g(|S_N|)S_N \to g(|S|)S \qquad \text{strongly in } [L^p_\#(\Omega)]^{d\times d} \text{ for all } p \in [1,\infty). \tag{38}$$

Finally, using (36) we have that, for all $N \ge 1$,

$$\int_\Omega |\nabla S_N g(|S_N|)|^2 \, \mathrm{d}x \le C \int_{\Omega\cap\{x;|S_N|\in \text{ supp } g\}} |\nabla S_N|^2 \, \mathrm{d}x \le C.$$

By combining this with (38) and using the reflexivity of $[W^{1,2}_\#(\Omega)]^{d\times d}$ we deduce that

$$g(|S_N|)S_N \rightharpoonup g(|S|)S \qquad \text{weakly in } [W^{1,2}_\#(\Omega)]^{d\times d}. \tag{39}$$

Hence, setting $v_N := P_N(g(|S_N|)v)$ in (6) with arbitrary $v \in C^1_\#(\Omega)$, we obtain the following identity

$$\int_\Omega g(|S_N|)S_N : \nabla v \, \mathrm{d}x + \int_\Omega S_N : (\nabla g(|S_N|) \otimes v) \, \mathrm{d}x = \int_\Omega P_N f \cdot g(|S_N|) v \, \mathrm{d}x. \tag{40}$$

Finally, we let $N \to \infty$ in (40) to obtain (34). Indeed, for the term on the right-hand side we use (37). For the first term on the left-hand side we use (38) and the second term on the left-hand side can be handled with the help of (39). □

6. Conclusion

This study contributes to the analysis of boundary-value problems describing the static state of implicitly constituted elastic solids, as formulated in (1). In this generality, there are no results known to the authors that are concerned with the analysis of the problem (1) and which involve the left Cauchy–Green deformation tensor B. The only results, obtained recently (some of them are presented in herein), concern a version of problem (1) where the tensor B is replaced by the linearized strain $D(u) = \varepsilon(u)$. Regarding this setting, the state of the art concerning the analysis of the relevant boundary-value problems is the following:

(a) Consider $-\text{div}\, T = 0$ with (2) in Ω and a prescribed traction over the whole boundary (i.e., $Tn = g$ on the boundary) in a special geometric setting (a cylinder $O \times (-\infty, \infty)$ with a planar cross-section $O \subset \mathbb{R}^2$) for a special deformation called anti-plane strain. One is then allowed to introduce the Airy stress function $U : O \to \mathbb{R}$ and the whole problem reduces to a scalar Dirichlet problem

$$-\text{div}\left(\frac{\nabla U}{(1+|\nabla U|^r)^{1/r}}\right) = 0 \quad \text{in } O, \qquad U = U_0 \quad \text{on } \partial O.$$

Note that the Neumann-type boundary condition for the original problem leads to a nonhomogeneous Dirichlet boundary condition for the Airy stress function over the entire boundary ∂O. In [1], the following results are established: if Ω is convex, then there is a unique weak solution to the problem for all $r \in (0, \infty)$. If Ω is nonconvex (such as a V-notch, for example,) with constant Dirichlet data on noncovex parts of the boundary, the existence of a unique weak solution is established for all $r \in (0, 2)$. Since it is known that there is a nonconvex set Ω such that a weak solution to (6) for $r = 2$ does not exist, the results concerning nonconvex (as well as convex) domains seem to be sharp.

(b) In this study, we have considered the system of partial differential equations $-\mathrm{div}\, T = f$ with (2) in any number of space dimensions; we have however confined ourselves to the spatially periodic setting. Relying on a constructive approximation method, based on a Fourier spectral method, we established the existence of a weak solution and its uniqueness for $r \in (0, 2/d)$. We further introduced the concept of renormalized solution and proved its existence for any $r \in (0, \infty)$ and obtained the condition on S that suffices to deduce that the renormalized solution is in fact a weak solution.

(c) In general situations, yet within the framework involving the linearized strain, most of the analytical problems concerning existence, uniqueness and stability of solutions are open. The only exception is the existence of a weak solution to (1) with nonhomogeneous Dirichlet data, i.e., $\Gamma_N = \emptyset$, for $r \in (0, 1/d)$; the proof of existence of a weak solution in this case will be presented in a forthcoming paper. There are no results we are aware of concerning the existence of renormalized solutions on general bounded domains, with either Dirichlet, or Neumann, or mixed boundary condition on the displacement u.

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Appendix

The purpose of this Appendix is to prove some auxiliary results, which are required in the arguments in the main body of the paper. In particular, we shall prove the inf-sup condition (5). We begin with the statement of Korn's inequality in the L^p norm, whose proof is also included for the sake of completeness, as we were unable to find it in the literature in the context of periodic boundary conditions. For $p = 2$ we explicitly calculate (an upper bound on) the constant in Korn's inequality in the L^2 norm, which then allows us to specify (a lower bound on) the constant $c_{\text{inf-sup}}$ appearing in the inf-sup condition (5).

Lemma A.1 (Korn's inequality in L^p). *Let $p \in (1, \infty)$, $d \geq 2$ and $\Omega := (0, 2\pi)^d$. There exists a positive constant c_p such that the following inequalities hold:*

$$\|\nabla v\|_{L^p(\Omega)} \leq c_p \left(\|D(v)\|_{L^p(\Omega)} + \|\text{div}\, v\|_{L^p(\Omega)}\right) \qquad \forall\, v \in [W^{1,p}_*(\Omega)]^d,$$

and, hence, also, with a possibly different constant c_p,

$$\|\nabla v\|_{L^p(\Omega)} \leq c_p \|D(v)\|_{L^p(v)} \qquad \forall\, v \in [W^{1,p}_*(\Omega)]^d.$$

Let, further, $D^{\text{dev}}(v) := D(v) - \frac{1}{d}(\text{div}\, v)\mathrm{I}$ denote the deviatoric part of $D(v)$, where I is the identity matrix in $\mathbb{R}^{d\times d}$; then, there exists a positive constant c_p such that

$$\|\nabla v\|_{L^p(\Omega)} \leq c_p \|D^{\text{dev}}(v)\|_{L^p(\Omega)} \qquad \forall\, v \in [W^{1,p}_*(\Omega)]^d.$$

Besides being dependent on p, the constant c_p (whose specific value may change from one line to the next) also depends on d, but we do not explicitly indicate that. In each case, the left-hand side of the inequality can be further bounded below by $C_p\|v\|_{W^{1,p}(\Omega)}$, where C_p is another positive constant dependent on p, but independent of v.

Proof. The statement in the penultimate sentence of the lemma is an immediate consequence of Poincaré's inequality for functions $v \in [W^{1,p}_*(\Omega)]^d$ (which, by definition, have zero integral over Ω); viz.,

$$C_p\|v\|_{W^{1,p}(\Omega)} \leq \|\nabla v\|_{L^p(\Omega)} \qquad \forall\, v \in W^{1,p}_*(\Omega).$$

The second stated inequality is an immediate consequence of the first inequality, based on the following argument: $\text{div}\, v = \text{tr}\, D(v)$; hence, by the Cauchy–Schwarz inequality for matrices,

$$|\text{div}\, v| = |\text{tr}\, D(v)| = |\mathrm{I} : D(v)| \leq |\mathrm{I}|\, |D(v)| = d^{1/2}\, |D(v)|,$$

and thus, also $\|\text{div}\, v\|_{L^p(\Omega)} \leq d^{1/2}\|D(v)\|_{L^p(\Omega)}$, so the second stated inequality is implied by the first.

In order to prove the two remaining inequalities we proceed as follows. As $[C^\infty_*(\overline{\Omega})]^d$ is, by definition, dense in $[W^{1,p}_*(\Omega)]^d$ for $p \in (1, \infty)$, it suffices to prove the inequalities for $v \in [C^\infty_*(\overline{\Omega})]^d$. For any such smooth v, we have that

$$\Delta v = 2\, \text{div}\, D(v) - \nabla(\text{div}\, v).$$

Because Δ has a well-defined inverse when considered as a mapping from $[C^\infty_*(\Omega)]^d$ into itself, we apply Δ^{-1} to both sides of the last equality and we then apply the gradient operator ∇ to both sides of the resulting equality, which then yields

$$\nabla v = 2(\nabla\Delta^{-1}\text{div} D(v)) - (\nabla\Delta^{-1}\nabla\text{div}\, v).$$

Here, ∇v is understood to mean the $d \times d$ matrix whose (i, j) entry is $(\nabla v)_{ij} = \frac{\partial}{\partial x_i} v_j$ for $i, j = 1, \ldots, d$. Let us also define

$$(\mathcal{F}v)(k) := \hat{v}(k) = \frac{1}{(2\pi)^d} \int_\Omega v(x)\, \mathrm{e}^{-\imath\, k\cdot x}\, \mathrm{d}x$$

so that $v = \mathcal{F}^{-1}\mathcal{F}v$, with

$$(\mathcal{F}^{-1}\mathcal{F}v)(x) = (\mathcal{F}^{-1}\hat{v})(x) = \sum_{k\in\mathbb{Z}^d} \hat{v}(k)\, \mathrm{e}^{\imath\, k\cdot x}.$$

When applied to matrix-functions, the transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are understood to be acting component-wise; thus, for example, $\mathcal{F}M$ for a matrix function $M = (m_{ij})_{i,j=1}^d$ means a matrix whose (i,j) entry is $\mathcal{F}m_{ij}$. A straightforward calculation yields that

$$\begin{aligned}\nabla v = \mathcal{F}^{-1}\mathcal{F}(\nabla v) &= 2\mathcal{F}^{-1}[\mathcal{F}(\nabla\Delta^{-1}\mathrm{div}D(v))] - \mathcal{F}^{-1}[\mathcal{F}(\nabla\Delta^{-1}\nabla\mathrm{div}\, v)]\\ &= 2\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D(v))\right] - \mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(\mathrm{div}\, v)\right].\end{aligned}$$

Here, in the first square bracket we have the product of the $d\times d$ matrix $(k\otimes k)/|k|^2$ with the matrix $\mathcal{F}(D(v))$, with $\mathcal{F}$ being applied component-by-component to the entries of the $d\times d$ matrix $D(v)$; while in the second square bracket we have the $d\times d$ matrix $(k\otimes k)/|k|^2$ post-multiplied by the scalar $\mathcal{F}(\mathrm{div}\, v)$.

Hence,

$$\begin{aligned}\|\nabla v\|_{L^p(\Omega)} &\le 2\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D(v))\right]\right\|_{L^p(\Omega)} + \left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(\mathrm{div}\, v)\right]\right\|_{L^p(\Omega)}\\ &\le c_p\left(\|D(v)\|_{L^p(\Omega)} + \|\mathrm{div}\, v\|_{L^p(\Omega)}\right),\end{aligned}$$

where in the transition to the last line we used that, thanks to Lizorkin's multiplier theorem, $\xi\in\mathbb{R}^d\setminus\{0\}\mapsto\frac{\xi_i\,\xi_m}{|\xi|^2}\in\mathbb{R}$ is a Fourier multiplier in $L^p(\mathbb{R}^d)$, $1<p<\infty$, and thus by De Leeuw's transference theorem (cf. Theorem 3.8 on p.260 of Stein & Weiss [19], or Theorem 3.4.2 and Remark 3.4.4 in Schmeisser & Triebel [18]), its restriction to $\mathbb{Z}^d\setminus\{0\}$, i.e., $k\in\mathbb{Z}^d\setminus\{0\}\mapsto\frac{k_i\,k_m}{|k|^2}\in\mathbb{R}$ is a Fourier multiplier in $L^p_\#(\Omega)$ (and hence in $L^p_*(\Omega)$), for all $i,m=1,\dots,d$. Since Lizorkin's multiplier theorem is usually formulated for $\mathbb{R}$-valued functions while here we are working with $\mathbb{R}^{d\times d}$-valued functions, we shall provide the details of the calculation for the first summand in the penultimate line above, but will then omit the details of similar subsequent calculations. It has to be borne in mind that the meaning of the L^p norm of a certain matrix function, say, $G : x\mapsto (G_{ij}(x))_{i,j=1}^d$ is that we take the standard L^p norm for scalar-valued functions of the matrix norm $|G|$ of G. In our case,

$$G_{ij} = \left(\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D(v))\right]\right)_{ij} = \sum_{m=1}^d \mathcal{F}^{-1}\left[\frac{k_i\,k_m}{|k|^2}\,\mathcal{F}((D(v))_{mj})\right].$$

Now, $|G|\le\sum_{i,j=1}^d|G_{ij}|$, and therefore

$$\|G\|_{L^p(\Omega)} \le \sum_{i,j=1}^d\sum_{m=1}^d\left\|\mathcal{F}^{-1}\left[\frac{k_i\,k_m}{|k|^2}\,\mathcal{F}((D(v))_{mj})\right]\right\|_{L^p(\Omega)}.$$

We then apply a combination of Lizorkin's theorem and De Leeuw's theorem to each of the d^3 summands, resulting in

$$\|G\|_{L^p(\Omega)} \le \sum_{i,j=1}^d\sum_{m=1}^d m_p\left(\frac{k_i\,k_m}{|k|^2}\right)\|(D(v))_{mj}\|_{L^p(\Omega)},$$

where $m_p(\cdot)$ is the multiplier norm in $L^p(\Omega)$. By letting $c_p := \max_{i,m=1,\dots,d} m_p\left(\frac{k_i\,k_m}{|k|^2}\right)$ and noting that, for each fixed $m,j\in\{1,\dots,d\}$, $|(D(v))_{mj}|\le|D(v)|$ and therefore $\|(D(v))_{mj}\|_{L^p(\Omega)}\le\||D(v)|\|_{L^p(\Omega)} = \|D(v)\|_{L^p(\Omega)}$, we have that

$$\|G\|_{L^p(\Omega)} \le d^3 c_p\|D(v)\|_{L^p(\Omega)},$$

and hence

$$\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D(v))\right]\right\|_{L^p(\Omega)} \le d^3 c_p\|D(v)\|_{L^p(\Omega)}.$$

Similarly,

$$\left\| \mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(\operatorname{div} v)\right]\right\|_{L^p(\Omega)} \leq \sum_{i,j=1}^{d} m_p\left(\frac{k_i\,k_j}{|k|^2}\right)\|\operatorname{div} v\|_{L^p(\Omega)} \leq d^2 c_p \|\operatorname{div} v\|_{L^p(\Omega)}.$$

The factors d^3 and d^2 are then absorbed into the symbol c_p, without further explicit indication of its dependence on d. This then proves the first stated inequality.

It remains to prove the third inequality in the statement of the lemma. To this end, we note that

$$\begin{aligned}\operatorname{div} D^{\mathrm{dev}}(v) &= \operatorname{div} D(v) - \frac{1}{d}\operatorname{div}\left((\operatorname{div} v)\,\mathrm{I}\right)\\ &= \frac{1}{2}\,\Delta v + \frac{1}{2}\nabla \operatorname{div} v - \frac{1}{d}\,\nabla\operatorname{div} v.\end{aligned}$$

Hence,

$$\operatorname{div}\operatorname{div} D^{\mathrm{dev}}(v) = \frac{1}{2}\,\Delta\operatorname{div} v + \frac{1}{2}\operatorname{div}\left(\nabla\operatorname{div} v\right) - \frac{1}{d}\operatorname{div}\left(\nabla(\operatorname{div} v)\right) = \frac{d-1}{d}\,\Delta\operatorname{div} v,$$

whereby

$$\operatorname{div} v = \frac{d}{d-1}\,\Delta^{-1}\operatorname{div}\operatorname{div} D^{\mathrm{dev}}(v).$$

Using this identity in the transition from the second to the third line in the chain of equalities below, and the definition of $D^{\mathrm{dev}}(v)$ in the transition from the first to the second line yields:

$$\begin{aligned}\Delta v &= 2\operatorname{div} D(v) - \nabla\operatorname{div} v\\ &= 2\operatorname{div}\left[D^{\mathrm{dev}}(v) + \frac{1}{d}\,(\operatorname{div} v)\,\mathrm{I}\right] - \nabla(\operatorname{div} v)\\ &= 2\operatorname{div} D^{\mathrm{dev}}(v) + \frac{2}{d}\operatorname{div}\left[\frac{d}{d-1}\,(\Delta^{-1}\operatorname{div}\operatorname{div} D^{\mathrm{dev}}(v))\,\mathrm{I}\right] - \nabla\left[\frac{d}{d-1}\,\Delta^{-1}\operatorname{div}\operatorname{div} D^{\mathrm{dev}}(v)\right],\end{aligned}$$

and therefore, by applying Δ^{-1} to both sides, and then ∇ to both sides of the resulting equality, we have that

$$\begin{aligned}\nabla v = 2\,\nabla\Delta^{-1}\operatorname{div} D^{\mathrm{dev}}(v) \;+ \frac{2}{d-1}\nabla\Delta^{-1}\operatorname{div}\left[(\Delta^{-1}\operatorname{div}\operatorname{div} D^{dev}(v))\,\mathrm{I}\right]\\ - \frac{d}{d-1}\nabla\Delta^{-1}\nabla[\Delta^{-1}\operatorname{div}\operatorname{div} D^{\mathrm{dev}}(v)].\end{aligned}$$

Thus,

$$\nabla v = \mathcal{F}^{-1}\mathcal{F}(\nabla v) = 2\,\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D^{\mathrm{dev}}(v))\right] + \frac{2-d}{d-1}\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\left(\frac{k\otimes k}{|k|^2} : \mathcal{F}(D^{\mathrm{dev}}(v))\right)\right],$$

which then implies, with $c_p := \max_{i,j=1,\dots,d} m_p\left(\frac{k_i\,k_j}{|k|^2}\right)$, that

$$\begin{aligned}\|\nabla v\|_{L^p(\Omega)} &\leq 2\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D^{\mathrm{dev}}(v))\right]\right\|_{L^p(\Omega)}\\ &\quad + \frac{d-2}{d-1}\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\left(\frac{k\otimes k}{|k|^2} : \mathcal{F}(D^{\mathrm{dev}}(v))\right)\right]\right\|_{L^p(\Omega)}\\ &= 2\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\,\mathcal{F}(D^{\mathrm{dev}}(v))\right]\right\|_{L^p(\Omega)}\\ &\quad + \frac{d-2}{d-1}\left\|\mathcal{F}^{-1}\left[\frac{k\otimes k}{|k|^2}\mathcal{F}\left[\mathcal{F}^{-1}\left(\frac{k\otimes k}{|k|^2} : \mathcal{F}(D^{\mathrm{dev}}(v))\right)\right]\right]\right\|_{L^p(\Omega)}\\ &\leq 2\,d^3 c_p\|D^{\mathrm{dev}}(v)\|_{L^p(\Omega)} + \frac{d-2}{d-1}d^2 c_p\left\|\mathcal{F}^{-1}\left(\frac{k\otimes k}{|k|^2} : \mathcal{F}(D^{\mathrm{dev}}(v))\right)\right\|_{L^p(\Omega)}\\ &\leq 2\,d^3 c_p\|D^{\mathrm{dev}}(v)\|_{L^p(\Omega)} + \frac{d-2}{d-1}d^4 c_p\|D^{\mathrm{dev}}(v)\|_{L^p(\Omega)},\end{aligned}$$

where in the transition to the penultimate line, and then again in the passage to the last line, we used that $k \in \mathbb{Z}^d \setminus \{0\} \mapsto k_i\, k_j/|k|^2 \in \mathbb{R}$ is a Fourier multiplier in $L^p_*(\Omega)$ for all $i, j = 1, \dots, d$ and all $p \in (1, \infty)$. After merging the two terms on the right-hand side in the last displayed line, and absorbing dependence on d into our notation for the constant c_p, we thus arrive at the third stated inequality. □

We shall now consider Lemma A.1 in the special case when $p = 2$, and will provide an elementary proof, which will allow us to explicitly compute the constant c_p in Korn's inequality stated in Lemma A.1 for $p = 2$, and thereby also the constant $c_{\text{inf-sup}}$ appearing in the inf-sup condition (5).

Lemma A.2 (Korn's inequality in L^2)**.** *We have that*

$$\|v\|_{L^2(\Omega)} \leq \sqrt{2}\, \|D(v)\|_{L^2(\Omega)} \qquad \forall\, v \in [W^{1,2}_*(\Omega)]^d,$$

and

$$\|v\|_{W^{1,2}(\Omega)} \leq 2\, \|D(v)\|_{L^2(\Omega)} \qquad \forall\, v \in [W^{1,2}_*(\Omega)]^d.$$

Proof. Let $v \in [C^\infty_*(\overline{\Omega})]^d$. Again, we shall use that the function v then has the Fourier series expansion

$$v(x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \hat{v}(k)\, \mathrm{e}^{ik\cdot x}, \quad \text{where} \quad \hat{v}(k) := \frac{1}{(2\pi)^d} \int_\Omega v(x)\, \mathrm{e}^{-ik\cdot x}\, \mathrm{d}x.$$

By partial integration in the transition from the third to the fourth line below, dropping the nonnegative term $\|\mathrm{div}\, v\|^2_{L^2(\Omega)}$ and using Parseval's identity in the transition from the fourth to the fifth line, and again in the last line, we have that

$$\begin{aligned}
\|D(v)\|^2_{L^2(\Omega)} &= \frac{1}{4} \sum_{i,j=1}^d \int_\Omega \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 \mathrm{d}x \\
&= \frac{1}{4} \sum_{i,j=1}^d \int_\Omega \left| \frac{\partial v_i}{\partial x_j} \right|^2 + 2 \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + \left| \frac{\partial v_j}{\partial x_i} \right|^2 \mathrm{d}x \\
&= \frac{1}{2} \sum_{i,j=1}^d \int_\Omega \left| \frac{\partial v_i}{\partial x_j} \right|^2 + \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\, \mathrm{d}x \\
&= \frac{1}{2} \left(\|\nabla v\|^2_{L^2(\Omega)} + \|\mathrm{div}\, v\|^2_{L^2(\Omega)} \right) \\
&\geq \frac{1}{2} (2\pi)^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^2 |\hat{v}(k)|^2 \\
&\geq \frac{1}{2} (2\pi)^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{v}(k)|^2 = \frac{1}{2} \|v\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus, by a density argument, we deduce the inequality

$$\|v\|_{L^2(\Omega)} \leq \sqrt{2}\, \|D(v)\|_{L^2(\Omega)} \qquad \forall\, v \in [W^{1,2}_*(\Omega)]^d.$$

Since both $\|D(v)\|^2_{L^2(\Omega)} \geq \frac{1}{2} \|\nabla v\|^2_{L^2(\Omega)}$ and $\|D(v)\|^2_{L^2(\Omega)} \geq \frac{1}{2} \|v\|^2_{L^2(\Omega)}$, we have, by adding these two inequalities and then taking the square root, the desired inequality:

$$\|v\|_{W^{1,2}(\Omega)} \leq 2\, \|D(v)\|_{L^2(\Omega)} \qquad \forall\, v \in [W^{1,2}_*(\Omega)]^d.$$

That completes the proof. □

We are now ready to prove the inf-sup condition (5). Given any $v_N \in V_N \setminus \{0\}$, we consider the function $T_N \in \Sigma_N \setminus \{0\}$ defined by $T_N = D(w_N)$, where $w_N \in V_N$ is the unique solution of the problem

$$(D(w_N), D(z_N)) = (v_N, z_N) \qquad \forall\, z_N \in V_N.$$

We note that, indeed, $T_N \neq 0$; for, if it were the case that $T_N = 0$, then we would have $(v_N, z_N) = 0$ for all $z_N \in V_N$, and hence $v_N = 0$, which would contradict our assumption that $v_N \in V_N \setminus \{0\}$.

The existence and uniqueness of w_N is a direct consequence of the Lax–Milgram theorem and Korn's inequality, as stated in Lemma A.2. Hence,

$$\begin{aligned} b(v_N, T_N) &:= (-v_N, \operatorname{div} T_N) = (-v_N, \operatorname{div} D(w_N)) = (D(v_N), D(w_N)) \\ &= (D(w_N), D(v_N)) = \|v_N\|^2_{L^2(\Omega)}. \end{aligned}$$

As $(-\mathrm{div}\ T_N, z_N) = (-\mathrm{div}\, D(w_N), z_N) = (D(w_N), D(z_N)) = (v_N, z_N)$ for all $z_N \in V_N$, we have, with $z_N = -\mathrm{div}\ T_N$, that

$$\|\mathrm{div}\ T_N\|^2_{L^2(\Omega)} = (-\mathrm{div}\ T_N, -\mathrm{div}\ T_N) = (v_N, -\mathrm{div}\ T_N) = b(v_N, T_N).$$

Further, by Lemma A.2,

$$\begin{aligned} \|T_N\|^2_{L^2(\Omega)} &= \|D(w_N)\|^2_{L^2(\Omega)} = (v_N, w_N) \leq \|v_N\|_{L^2(\Omega)} \|w_N\|_{L^2(\Omega)} \\ &\leq \sqrt{2}\, \|v_N\|_{L^2(\Omega)} \|D(w_N)\|_{L^2(\Omega)} = \sqrt{2}\, \|v_N\|_{L^2(\Omega)} \|T_N\|_{L^2(\Omega)}, \end{aligned}$$

whereby

$$\|T_N\|^2_{L^2(\Omega)} \leq 2\, \|v_N\|^2_{L^2(\Omega)} = 2\, b(v_N, T_N).$$

Now, summing this last inequality and the equality

$$\|\operatorname{div} T_N\|^2_{L^2(\Omega)} = b(v_N, T_N) = \|v_N\|^2_{L^2(\Omega)}$$

yields that

$$\begin{aligned} b(v_N, T_N) &\geq \tfrac{1}{3}\, \|T_N\|^2_{H(\mathrm{div};\Omega)} \\ &\geq \tfrac{1}{3}\, \|T_N\|_{H(\mathrm{div};\Omega)} \|\operatorname{div} T_N\|_{L^2(\Omega)} \\ &= \tfrac{1}{3}\, \|T_N\|_{H(\mathrm{div};\Omega)} [b(v_N, T_N)]^{\frac{1}{2}} \\ &= \tfrac{1}{3}\, \|T_N\|_{H(\mathrm{div};\Omega)} \|v_N\|_{L^2(\Omega)}. \end{aligned}$$

Thus we have shown that for each $v_N \in V_N \setminus \{0\}$ there exists a $T_N \in \Sigma_N$ such that

$$b(v_N, T_N) \geq \tfrac{1}{3}\, \|T_N\|_{H(\mathrm{div};\Omega)} \|v_N\|_{L^2(\Omega)}.$$

This implies that

$$\sup_{T_N \in \Sigma_N \setminus \{0\}} \frac{b(v_N, T_N)}{\|T_N\|_{H(\mathrm{div};\Omega)}} \geq \tfrac{1}{3}\, \|v_N\|_{L^2(\Omega)} \qquad \forall\, v_N \in V_N,$$

and hence,

$$\inf_{v_N \in V_N \setminus \{0\}} \sup_{T_N \in \Sigma_N \setminus \{0\}} \frac{b(v_N, T_N)}{\|v_N\|_{L^2(\Omega)} \|T_N\|_{H(\mathrm{div};\Omega)}} \geq \tfrac{1}{3}.$$

We thus deduce that the inf-sup condition (5) holds, with $c_{\text{inf-sup}} \geq 1/3$.

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