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WEIGHTED INTEGRAL TECHNIQUES AND $\mathcal{C}^\alpha$-ESTIMATES FOR A CLASS OF ELLIPTIC SYSTEMS WITH p-GROWTH

MIROSLAV BULÍČEK, JENS FREHSE, AND MARK STEINHAUER

Abstract. We consider weak solutions to nonlinear elliptic systems in a $W^{1,p}$-setting which arise as Euler - Lagrange equations to certain variational integrals plus pollution term and/or we consider minimizers to a variational problem. The solutions are assumed to be stationary in the sense that the differential of the variational integral vanishes with respect to variations of the independent *and* dependent variables. We impose new structural conditions on the nonlinearities which yield $\mathcal{C}^\alpha$-regularity and $\mathcal{C}^\alpha$-estimates for the solutions. These structure conditions cover variational integrals like $\int F(\nabla u)\, dx$ with potential $F(\nabla u) := \tilde{F}(Q_1(\nabla u), \dots, Q_N(\nabla u))$ and positively definite quadratic forms Q_i in ∇u defined as $Q_i(\nabla u) = \sum_{\alpha\beta} a_i^{\alpha\beta} \nabla u^\alpha \cdot \nabla u^\beta$. A simple example consists in $\tilde{F}(\xi_1, \xi_2) := |\xi_1|^{\frac{p}{2}} + |\xi_2|^{\frac{p}{2}}$ or $\tilde{F}(\xi_1, \xi_2) := |\xi_1|^{\frac{p}{4}} |\xi_2|^{\frac{p}{4}}$. Since the quadratic forms Q_i need not to be linearly dependent our result covers a class of *nondiagonal*, possibly *nonmonotone* elliptic systems. As a by product we also prove a kind of the Liouville theorem. As a new analytical tool we use a new weighted integral technique with singular weights in an L^p-setting for the proof and establish a weighted hole-filling inequality in a setting where Green-function techniques are not available.

1. Introduction and statement of the result

This paper deals with nonlinear elliptic system of the form

$$-\operatorname{div} F_\eta(u, \nabla u) + F_u(u, \nabla u) = b(x, u, \nabla u) \tag{1.1}$$

that is supposed to be satisfied in an open set $\Omega \subset \mathbb{R}^d$. The left hand side of (1.1) is the Euler operator of a variational integral

$$J(u) := \int_\Omega F(u, \nabla u) \tag{1.2}$$

with p-growth (with $p \in (1, \infty)$) in the gradient of an unknown $u : \Omega \to \mathbb{R}^N$. The right hand side of (1.1) is a pollution term with $b : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^N$ being a Carathéodory mapping which satisfies certain growth assumptions (see below for precise formulation). The potential $F : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is supposed to be a $\mathcal{C}^1$

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function which has the p-growth in the second variable and through the paper we employ the notation

$$F_\eta(u,\eta) := \frac{\partial F(u,\eta)}{\partial \eta} : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N},$$
$$F_u(u,\eta) := \frac{\partial F(u,\eta)}{\partial u} : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^{N}.$$

In particular, we also use the following abbreviations

$$F_{\eta_i^\alpha}(u,\eta) := \left(\frac{\partial F(u,\eta)}{\partial \eta}\right)^{\alpha,i} := \frac{\partial F(u,\eta)}{\partial \eta_i^\alpha},$$
$$F_{u^\alpha}(u,\eta) := \left(\frac{\partial F(u,\eta)}{\partial u}\right)^{\alpha} := \frac{\partial F(u,\eta)}{\partial u^\alpha}.$$

The paper focuses on new weighted estimates for a solution to (1.1) and/or to minimizers of (1.2) from which we finally deduce the everywhere Hölder continuity of the solution and the Liouville theorem for the system (1.1). This paper also completes and extends the results based on the weighted estimates obtained in [5, 6], where however the simpler cases were solved - in [5] the authors considered the potential F being u independent and in [6] only the quadratic growth, i.e. $p = 2$ is considered.

In order to simply describe the main novelties of the paper, we first (and very roughly) describe the structural assumptions on the potential F, which will be later described in more details. It is worth of noticing that we do not consider any convexity of F with respect to the second variable here and we replace it just by the standard p-coercivity and p-growth assumption on F. Besides them we impose in addition two following conditions.
(i) a **one sided condition**, i.e.,

$$F_u(u,\eta)\cdot u \geq -K$$

and related generalizations;
(ii) a type of generalized **splitting condition** which allow us to treat potentials of the form:

$$F^1(u,\eta) := \sum_{i=1}^k a^i(u)|\eta|_i^p, \tag{1.3}$$

$$F^2(u,\eta) := a(u)\prod_{i=1}^k |\eta|_i^{p_i}, \qquad p_i\in\mathbb{R},\ \sum_{i=1}^k p_i = p, \tag{1.4}$$

for some $k \in \mathbb{N}$ and for the i's norm defined through

$$|\eta|_i^2 := \sum_{\alpha,\beta=1}^N \sum_{l,m=1}^d B_i^{\alpha\beta} h_{lm}\eta_l^\alpha \eta_m^\beta,$$

where, the matrices $B_i \in \mathbb{R}^{N\times N}$ are assumed to by symmetric and positively definite as well as the matrix $h \in \mathbb{R}^{d\times d}$. Notice here, that while there B_i can vary for different is the matric h is the same for all i-norms. In principle, we could also consider h to be x-dependent but for simplicity we omit such a generalization here and we refer the interested reader to [5] for detailed comments.

In case $k = 1$ we will refer to (1.3) or (1.4) as to the potential having the Uhlenbeck[1] structure. However, since matrices B_i need not to commute the potential F given by (1.3)–(1.4) can be very complex (and for (1.4) even non-convex in the second variable) and consequently the system (1.1) can be highly non-diagonal (or even non-elliptic) and moreover "far away" from the Uhlenbeck like structure, which is (up to small perturbations) the only case of a non-diagonal operator where the full regularity and other important qualitative properties of the corresponding elliptic boundary value problem are available, see [25] or [17, 18] for related generalizations. On the other hand for general elliptic systems and/or potentials F even being u-independent, one cannot expect the everywhere regularity theorem as was shown in [23] for solution being not $\mathcal{C}^1$ and in [26] for unbounded solutions. From this background it is of interest to find other classes of non-diagonal principal part where everywhere regularity theorems (e.g. everywhere Höder continuity theorem, which is one of the results proved here), Liouville theorems, etc., can be established and the potentials given in (1.3)–(1.4) may serve as a prototype examples.

Since the term F_u in the equation may still have the critical p-growth with respect to η, the $\mathcal{C}^\alpha$-regularity of a solution may fail. The so-called one sided condition is an additional condition on the structure of F that allows us to obtain regularity of the solution (combined with further assumptions). In applications to differential geometry (in that case usually $p = 2$ and F is of the Uhlenbeck structure), the one sided condition has a geometrical interpretation. Moreover, it is known for F not satisfying such a condition the solution or minimizer may not be continuous, see [8, 12, 13, 14]. Further, the one sided condition occurs also in Bellman systems of stochastic differential games with discount control, see [2, 3]. In addition, for *non-variational* problems, even for $d = p = 2$, the one sided condition need not be sufficient for the Hölder continuity of the solution, cf. the example in [1]. In such cases to obtain Hölder continuity of the solution one need to assume a priori more structural information about the solution and a prototype example is the so-called angle condition of Wiegner [27, 29, 28, 30]. On the other hand, in variational case this angle condition is **not** needed, as one can see from our present paper. Furthermore, the equation are much more general (p-growth, possible non-convexity, generalized splitting condition rather than the diagonallity of the principal part).

1.1. **Assumptions on the data.** First, we specify the precise structural assumptions imposed on the potential F and we also discuss them in context of the prototype examples (1.3)–(1.4). In the rest of this section we assume that $\alpha_0, \alpha_0^*, \delta_A$ are given strictly positive constants and that $\delta_0 \geq 0$. The starting assumptions are the standard coercivity, growth and smoothness assumption on F, namely

$$F \in \mathcal{C}^1(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{d\times N}), \tag{1.5}$$

$$\alpha_0|\eta|^p - \alpha_0^* \leq F(u,\eta) \leq \alpha_0^*(|\eta|^p + 1) \qquad \text{for all } (u,\eta) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}. \tag{1.6}$$

Note that in the case when F is given by (1.3)–(1.4), the assumptions (1.5)–(1.6) reduces to condition that $a^i \in \mathcal{C}^1$ are strictly nonnegative bounded functions. Fore sure, the assumptions (1.5)-(1.6) are not sufficient for proving further qualitative results for the solution u as is well demonstrated in [26, 23, 15] for counterexamples

[1] We call it Uhlenbeck according to the first proof of the regularity of systems with F having the structure (1.3) with $k = 1$ authored by Uhlenbeck [25].

for even convex potentials. Therefore, inspired by introduction we introduce a generalized version of the one-sided condition i), namely

$$F_\eta(u,\eta)\cdot\eta + F_u(u,\eta)\cdot u \ge \alpha_0(\delta_0+|\eta|^2)^{\frac{p-2}{2}}|\eta|^2 - \alpha_0^* \tag{1.7}$$

for all $(u,\eta)\in\mathbb{R}^N\times\mathbb{R}^{d\times N}$. To illustrate this assumption for F given by (1.3)–(1.4) we first evaluate F_η and F_u. Thus, a simple algebraic manipulation gives

$$F^1_{\eta^\alpha_l}(u,\eta) = p\sum_{i=1}^k\sum_{\beta=1}^N\sum_{m=1}^d a^i(u)|\eta|_i^{p-2}B_i^{\alpha\beta}h_{lm}\eta^\beta_m, \tag{1.8}$$

$$F^2_{\eta^\alpha_l}(u,\eta) = a(u)\sum_{i=1}^k\sum_{\beta=1}^N\sum_{m=1}^d p_i|\eta|_i^{p_i-2}B_i^{\alpha\beta}h_{lm}\eta^\beta_m\prod_{j=1,j\neq i}^k|\eta|_j^{p_j}, \tag{1.9}$$

$$F^1_{u^\alpha}(u,\eta) = \sum_{i=1}^k a^i_{u^\alpha}(u)|\eta|_i^p, \qquad F^2_{u^\alpha}(u,\eta) = a_{u^\alpha}(u)\prod_{i=1}^k|\eta|_i^{p_i}. \tag{1.10}$$

Next, it is evident that in both cases we have that

$$F^{1,2}_\eta(u,\eta)\cdot\eta = pF^{1,2}(u,\eta) \tag{1.11}$$

and therefore for such potentials the assumption (1.7) reduces to

$$a^i(u) + pa^i_u(u)\cdot u \ge \alpha_0. \tag{1.12}$$

Next, we introduce the generalized splitting condition (related to ii)). We assume that there exists a symmetric matric-valued function $A:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}^{N\times N}$, a matrix-valued function $H:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}^{N\times d}$ and positively definite symmetric matrix $h\in\mathbb{R}^{d\times d}$ such that

$$F_{\eta^\alpha_l} = \sum_{\beta=1}^N\sum_{m=1}^d A^{\alpha\beta}(u,\eta)h_{lm}\eta^\beta_l + H^\alpha_i(u,\eta) \tag{1.13}$$

for all $(u,\eta)\in\mathbb{R}^N\times\mathbb{R}^{d\times N}$, all $l=1,\dots,d$ and all $\alpha=1,\dots,N$ and we assume that

$$|H(u,\eta)| \le \alpha_0^*(1+|\eta|)^{p-1-\delta_A}. \tag{1.14}$$

In addition, we require that A is uniformly p-positively definite, that means

$$\sum_{\alpha,\beta=1}^N A^{\alpha\beta}(u,\eta)\mu^\alpha\mu^\beta \ge \alpha_0(|\eta|^2+\delta_0)^{\frac{p-2}{2}}|\mu|^2, \quad |A(u,\eta)|\le\alpha_0^*(\delta_0+|\eta|^2)^{\frac{p-2}{2}} \tag{1.15}$$

uniformly with respect to the argument of u, μ and η. Note that (1.13) holds for F given by (1.3)–(1.4) with H being identically zero. Indeed defining $A_{1,2}$ by

$$A_1^{\alpha\beta} := p\sum_{i=1}^k a^i(u)|\eta|_i^{p-2}B_i^{\alpha\beta},$$

$$A_2^{\alpha\beta} := a(u)\sum_{i=1}^k p_i|\eta|_i^{p_i-2}B_i^{\alpha\beta}\prod_{j=1,j\neq i}^k|\eta|_j^{p_j}$$

we see that (1.13) holds. It is also easily follows from positive definiteness of B_i that (1.15) holds also for A_1. Moreover, for A_2 it is valid as well in case that $p_i\ge 0$ for all $i=1,\dots,k$. Even more, if some p_i is negative the validity of (1.15) still may hold under some additional hypothesis on the structure of B_i. Moreover, due to the presence of the pollution term H in (1.13) we can even consider the case when B_i depends on η and then besides the ellipticity condition we require that

$$|B_\eta(\eta)| \le \alpha_0^*(1+|\eta|)^{-1-\delta_A} \tag{1.16}$$

which automatically guarantee the validity of (1.13) and (1.14). Variational integrals with p-growth and splitting condition have been consider in wide literature, we refer here to [10] for the proof of partial regularity of the solution and we would like to also mention the original results [11]. For more general overview of known results for nonlinear elliptic systems, we refer the interested reader to [22, 12] and the references therein.

The last restriction we impose on F is related to the p-growth and p-coercivity also for F_η, which however seems to be very natural and valid for most potentials. Hence we assume in the paper that for all $(u,\eta) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ the following holds

$$-\alpha_0^* + \alpha_0(\delta_0 + |\eta|^2)^{\frac{p-2}{2}}|\eta|^2 \le F_\eta(u,\eta)\cdot\eta \le pF(u,\eta) + E(u,\eta) \tag{1.17}$$

with E satisfying

$$|E(u,\eta)| \le C(1+|\eta|)^{p-\delta_A} \tag{1.18}$$

for all $(u,\eta) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$. Again note that (1.17) is trivially valid for F^1 and F^2 even with $E \equiv 0$ (see (1.11)) and one can observe that the presence of p in front of F in (1.17) is a natural setting. Moreover, the presence of E allows us to consider more general structure than (1.3)–(1.4) just by adding some lower order terms or just by considering B dependent on η and satisfying (1.16). To complete the set of assumptions on F we add there also the natural growth conditions for F_η and F_u

$$|F_\eta(u,\eta)|^{p'} + |F_u(u,\eta)| \le \alpha_0^*(1+|\eta|^p) \qquad \text{for all } (u,\eta) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}. \tag{1.19}$$

Having all these assumptions on F we finally introduce the growth condition for b, i.e., we require that it satisfies the following

$$|b(x,u,\eta)| \le \alpha_0^*(1+|\eta|^{p-1-\delta_A}) \qquad \text{for all } (x,u,\eta) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}. \tag{1.20}$$

A lot of alternative conditions with better growth in η is possible. We consider the pollution term in order to destroy the variational structure and thus to exclude the use of arguments based on minimization properties. However, it is also of interest to consider the minimizers or just to simply set $b \equiv 0$.

1.2. **Statement of the results.** In this subsection we state all main results of the paper. Before doing it, we recall some important notions needed in the paper. First, for simplicity we denote $D_k := \frac{\partial}{\partial x_k}$. Since we deal with a weak solution to (1.1), which in principle do not need to have certain usual qualitative properties, we need to add such properties to them a priori. However, it appears (and is inspired by [5]) that the only property is the so-called the Noether equation, which has the form

$$-\operatorname{div}(F_\eta(u,\nabla u)\cdot D_k u) + D_k F(u,\nabla u) = b(x,u,\nabla u)\cdot D_k u, \qquad \text{for all } k = 1,\dots,d,$$

or in a weak formulation

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_i^\alpha}(u,\nabla u) D_i\psi_j D_j u^\alpha - F(u,\nabla u)\operatorname{div}\psi \, dx \\
&\qquad = \int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{N} b^\alpha(\cdot,u,\nabla u)\psi_i D_i u^\alpha \, dx \qquad \text{for all } \psi\in\mathcal{D}(\Omega;\mathbb{R}^d).
\end{aligned} \tag{1.21}
$$

Note that the Noether equation can be formally derived from (1.1) by multiplying by $D_k u$, but in general cannot be derived rigorously due to the low regularity of u. On the other hand, in case we deal with minimizers to (1.2), we can show also the validity of (1.21) without any a priori knowledge (see [4, 6]). To be more precise, we say that a solution $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer if for all $\varphi\in\mathcal{D}(\Omega;\mathbb{R}^N)$ it satisfies

$$
\int_{\operatorname{supp}\varphi} F(u,\nabla u)\, dx \le \int_{\operatorname{supp}\varphi} F(u+\varphi,\nabla u+\nabla\varphi) - b(x,u,\nabla u)\cdot\varphi \, dx. \tag{1.22}
$$

Note that in case b depends only on x it is a standard notion to minimizer of the variational problem

$$
\int_\Omega F(u,\nabla u) - b\cdot u \, dx.
$$

It is worth of noticing that the Noether equation as an additional condition plays an important in regularity theory of harmonic mappings, cf. [7] or [21] but can be also used for further investigation of qualitative properties to variational but non-coercive problems, see [24].

Finally, since we have to deal with possible unbounded solutions, we need to add some additional conditions. Thus, either we assume some more restrictive assumption on F or we will assume some critical explosion rate for the mean value for the given solution u. To be more precise, we say that u satisfies the ln condition in Ω if there exists $C_{\ln}$ such that for all $B_R(x_0)\subset\Omega$ and all $R\le\frac12$

$$
\left| \int_{B_R(x_0)} u \, dx \right| \le C_{\ln} |\ln R|^{\min(\frac12,\frac1{p'})} R^d \tag{1.23}
$$

Another possibility how to avoid a possible explosion of the solution is to assume more restrictive condition on F. Therefore, in case (1.23) is not valid we will need that for all $\alpha = 1,\dots,N$ either

$$
F_{u^\alpha}(u,\eta)u^\alpha \ge -\frac{\alpha_0}{2N}|\eta|^p - \alpha_0^* \qquad \text{and} \quad |F_{u^\alpha}(u,\eta)| \le a(u^\alpha)(1+|\eta|^p), \tag{1.24}
$$

or

$$
F_{u^\alpha}(u,\eta)u^\alpha \le a(u^\alpha)(1+|\eta|^p) \qquad \text{and} \quad |F_{u^\alpha}(u,\eta)| \le a(u^\alpha)(1+|\eta|^p), \tag{1.25}
$$

where a is a nonnegative continuous function fulfilling

$$
a(s)\to 0 \qquad \text{if } s\to\pm\infty.
$$

Note that (1.24) or (1.25) may be viewed as a no further restriction (see (1.7) where the growth of $F_u\cdot u$ is considered) and therefore (1.24) or (1.25) represents a kind of small oscillation property for large values of u.

Therefore in what follows, we state all theorem either for minimizers, or for just weak solution satisfying in addition certain regularity or satisfying (1.21) and we shall also assume either (1.25) or (1.23). The first result of the paper is the local everywhere Hölder continuity result.

Theorem 1.1 (Local Hölder continuity)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let F satisfy* (1.5)–(1.7), (1.13)–(1.15) *and* (1.17)–(1.19) *and let b satisfy* (1.20). *Assume that $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ is a weak solution to* (1.1). *Then there exists $\alpha \in (0,1)$ depending only on $\alpha_0, \alpha_0^*, \delta_0$ such that for any $\Omega_0 \subset\subset \Omega$*

$$\|u\|_{\mathcal{C}^\alpha(\Omega_0)} \le K, \tag{1.26}$$

provided that one of the following holds:

1) *u is a minimizer, i.e., satisfies* (1.22), *and either F satisfies* (1.24) *or* (1.25), *or the condition* (1.23) *holds. Then $K = K(\|u\|_{1,p}, \Omega_0, \alpha_0, \alpha_0^*, \delta_A, C_{\ln})$ and in case we consider* (1.24) *or* (1.25) *then K does not depend on $C_{\ln}$.*
2) *u is continuous and either u satisfies* (1.21) *or $u \in W^{1,p+1} \cap W^{2,\frac{p+1}{2}}$. Then $K = K(\|u\|_{1,p}, C_{\ln}, \Omega_0, \alpha_0, \alpha_0^*, \delta_A)$.*

Furthermore, in case $p = 2$ the constant K does not depend on $C_{\ln}$.

We would like to point our here, that the condition 1) states the Hölder continuity for any weak solution being minimizer with the bound K depending only on *known* data and 2) is though to be used for proving the uniform $\mathcal{C}^\alpha$ estimates for a regular approximative problem which then lead to the existence for ate least one Hölder continuous solution in case that F is convex with respect to η and if L^∞ a priori bound is available.

The second theorem states the the Hölder continuity of the solution near the boundary.

Theorem 1.2 (Boundary regularity)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with $\mathcal{C}^{1,1}$ boundary. Let F satisfy* (1.5)–(1.7), (1.13)–(1.15) *and* (1.17)–(1.19) *and let b satisfy* (1.20). *Assume that $u \in W^{1,p}_0(\Omega;\mathbb{R}^N)$ is a weak solution to* (1.1). *Then there exists $\varepsilon, \alpha > 0$ such that such that*

$$\|u\|_{\mathcal{C}^\alpha(\Omega_\varepsilon)} \le K, \qquad \Omega_\varepsilon := \{x \in \Omega;\ dist\ (x, \partial\Omega) < \varepsilon\}. \tag{1.27}$$

provided that one of the following holds:

1) *u is a minimizer, i.e., satisfies* (1.22). *Then $K = K(\|u\|_{1,p}, \alpha_0, \alpha_0^*, \delta_A)$.*
2) *u is continuous and either u satisfies* (1.21) *or $u \in W^{1,p+1} \cap W^{2,\frac{p+1}{2}}$. Then $K = K(\|u\|_{1,p}, \alpha_0, \alpha_0^*, \delta_A)$.*

It should be mention here that while in Theorem 1.1 we require either some a priori knowledge about the possible blow up of mean values or we assumed further conditions on F (namely (1.24) and (1.25)), in Theorem 1.2 such restrictions are *not needed*. It is due to the fact, that we already fixed $u = 0$ at the boundary $\partial\Omega$. Moreover, the corresponding estimate represented by K in (1.27) does not depends on L^∞ bound of the solution.

The last theorem we prove is the Liouville type theorem.

Theorem 1.3 (Liouville theorem)**.** *Let F satisfy* (1.5)–(1.7), (1.13)–(1.15), (1.17)–(1.19) *and let $F(u, \lambda\eta) = \lambda^p F(u, \eta)$ for all u, η and all $\lambda > 0$. Assume that $u \in W^{1,p}_{loc}(\mathbb{R}^d;\mathbb{R}^N)$ is a weak solution to* (1.1) *with $b \equiv 0$. Then u is a constant provided that one of the following holds:*

1) *u is a minimizer, i.e., satisfies* (1.22)*, F satisfies* (1.25) *and that there exists a constant $C>0$ such that (BMO property at infinity)*

$$\int_{B_R(0)} |u-\overline{u}_{B_R(0)}|^p \le CR^d \qquad \text{for all } R>1. \tag{1.28}$$

2) *u is bounded and satisfies* (1.21) *for all $\psi \in \mathcal{D}(\mathbb{R}^d;\mathbb{R}^d)$.*

Note here that in case $p \ge d$, the proof of Theorem 1.3 is standard and based on the testing by solution and using the one sided condition. Therefore the main novelty for this setting consist in case $p<d$, where we are able to replace the assumption on the decay of ∇u at infinity by some corresponding weighted estimates for $|\nabla u|^p$. Furthermore, although the condition (1.28) does not fit well to the one sided condition, the Liouville theorem still holds. The importance of a such theorem also follows from the results in [19, 20, 16], where the very close relation between the Liouville theorem and the $\mathcal{C}^{1,\alpha}$ regularity of the solution is investigated in the setting when one assumes that ∇u is bounded. Here, we consider the case with one derivative less and therefore Theorem 1.3 is related to $\mathcal{C}^\alpha$ regularity and can be further used for an indirect approach, where as the comparison problem one take the potentials fulfilling assumptions of Theorem 1.3.

Finally, in case we assume that F is uniformly p-convex in the second variable, one can use the standard difference quotient technique and the interpolation theorem and to prove the following

Corollary 1.1. *Let all assumptions of Theorem 1.1 be satisfied and assume that b is bounded. In addition, let F satisfy*

$$\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{d} \frac{\partial F(u,\eta)}{\partial \eta_i^\alpha \partial \eta_j^\beta}\mu_i^\alpha \mu_j^\alpha \ge C(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\mu|^2.$$

Then

$$(\delta_0+|\nabla u|^2)^{\frac{p}{4}} \in W^{1,2}_{loc}(\Omega).$$

Furthermore, there exists $\varepsilon>0$ such that

$$u \in W^{1,q}_{loc}(\Omega;\mathbb{R}^N) \qquad \text{for } q := p+2+\varepsilon.$$

In addition, if F has the Uhlenbeck structure then $u \in \mathcal{C}^{1,\alpha}_{loc}(\Omega;\mathbb{R}^N)$ for some $\alpha>0$.

The rest of the paper is devoted to the proof of Theorem 1.1–1.3. For sake of simplicity we prove all results only for the case $h_{ij} := \delta_{ij}$, where δ is the Kronecker symbol and we refer the interested reader to [5] for the generalized method that is able to capture the general case. In Section 2, we provide a general scheme for proving the Hölder continuity of the solution (without proofs) and we restrict ourselves to the simplest Uhlenbeck setting. Next, in Section 3, we mostly recall two standard results for minimizers, namely the Caccioppoli inequality and consequently the reverse Hölder inequality. Section 4 is devoted to the estimates based on the use of the Noether equation and can be understood as a generalization of the monotonicity formula used in the theory of harmonic mappings. Section 5 is devoted to the estimates based on the one sided condition that finally lead to the proof of the VMO property of the solution u in Section 6. Finally, in Section 7 we give the complete proofs of Theorem 1.1–1.2 and Section 8 is devoted to the Liouville theorem. Last, in Appendix we provide several examples of the structural assumptions on F that leady either to the L^∞ bound or to sharp bound for the constant $C_{\ln}$ used in the assumption of Theorem 1.1 and in addition there is shown that in case $p=2$ the constant $C_{\ln}$ can be bounded in terms of other data.

We end this subsection by introducing all necessary notations related to the localization procedure used further in the paper. Therefore, for any $x_0 \in \mathbb{R}^d$, any $R>0$ and any $\Omega \subset \mathbb{R}^d$, we define

$$\begin{aligned} B_R(x_0) &:= \{x \in \mathbb{R}^d; |x-x_0| \le R\}, & B_R^\Omega(x_0) &:= B_R(x_0) \cap \Omega, \\ A_R(x_0) &:= B_{2R}(x_0) \setminus B_R(x_0), & A_R^\Omega(x_0) &:= A_R(x_0) \cap \Omega \end{aligned}$$

In addition, we introduce the cut-off function τ_R as

$$\tau_R(s) := \tau(s/R),$$

where τ is a smooth nonnegative non-increasing function being equal to one in the interval $[0,1]$ and identically equal to zero in $[2,\infty)$.

2. Structure of the proof in the Uhlenbeck setting

In this section we provide a sketch of the proof in the simplest case when the potential F is given by

$$F(u,\nabla u) = \frac{1}{p}\, a(u)[|\nabla u|^2 + \delta_0]^{\frac{p}{2}}.$$

First, for sufficiently smooth solution or for any minimizer the Noether identity has the form.

Lemma 2.1. *Assume the hypothesis of Theorem 1.1. Then u satisfies*

$$\begin{aligned} &\sum_{\alpha=1}^N \sum_{i,j=1}^d \int_\Omega a(u)(|\nabla u|^2+\delta_0)^{p-2} D_i u^\alpha D_j u^\alpha D_i \varphi_j \, dx \\ &\qquad - \frac{1}{p}\int_\Omega a(u)(|\nabla u|^2+\delta_0)^p \operatorname{div}\varphi \, dx = \sum_{\alpha=1}^N \sum_{i=1}^d \int_\Omega b^\alpha D_i u^\alpha \varphi_i \, dx. \end{aligned} \tag{2.1}$$

for all $\varphi \in \mathcal{D}(\Omega;\mathbb{R}^d)$.

The identity (2.1) is further used to provide a weighted estimate. Setting

$$\varphi_i = \frac{x_i \tau}{|x|^{d-p-\varepsilon}}$$

in (2.1) where $\tau \in \mathcal{D}(\Omega)$ is a nonnegative function, we deduce

Lemma 2.2. *Assume that assumptions of Theorem 1.1 hold. Then the solution u in Lemma 2.1 satisfies*

$$\begin{aligned} &\int_\Omega \frac{\varepsilon |\nabla u|^p \tau}{|x|^{d-p-\varepsilon}} + \frac{(|\nabla u|^2+\delta_0)^{\frac{p-2}{2}} |\nabla u \cdot x|^2 \tau}{|x|^{d-p+2-\varepsilon}} \, dx \\ &\qquad \le C \int_\Omega \frac{(1+|\nabla u|^p)(|\nabla \tau| + \tau)}{|x|^{d-p-1-\varepsilon}} \, dx. \end{aligned} \tag{2.2}$$

The approach is related to the monotonicity method from harmonic mapping theory, however we work with the term that is related to the boundary integral in the standard theory. Thus, setting in (2.2) $\tau = 1$ in a ball $B_R(0)$ and $\tau = 0$ outside the ball $B_{2R}(0)$ and using the fact that the last term is of the lower order, we have the weighted estimate (for some fixed $\beta > 0$ depending only on data)

$$\int_{B_R} \frac{(|\nabla u|^2+\delta_0)^{\frac{p-2}{2}}|\nabla u \cdot x|^2}{|x|^{d-p+2}}\,dx \le C\left(R^\beta + \int_{B_{2R}\setminus B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx\right). \tag{2.3}$$

In an intermediate step one can establish an L^∞-bound for u or an estimate $\int_Q e^{|u|^q}\,dx \le K$ which is not hard to prove, say by the Moser method - using the additional coercivity (1.12). We also refer to appendix, where such a procedure is described in detail for certain special form of F.

In the second step we test (1.1) by $u\tau$ and by using the one sided condition we establish

Lemma 2.3. *The solution in Lemma 2.1 satisfies*

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le C\left(R^\beta + \int_{B_{2R}\setminus B_R} \frac{(|\nabla u|^2+\delta_0)^{\frac{p-2}{2}}|\nabla u \cdot x||u|}{|x|^{d-p+2}}\,dx\right) \tag{2.4}$$

with some $\beta > 0$.

If an L^∞-estimate for u is available we conclude from Lemma 2.1 and Lemma 2.2 a non-homogeneous hole-filling inequality for the quantity

$$G := \frac{(|\nabla u|^2+\delta_0)^{\frac{p-2}{2}}|\nabla u \cdot x|^2}{|x|^{d-p+2}},$$

which is of the form

Lemma 2.4. *The solution u satisfies*

$$\int_{B_R} G\,dx \le C\left(\int_{B_{4R}\setminus B_R} G\,dx\right)^{\frac{1}{2}} + KR^\beta. \tag{2.5}$$

From [9] we know that this implies the logarithmic Morrey estimate

$$\int_{B_R} G\,dx \le K|\ln R|^{-\gamma}, \tag{2.6}$$

with some $\gamma > 0$. This implies uniform smallness for $\int_{B_R} G\,dx$ as $R \to 0$. Concerning the proof of Lemma 2.4, we may replace in Lemma 2.3 B_R by B_{2R}, estimate the right hand side of (2.4) by using the Hölder inequality and apply Lemma 2.2 to obtain Lemma 2.4.

Finally, from the uniform smallness (2.6), it is possible to derive a uniform estimate for the Hölder norm of the solution provided it is smooth enough. This is usually done by Campanato-like technique, which however cannot be used because we do not have a proper comparison problem. Therefore, we present an alternative way based on a global hole-filling technique. The starting point is the Caccioppoli inequality, which may be derived from (1.1) by testing $(u-c)\tau$ and by using the reverse Hölder inequality and the BMO-estimate coming from (2.3)

Lemma 2.5. *The solution u satisfies*

$$\begin{aligned}\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}} &\le CR^\beta + C\left(\int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\,dx\right)^{1+\beta}\\ &\quad + \int_{B_{2R}\setminus B_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u \cdot x|^2}{|x|^{d-p+2}}\,dx.\end{aligned} \tag{2.7}$$

Hence, combining (2.3) and (2.7) we can find $K \gg 1$ such that defining

$$A(R) := \int_{B_R} \frac{|\nabla u|^p}{R^{d-p}} + \frac{K(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u \cdot x|^2}{|x|^{d-p+2}} \, dx$$

and using the hole-filling technique we obtain

Lemma 2.6. *The solution u satisfies*

$$A(R) \le CR^{\beta} + \frac{1}{2} A(2R) + C(A(2R))^{1+\beta}. \tag{2.8}$$

Finally, using the uniform smallness (2.6), we can iterate in (2.8) and observe that

$$A(R) \le CR^{\gamma}$$

for some $\gamma > 0$. Consequently, due to the Morrey lemma we conclude

Corollary 2.1. *The solution u satisfies*

$$\|u\|_{\mathcal{C}^{\frac{\gamma}{p}}} \le K$$

with uniform bound K depending only on the data.

We would like to finish this introductory part by recalling that for the simplest prototype case, the more efficient method could be used leading finally to the full regularity of the solution. However, the main purpose was to demonstrate our new technique on the simplest most understandable case in order to simplify the further reading of the paper.

3. Caccioppoli inequality for minimizers and its consequences

This section is devoted to the standard properties of minimizers and we refer to [12] for detailed proof. Since we have in (1.1) the possible pollution term b, we quickly repeat the standard proofs in this subsection.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, F satisfy* (1.6) *and b satisfy* (1.20). *Assume that $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ satisfies* (1.22). *Then there exists a constant C depending only on α_0, α_0^* such that for all $x_0 \in \Omega$ and all $R > 0$ fulfilling $B_{2R}(x_0) \subset \Omega$ we have*

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{R^d} \, dx \le C + C \int_{B_{2R}(x_0)} \frac{|u - \bar{u}_{2R}|^p}{R^{d+p}} \, dx, \tag{3.1}$$

where $\bar{u}_{2R}$ denotes the mean value over the ball $B_{2R}(x_0)$. Moreover, if Ω is a Lipschitz domain and $u \in W^{1,p}_0(\Omega; \mathbb{R}^N)$ then for all $x_0 \in \Omega$ and all $R > 0$ such that $B_{2R}(x_0) \not\subseteq \Omega$ there holds

$$\int_{B^{\Omega}_R(x_0)} \frac{|\nabla u|^p}{R^d} \, dx \le C + C \int_{B^{\Omega}_{2R}(x_0)} \frac{|u|^p}{R^{d+p}} \, dx. \tag{3.2}$$

Proof. The proof follows line by line the proof of [12, Theorem 3.1, page 159]. Hence, for any $t < s < 2R$ we find $\eta \in \mathcal{D}(B_s(x_0))$ such that $\eta \equiv 1$ in $B_t(x_0)$ and $|\nabla \eta| \le \frac{C}{s-t}$. Then we define $\varphi := \eta(\tilde{u} - u)$ where

$$\tilde{u} := \begin{cases} \bar{u}_{2R} & \text{if } B_{2R}(x_0) \subset \Omega, \\ 0 & \text{otherwise} \end{cases}$$

and use such a φ in (1.22). Consequently, we get (we omit writing x_0 in $B_s(x_0)$ in what follows)

$$\begin{aligned}\int_{B_s^\Omega} F(u,\nabla u)\,dx \le \int_{B_s^\Omega} F(u-\eta(u-\tilde{u}),\nabla(u-\eta(u-\tilde{u})))\,dx \\ +\int_{B_s^\Omega} b(x,u,\nabla u)\cdot\eta(u-\tilde{u})\,dx.\end{aligned}\tag{3.3}$$

Thus, using (1.6) and (1.20) we deduce that

$$\begin{aligned}\int_{B_s^\Omega} \alpha_0|\nabla u|^p-\alpha_0^*\,dx &\le C\alpha_0^*\int_{B_s^\Omega} 1+(1-\eta)^p|\nabla u|^p+|\nabla\eta|^p|u-\tilde{u}|^p\,dx\\ &+\int_{B_s^\Omega}\alpha_0^*(1+|\nabla u|)^{p-1}|u-\tilde{u}|\,dx.\end{aligned}$$

Hence, using the Young inequality to absorb the part of the last term to the left hand side, using the definition of η we find that

$$\int_{B_t^\Omega}|\nabla u|^p\,dx \le C\left(R^d+\int_{B_s^\Omega\setminus B_t^\Omega}|\nabla u|^p\,dx+\int_{B_{2R}}\frac{|u-\tilde{u}|^p}{(s-t)^p}+|u-\tilde{u}|^p\,dx\right).$$

Thus, using [12, Lemma 3.1, page 161] we finally conclude (3.1) and (3.2). □

We end this short subsection be recalling the reverse Hölder inequality which directly follows from Lemma 3.1 and the Poincaré inequality (see also [12, Chapter V]) and therefore we state it here without proof.

Lemma 3.2. *Let $\Omega\subset\mathbb{R}^d$ be an open set, F satisfy* (1.6) *and b satisfy* (1.20)*. Assume that $u\in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ satisfies* (1.22)*. Then there exist constants $C,\varepsilon>0$ depending only on α_0,α_0^* such that for all $x_0\in\Omega$ and all $R\in(0,1)$ fulfilling $B_{2R}(x_0)\subset\Omega$ we have*

$$\left(\int_{B_R(x_0)}\frac{|\nabla u|^{p+\varepsilon}}{R^d}\,dx\right)^{\frac{1}{p+\varepsilon}}\le C+C\left(\int_{B_{2R}(x_0)}\frac{|\nabla u|^p}{R^d}\,dx\right)^{\frac{1}{p}}.\tag{3.4}$$

Moreover, if Ω is a Lipschitz domain and $u\in W^{1,p}_0(\Omega;\mathbb{R}^N)$ then there exists $R_H>0$ such that for all $x_0\in\Omega$ and all $R\in(0,R_H)$ such that $B_{2R}(x_0)\nsubseteq\Omega$ there holds

$$\left(\int_{B_R^\Omega(x_0)}\frac{|\nabla u|^{p+\varepsilon}}{R^d}\,dx\right)^{\frac{1}{p+\varepsilon}}\le C+C\left(\int_{B_{2R}^\Omega(x_0)}\frac{|\nabla u|^p}{R^d}\,dx\right)^{\frac{1}{p}}.\tag{3.5}$$

4. Weighted estimates based on the use of Noether's equation

In this section we derive uniform a priori estimates for any sufficiently smooth solution to (1.1), in particular for those satisfying (1.21). Before doing so, we recall the following Lemma, which is related to Lemma 2.1.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^d$ be an open set and let F satisfy* (1.5)–(1.6) *and* (1.19) *and b satisfy* (1.20)*. Assume that $u\in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ is a weak solution to* (1.1)*. Then* (1.21) *holds provided that either u is a minimizers, i.e., it satisfies* (1.22)*, or u has the additional regularity $u\in W^{1,p+1}\cap W^{2,\frac{p+1}{2}}(\Omega;\mathbb{R}^N)$. Moreover, if $u\in W^{1,p}_0(\Omega;\mathbb{R}^N)$ and $\Omega\in\mathcal{C}^{1,1}$ then* (1.21) *holds for all $\varphi\in\mathcal{C}^{0,1}(\Omega;\mathbb{R}^N)$ such that $\varphi\cdot n=0$ on $\partial\Omega$, where n denotes the unit normal vector on $\partial\Omega$.*

Proof. We do not provide the proof here but refer to our previous results [4, 6], where Lemma 4.1 is proved in detail. □

The next step is to deduce a weighted local estimate from (1.21). Thus, we introduce a generalization of Lemma 2.2 based on the assumptions (1.13), i.e., p-growth, and (1.17), i.e., the spitting condition. For simplicity (and as was announced in the introduction) we focus only on the case, where $h_{lm} = \delta_{lm}$ in (1.13) and we refer the interested reader to [5] for details with general h.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set, F satisfy (1.5)–(1.6) and (1.13)–(1.15), (1.17)–(1.19) with $\delta_A \le 1$ and b satisfy (1.20). Then there exists $R_0 > 0$ such that for any $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ satisfying the Noether equation (1.21), all $\gamma \in [p,d]$, all $x_0 \in \Omega$ and all $R \in (0,R_0)$ such that $B_{2R}(x_0) \subset \Omega$ we have the following uniform estimate*

$$\begin{aligned}\int_{B_R(x_0)} \frac{(\gamma-p)|\nabla u|^p}{|x|^{d-\gamma}} + \frac{(d-\gamma)(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|(x-x_0)\cdot\nabla u|^2}{|x-x_0|^{d-\gamma+2}}\,dx \\ \le CR^{\gamma-p+\delta_A} + C\int_{A_R(x_0)} \frac{|\nabla u|^p}{|x-x_0|^{d-\gamma}}\,dx,\end{aligned} \tag{4.1}$$

where C depends only on $\alpha_0, \alpha_0^*, \delta_A$. In particular, we can conclude

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{|x-x_0|^{d-\gamma}R^{\gamma-p}}\,dx \le \frac{C}{\gamma-p}\left(dist\,(x_0,\partial\Omega))^{\delta_A} + \frac{\|\nabla u\|^p}{(dist\,(x_0,\partial\Omega))^{d-p}}\right). \tag{4.2}$$

and

$$\|u\|_{BMO_{loc}(\Omega),} \le C(\|u\|_{1,p}). \tag{4.3}$$

Moreover, if $\Omega \in \mathcal{C}^{1,1}$ and $u \in W^{1,p}_0(\Omega;\mathbb{R}^N)$ then for arbitrary $x_0 \in \partial\Omega$ and arbitrary $R \in (0,R_0)$ there holds

$$\begin{aligned}\int_{B^\Omega_R(x_0)} \frac{(\gamma-p)|\nabla u|^p}{|x|^{d-\gamma}} + \frac{(d-\gamma)(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|(x-x_0)\cdot\nabla u|^2}{|x-x_0|^{d-\gamma+2}}\,dx \\ \le CR^{\gamma-p+\delta_A} + C\int_{A^\Omega_R(x_0)} \frac{|\nabla u|^p}{|x-x_0|^{d-\gamma}}\,dx.\end{aligned} \tag{4.4}$$

Proof. We give here the proof only for the sake of completeness. Therefore we proceed here only formally and for the rigorous justification we refer to [5, 6]. First, to simplify the proof we assume that $x_0 = 0$. For others x_0 the proof is the same. The proof of (4.1) is based on using

$$\psi(x) := \frac{x\tau_R^p(|x|)}{|x|^{d-\gamma}}$$

as a test function in (1.21). Note that $R > 0$ is assumed such that ψ has a compact support in $B_{2R} \subset \Omega$. By a simple computation we observe that

$$\begin{aligned} D_j\psi_i &= \frac{\delta_{ij}\tau_R^p(|x|)}{|x|^{d-\gamma}} - (d-\gamma)\frac{x_ix_j\tau_R^p(|x|)}{|x|^{d-\gamma+2}} + p\frac{x_ix_j\tau_R^{p-1}(|x|)\tau_R'(|x|)}{|x|^{d-\gamma+1}}, \\ \operatorname{div}\psi &= \frac{\gamma\tau_R^p(|x|)}{|x|^{d-\gamma}} + p\frac{\tau_R^{p-1}(|x|)\tau_R'(|x|)}{|x|^{d-\gamma-1}}. \end{aligned} \tag{4.5}$$

Next, we evaluate all terms in (1.21) with our ψ. To shorten the formulae we omit writing the dependence of τ_R on $|x|$ in what follows. Hence, using (4.5) we get that

$$\int_\Omega F(u,\nabla u)\operatorname{div}\psi\,dx = \int_\Omega F(u,\nabla u)\left(\frac{\gamma\tau_R^p}{|x|^{d-\gamma}} + p\frac{\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma-1}}\right)\,dx. \tag{4.6}$$

Similarly, using (4.5) again and using the splitting assumption (1.13) we get that

$$\begin{aligned}
&\int_\Omega \left(\sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(u,\nabla u)D_i\psi_j D_j u^\alpha\right)\,dx\\
&=\int_\Omega \sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(u,\nabla u)D_j u^\alpha \frac{\delta_{ij}\tau_R^p}{|x|^{d-\gamma}}\,dx\\
&\quad -\int_\Omega \sum_{i,j=1}^d\sum_{\alpha=1}^N F_{\eta_i^\alpha}(u,\nabla u)D_j u^\alpha\left((d-\gamma)\frac{x_ix_j\tau_R^p}{|x|^{d-\gamma+2}} - p\frac{x_ix_j\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma+1}}\right)\,dx\\
&=\int_\Omega \frac{F_\eta(u,\nabla u)\cdot\nabla u\tau_R^p}{|x|^{d-\gamma}}\,dx\\
&\quad -\int_\Omega \sum_{\alpha,\beta=1}^N \frac{A^{\alpha\beta}(u,\nabla u)(x\cdot\nabla u^\alpha)(x\cdot\nabla u^\beta)}{|x|^{d-\gamma+2}}\left((d-\gamma)\tau_R^p - p|x|\tau_R^{p-1}\tau_R'\right)\,dx\\
&\quad -\int_\Omega \sum_{i,j=1}^d\sum_{\alpha=1}^N H_i^\alpha(u,\nabla u)D_j u^\alpha\left((d-\gamma)\frac{x_ix_j\tau_R^p}{|x|^{d-\gamma+2}} - p\frac{x_ix_j\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma+1}}\right)\,dx.
\end{aligned}$$

Thus, using these identities in (1.21) and moving the terms with corresponding signs on the one side we deduce that

$$\begin{aligned}
&-p\int_\Omega \sum_{\alpha,\beta=1}^N A^{\alpha\beta}(u,\nabla u)\frac{(x\cdot\nabla u^\alpha)(x\cdot\nabla u^\beta)\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma+1}}\,dx\\
&\quad+\gamma\int_\Omega \frac{F(u,\nabla u)\tau_R^p}{|x|^{d-\gamma}}\,dx\\
&\quad+(d-\gamma)\int_\Omega \sum_{\alpha,\beta=1}^N A^{\alpha\beta}(u,\nabla u)\frac{(x\cdot\nabla u^\alpha)(x\cdot\nabla u^\beta)\tau_R^p}{|x|^{d-\gamma+2}}\,dx\\
&=\int_\Omega \frac{F_\eta(u,\nabla u)\cdot\nabla u\tau_R^p}{|x|^{d-\gamma}} - \frac{pF(u,\nabla u)\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma-1}}\,dx\\
&\quad-\int_\Omega \sum_{i,j=1}^d\sum_{\alpha=1}^N H_i^\alpha(u,\nabla u)D_j u^\alpha\left((d-\gamma)\frac{x_ix_j\tau_R^p}{|x|^{d-\gamma+2}} - p\frac{x_ix_j\tau_R^{p-1}\tau_R'}{|x|^{d-\gamma+1}}\right)\,dx\\
&\quad-\int_\Omega \sum_{\alpha=1}^N \frac{b^\alpha(\cdot,u,\nabla u)(x\cdot\nabla u^\alpha)\tau_R^p}{|x|^{d-\gamma}}\,dx.
\end{aligned}\tag{4.7}$$

First, since τ_R is nonnegative and non-increasing we see, after using (1.15), that the first integral on the left hand side is nonnegative and therefore we neglect it in what follows. Next, to bound the first integral on the right hand side, we use (1.17)–(1.18) for the first term and (1.6) for the second one, together with the fact that $|\tau_R'| \le C/R$ and that $\nabla \tau_R$ is supported in A_R. For last two integrals we use (1.14) and (1.20) to get the resulting inequality

$$
\begin{aligned}
&\gamma \int_\Omega \frac{F(u,\nabla u)\tau_R^p}{|x|^{d-\gamma}}\,dx \\
&\qquad + (d-\gamma)\int_\Omega \sum_{\alpha,\beta=1}^{N} A^{\alpha\beta}(u,\nabla u)\frac{(x\cdot\nabla u^\alpha)(x\cdot\nabla u^\beta)\tau_R^p}{|x|^{d-\gamma+2}}\,dx \\
&\le p\int_\Omega \frac{F(u,\nabla u)\tau_R^p}{|x|^{d-\gamma}}\,dx + C\int_{A_R}\frac{(1+|\nabla u|)^p}{|x|^{d-\gamma}}\,dx \\
&\qquad + \int_\Omega \frac{(1+|\nabla u|)^{p-\delta_A}\tau_R^p}{|x|^{d-\gamma}}\,dx + \int_\Omega \frac{(1+|\nabla u|)^p\tau_R^p}{|x|^{d-\gamma-1}}\,dx.
\end{aligned}
\tag{4.8}
$$

Thus, absorbing the first integral on the right hand side by the first one on the left hand side, using (1.6), (1.15), the fact that $\gamma \ge p$ and the properties of τ_R we conclude the final formula

$$
\begin{aligned}
&(\gamma-p)\int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma}}\,dx + (d-\gamma)\int_{B_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|x\cdot\nabla u|^2}{|x|^{d-\gamma+2}}\,dx \\
&\le C\left(R^\gamma + \int_{A_R}\frac{|\nabla u|^p}{|x|^{d-\gamma}}\,dx + \int_\Omega \frac{|\nabla u|^{p-\delta_A}\tau_R^p}{|x|^{d-\gamma}}\,dx + \int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma-1}}\,dx\right).
\end{aligned}
\tag{4.9}
$$

Next, to estimate the third term on the right hand side, we assume for simplicity that $\delta_A \le 1$ and with the help of the Young inequality we get

$$
\begin{aligned}
\frac{|\nabla u|^{p-\delta_A}}{|x|^{d-\gamma}} &= \left(\frac{|\nabla u|^p}{|x|^{d-\gamma-\delta_A}}\right)^{\frac{p-\delta_A}{p}} \frac{1}{|x|^{\frac{\delta_A(d-\gamma+p-\delta_A)}{p}}} \\
&\le \frac{|\nabla u|^p}{|x|^{d-\gamma-\delta_A}} + \frac{1}{|x|^{d-\gamma+p-\delta_A}}.
\end{aligned}
$$

Hence substituting this relation into (4.9) and assuming that $R, \delta_A \le 1$ we deduce

$$
\begin{aligned}
&(\gamma-p)\int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma}}\,dx + (d-\gamma)\int_{B_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|x\cdot\nabla u|^2}{|x|^{d-\gamma+2}}\,dx \\
&\le C\left(R^{\gamma-p+\delta_A} + \int_{A_R}\frac{|\nabla u|^p}{|x|^{d-\gamma}}\,dx + \int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma-\delta_A}}\,dx\right).
\end{aligned}
\tag{4.10}
$$

Finally, we find the maximal radius R_0 such that for all $R \le R_0$ we have that $CR^{\delta_A} \le \delta_A/2$. Consequently, for all $\gamma \ge p+\delta_A$ we can conclude from (4.10) that

$$
\begin{aligned}
&\delta_A \int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma}}\,dx \\
&\le C\left(R^{\gamma-p+\delta_A} + \int_{A_R}\frac{|\nabla u|^p}{|x|^{d-\gamma}}\,dx\right) + CR^{\delta_A}\int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma}}\,dx,
\end{aligned}
\tag{4.11}
$$

which implies (by absorbing the last integral to the left hand side) that for all $\gamma \ge p+\delta_A$ we have

$$
\int_\Omega \frac{|\nabla u|^p\tau_R^p}{|x|^{d-\gamma}}\,dx \le C(\delta_A)\left(R^{\gamma-p+\delta_A} + \int_{A_R}\frac{|\nabla u|^p}{|x|^{d-\gamma}}\,dx\right).
\tag{4.12}
$$

Consequently, using this estimate and the fact that $\gamma \geq p$ we deduce that

$$\int_\Omega \frac{|\nabla u|^p \tau_R^p}{|x|^{d-\gamma-\delta_A}}\, dx \leq C\left(R^{\gamma-p+2\delta_A} + \int_{A_R} \frac{|\nabla u|^p}{|x|^{d-\gamma-\delta_A}}\, dx\right). \tag{4.13}$$

Hence, substituting this relation into (4.10) we derive (4.1).

Next, we show the estimate (4.2). We start with the following observation

$$\int_{B_R(0)} \frac{|\nabla u|^p}{R^{d-p}}\, dx \leq C \sup_{x_0 \in B_{2R}(0)} \int_{B_{4R}(x_0)} \frac{|\nabla u|^{p-2}|\nabla u \cdot (x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx. \tag{4.14}$$

Indeed, for any $x \in B_1(0)$ we have the point-wise estimate[2]

$$|\nabla v(x)|^2 \leq \sum_{i=1}^d \frac{|\nabla v(x) \cdot (x-2e_i)|^2}{|x-2e_i|^2},$$

where e_i denotes the unit vector in the i-th direction. Consequently, we get

$$\int_{B_1(0)} |\nabla v(x)|^p \leq \sum_{i=1}^d \int_{B_1(0)} \frac{|\nabla v(x)|^{p-2}|\nabla v(x)\cdot(x-2e_i)|^2}{|x-2e_i|^2}\, dx. \tag{4.15}$$

Next, for a given $u \in W^{1,p}(B_{8R}(0))$ we define v as

$$v(x) := u(Rx)$$

and by using (4.15) we deduce that

$$\int_{B_1(0)} |\nabla u(Rx)|^p\, dx \leq \sum_{i=1}^d \int_{B_1(0)} \frac{|\nabla u(Rx)|^{p-2}|\nabla u(Rx)\cdot(x-2e_i)|^2}{|x-2e_i|^2}\, dx \tag{4.16}$$

and by standard substitution and dividing by R^{d-p} we get that

$$\begin{aligned}\int_{B_R(0)} \frac{|\nabla u(x)|^p}{R^{d-p}}\, dx &\leq \sum_{i=1}^d \int_{B_R(0)} \frac{|\nabla u(x)|^{p-2}|\nabla u(x)\cdot(x-2Re_i)|^2}{R^{d-p}|x-2Re_i|^2}\, dx \\ &\leq \sum_{i=1}^d C\int_{B_R(0)} \frac{|\nabla u(x)|^{p-2}|\nabla u(x)\cdot(x-2Re_i)|^2}{|x-2Re_i|^{d-p+2}}\, dx\end{aligned} \tag{4.17}$$

and (4.14) follows. Since

$$\begin{aligned}&\int_{B_{4R}(x_0)} \frac{|\nabla u|^{p-2}|\nabla u\cdot(x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx \leq \int_{B_{4R}^{\{|\nabla u|\leq\delta_0\}}(x_0)} \dots + \int_{B_{4R}^{\{|\nabla u|>\delta_0\}}(x_0)} \dots \\ &\leq C(p)\delta_0^p R^p + C\int_{B_{4R}(x_0)} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot(x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx \\ &\leq C(p)\delta_0^p R^p + C\int_{B_{R^*}(x_0)} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot(x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx,\end{aligned}$$

where R^* is the largest ball such that $B_{2R^*}(x_0) \subset \Omega$. Consequently, using this estimate in (4.14) and applying (4.1) with $\gamma = p$ we deduce (4.2) with $\gamma = d$. To prove it for general γ it is enough to combine (4.1) and (4.2) once again. The estimate (4.3) is the an easy consequence of (4.2) and the Poincaré inequality.

---

[2]It is a consequence of the fact that for any $x \in B_1$ we have $|(x-e_i)\cdot(x-e_j)| \leq (1-\delta)|x-e_i||x-e_j|$ for $i \neq j$ and therefore $\{x-e_i\}_{i=1}^d$ forms a basis.

The relation (4.4) is derived in a similar manner but one need to correct a test function in such a way that the normal component is zero on $\partial\Omega$. Such a procedure is easy in case of the flat boundary, see [5], while in case of general boundary it is more technical. Therefore we refer the interested reader to [6] where the problem is solved for $p = 2$ and for general growth condition we refer to [4]. $\square$

5. Use of the one-sided condition

This section is devoted to the second fundamental estimate which is a consequence of the one-sided condition (1.7) and which play the important role in the proof of the VMO-property and also replaces the standard Caccioppoli inequality from Section 3 by its another version more suitable for proving the main result of the paper. Thus, the key lemma of this section, which is related to Lemma 2.3, is the following.

Lemma 5.1. *Let Ω be an open set, F satisfy (1.5)–(1.7), (1.13)–(1.15) and (1.19), and b satisfy (1.20). Then there exists a constant $C > 0$ depending only on α_0, α_0^* and δ_A such that for any $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ solving (1.1), any $x_0 \in \Omega$, any $R > 0$ such that $B_{2R}(x_0) \subset \Omega$ and any $c \in \mathbb{R}^N$ the following inequalities hold*

$$
\begin{aligned}
\int_\Omega \frac{|\nabla u|^p \tau_R^p}{R^{d-p}}\, dx \le CR^p + C \int_{B_{2R}(x_0)} \frac{|u-c|^p}{R^{d-p}} + \frac{|u-c|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}+d-p}}\, dx \\
+ C \int_{B_{2R}(x_0)} \frac{F_u(u,\nabla u)\cdot c}{R^{d-p}}\, dx + C I_{R,x_0}^{\alpha} Y_{R,x_0}^{1-\alpha},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\int_\Omega \frac{\alpha_0 |\nabla u|^p \tau_R^p}{2R^{d-p}}\, dx \le CR^p + C \int_{B_{2R}(x_0)} \frac{|u-c|^p}{R^{d-p}} + \frac{|u-c|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}+d-p}}\, dx \\
+ C I_{R,x_0}^{\alpha} Y_{R,x_0}^{1-\alpha} - \int_\Omega \frac{F_u(u,\nabla u)\cdot (u-c)\tau_R^p}{R^{d-p}}\, dx,
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
I_{R,x_0} &:= \int_{A_R(x_0)} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u \cdot (x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx, \\
Y_{R,x_0} &:= \int_{B_{2R}(x_0)} \frac{|u-c|^p}{R^d}\, dx, \\
\alpha &:= \min\left(\frac{p}{p+2}, \frac{1}{p'}\right).
\end{aligned}
\tag{5.3}
$$

In addition, if Ω is Lipschitz and $u \in W^{1,p}_0(\Omega; \mathbb{R}^N)$ then (5.1)–(5.2) hold with $c \equiv 0$ for any $x_0 \in \mathbb{R}^d$ and any $R > 0$ after redefining $u \equiv 0$ outside Ω.

The next result is the so-called Caccioppoli inequality, where however the better regularity of u is required a priori.

Proof. For simplicity, we show the result only for $x_0 = 0$ and in what follows we omit writing x_0. We test (1.1) by

$$
\frac{(u-c)\tau_R^p(|x|)}{1+\varepsilon|u|},
$$

where $\varepsilon > 0$ is arbitrary and τ_R denotes the standard cut-off function. Note that such a setting is possible since the test function is even in L^∞ and vanishing at the boundary. Hence after a standard manipulation we get the following identity

$$\begin{aligned}
&\int_\Omega \frac{(F_\eta(u,\nabla u)\cdot\nabla u + F_u(u,\nabla u)\cdot u)\tau_R^p}{1+\varepsilon|u|}\,dx \\
&= \int_\Omega \frac{b(\cdot,u,\nabla u)\cdot(u-c)\tau_R^p}{1+\varepsilon|u|}\,dx + \int_\Omega \frac{F_u(u,\nabla u)\cdot c\tau_R^p}{1+\varepsilon|u|}\,dx \\
&\quad - p\int_\Omega \sum_{\alpha=1}^N\sum_{i=1}^d \frac{F_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)x_i\tau_R^{p-1}\tau_R'}{|x|(1+\varepsilon|u|)}\,dx \\
&\quad + \int_\Omega \sum_{\alpha=1}^N\sum_{i=1}^d \frac{F_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)\varepsilon D_i|u|\tau_R^p}{(1+\varepsilon|u|)^2}\,dx.
\end{aligned}\tag{5.4}$$

Our first goal is to let $\varepsilon\to 0_+$. Note that in all terms on the right hand side it is possible by a simple use of the Lebesgue dominated convergence theorem and by the use of the growth assumptions (1.19) and (1.20). For the term on the left hand side we use the one sided condition (1.7) to deduce that

$$\frac{(F_\eta(u,\nabla u)\cdot\nabla u + F_u(u,\nabla u)\cdot u)\tau_R^p}{1+\varepsilon|u|} \ge \frac{\alpha_0(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|^2\tau_R^p - \alpha_0^*\tau_R^p}{1+\varepsilon|u|}. \tag{5.5}$$

Thus we see that the integrand in (5.4) is bounded from below and therefore we can use the Fatou lemma letting $\varepsilon\to 0_+$ we deduce that

$$\begin{aligned}
&\int_\Omega \alpha_0(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|^2\tau_R^p - \alpha_0^*\tau_R^p\,dx \\
&\le \int_\Omega (F_\eta(u,\nabla u)\cdot\nabla u + F_u(u,\nabla u)\cdot u)\tau_R^p\,dx \\
&\le \int_\Omega b(\cdot,u,\nabla u)\cdot(u-c)\tau_R^p\,dx + \int_\Omega F_u(u,\nabla u)\cdot c\tau_R^p\,dx \\
&\quad - p\int_\Omega \sum_{\alpha=1}^N\sum_{i=1}^d \frac{F_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)x_i\tau_R^{p-1}\tau_R'}{|x|}\,dx.
\end{aligned}\tag{5.6}$$

Next, we split the above inequality onto two cases, the first related to (5.1) and the second related to (5.2). Hence, using (1.17) and a simple algebraic manipulation together with the properties of τ_R, we get the following inequalities

$$\begin{aligned}
\int_\Omega \alpha_0|\nabla u|^p\tau_R^p &\le CR^d + \int_\Omega F_u(u,\nabla u)\cdot c\tau_R^p\,dx \\
&+ \int_\Omega b(\cdot,u,\nabla u)\cdot(u-c)\tau_R^p - \sum_{\alpha=1}^N\sum_{i=1}^d \frac{pF_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)x_i\tau_R^{p-1}\tau_R'}{|x|}\,dx
\end{aligned}\tag{5.7}$$

which corresponds to (5.1) and

$$\begin{aligned}
\int_\Omega \alpha_0|\nabla u|^p\tau_R^p &\le CR^d + \int_\Omega F_u(u,\nabla u)\cdot(c-u)\tau_R^p\,dx \\
&+ \int_\Omega b(\cdot,u,\nabla u)\cdot(u-c)\tau_R^p - \sum_{\alpha=1}^N\sum_{i=1}^d \frac{pF_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)x_i\tau_R^{p-1}\tau_R'}{|x|}\,dx,
\end{aligned}\tag{5.8}$$

which is related to (5.2). Next,we focus on the estimate with b. Thus, using (1.20) and the Young inequality, we get

$$\begin{aligned}\int_\Omega b(\cdot,u,\nabla u)\cdot(u-c)\tau_R^p &\le C\int_\Omega (1+|\nabla u|)^{p-1}|u-c|\tau_R^p\,dx\\ &\le \frac{\alpha_0}{4}\int_\Omega |\nabla u|^p\tau_R^p\,dx + CR^d + C\int_\Omega |u-c|^p\tau_R^p.\end{aligned}\tag{5.9}$$

Finally, we focus on the term with F_η. Using the assumptions on F_η (1.13)–(1.14) and the properties of τ_R we observe that

$$\begin{aligned}&-\int_\Omega \sum_{\alpha=1}^N\sum_{i=1}^d \frac{pF_{\eta_i^\alpha}(u,\nabla u)(u^\alpha-c^\alpha)x_i\tau_R^{p-1}\tau_R'}{|x|}\,dx\\ &\le C\int_{A_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^2}\,dx\\ &\quad + C\int_{A_R}\frac{(1+|\nabla u|)^{p-1-\delta_A}|u-c|\tau_R^{p-1}}{|x|}\,dx,\end{aligned}\tag{5.10}$$

Next, for the second integral, we use the Young inequality to deduce that

$$\begin{aligned}&C\int_{A_R}\frac{(1+|\nabla u|)^{p-1-\delta_A}|u-c|\tau_R^{p-1}}{|x|}\,dx\\ &\le \frac{\alpha_0}{8}\int_\Omega |\nabla u|^p\tau_R^p + CR^d + C\int_{B_{2R}}\frac{|u-c|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}}}\,dx.\end{aligned}\tag{5.11}$$

Consequently, we substitute (5.9)–(5.11) into (5.7)–(5.8) and divide the resulting inequalities by R^{d-p} to observe that

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p\tau_R^p}{R^{d-p}}\,dx &\le CR^p + C\int_{A_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^{d-p+2}}\,dx\\ &\quad + C\int_{B_{2R}}\frac{|u-c|^p}{R^{d-p}}+\frac{|u-c|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}+d-p}}+\frac{F_u(u,\nabla u)\cdot c}{R^{d-p}}\,dx\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}\int_\Omega \frac{3\alpha_0|\nabla u|^p\tau_R^p}{4R^{d-p}}\,dx &\le CR^p + C\int_{A_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^{d-p+2}}\,dx\\ &+ C\int_{B_{2R}(x_0)}\frac{|u-c|^p}{R^{d-p}}+\frac{|u-c|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}+d-p}}\,dx - \int_\Omega \frac{F_u(u,\nabla u)\cdot(u-c)\tau_R^p}{R^{d-p}}\,dx.\end{aligned}\tag{5.13}$$

Thus, comparing (5.12)–(5.13) with (5.1)–(5.2) we see that all we need is that

$$\begin{aligned}&C\int_{A_R}\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^{d-p+2}}\,dx\\ &\le \int_\Omega \frac{\alpha_0|\nabla u|^p\tau_R^p}{4R^{d-p}}\,dx + CI_R^\alpha Y_R^{1-\alpha} + CR^p,\end{aligned}\tag{5.14}$$

where Y_R, I_R and α are given through (5.3). To prove (5.14), we first use the Hölder inequality to observe that

$$\begin{aligned}
&C\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^{d-p+2}}\,dx\\
&\qquad \le CY_R^{\frac{1}{p}}\left(\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{(p-2)p'}{2}}|\nabla u\cdot x|^{p'}\tau_R^{p}}{|x|^{d-pp'+2p'}}\,dx\right)^{\frac{1}{p'}}.
\end{aligned} \tag{5.15}$$

Finally, we estimate the last term. First if $p\in(1,2]$ we have that $p'\ge 2$ and consequently

$$\begin{aligned}
&\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}p'}|\nabla u\cdot x|^{p'}\tau_R^{p}}{|x|^{d-pp'+2p'}}\,dx\\
&\qquad =\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}p'}|\nabla u\cdot x|^{p'-2}|\nabla u\cdot x|^2\tau_R^{p}}{|x|^{d-pp'+2p'}}\,dx\\
&\qquad \le\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}p'+\frac{p'-2}{2}}|\nabla u\cdot x|^2}{|x|^{d-pp'+2p'-p'+2}}\,dx = I_R.
\end{aligned} \tag{5.16}$$

Hence, substituting this inequality into (5.15), we see for $p\in(1,2]$ the inequality (5.14) is valid and therefore the proof is finished for such a range of p's. Next, we focus ont the case $p>2$ (and consequently $p'<2$). Using the Hölder inequality we get that

$$\begin{aligned}
&\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}p'}|\nabla u\cdot x|^{p'}\tau_R^{p}}{|x|^{d-pp'+2p'}}\,dx\\
&\le\int_{A_R} \left(\frac{(\delta_0+|\nabla u|^2)^{\frac{p}{2}}\tau_R^{p}}{|x|^{d-p}}\right)^{\frac{p-2}{2(p-1)}}\left(\frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x|^2}{|x|^{d-p+2}}\right)^{\frac{p'}{2}}\,dx\\
&\le\left(\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p}{2}}\tau_R^{p}}{|x|^{d-p}}\,dx\right)^{\frac{p-2}{2(p-1)}} I_R^{\frac{p'}{2}}\\
&\le C\left(R^p+\int_{\Omega}\frac{|\nabla u|^p\tau_R^p}{R^{d-p}}\,dx\right)^{\frac{p-2}{2(p-1)}} I_R^{\frac{p'}{2}}.
\end{aligned} \tag{5.17}$$

Finally, substituting this inequality into (5.15) and using the Young inequality we observe that

$$\begin{aligned}
&C\int_{A_R} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x||u-c|\tau_R^{p-1}}{|x|^{d-p+2}}\,dx\\
&\qquad\le CY_R^{\frac{1}{p}}\left(R^p+\int_{\Omega}\frac{|\nabla u|^p\tau_R^p}{R^{d-p}}\,dx\right)^{\frac{p-2}{2p}} I_R^{\frac{1}{2}}\\
&\qquad\le CR^p+\frac{\alpha_0}{4}\int_{\Omega}\frac{|\nabla u|^p\tau_R^p}{R^{d-p}}\,dx + CY_R^{\frac{2}{p+2}} I_R^{\frac{p}{p+2}}
\end{aligned}$$

and by using the definition of α in (5.3), we see that (5.14) follows and thus the proof is complete. $\square$

6. Smallness of the p-Dirichlet integral - VMO property

This section is devoted to the first essential estimate - the smallness of the Dirichlet integral, which plays an important role in further analysis due to the dependence of F on u. Note that in case that F is u independent such an estimate can be avoided and one can proceed directly as was shown in [5]. The first result, which is related to Lemma 2.4, is focused on the case when a weak solution satisfies also the Noether equation (1.21) and we control the explosion rate of the mean value by (1.23).

Lemma 6.1. *Let Ω be an open set, F satisfy* (1.5)–(1.7), (1.13)–(1.15) *and* (1.19), *and b satisfy* (1.20). *Then there exist constants $C, R_0 > 0$ depending only on $\alpha_0, \alpha_0^*, \delta_A$ and $C_{\ln}$ such that for any $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^N)$ solving* (1.1) *and satisfying* (1.21) *and* (1.23), *for any $x_0 \in \Omega$ and for any $R \in (0, R_0)$ such that $B_{2R}(x_0) \subset \Omega$ the following estimate holds*

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le CR^{\delta_A} + \frac{C\ln\ln|\ln R_{**}|}{\ln\ln|\ln R|}\int_{B_{R_{**}}(x_0)} \frac{|\nabla u|^p}{R_{**}^{d-p}}, \tag{6.1}$$

where R_{**} is the largest number such that $B_{2R_{**}}(x_0) \subset \Omega$.

Proof. We again prove the result only for $x_0 \equiv 0$ and omit writing the dependence on x_0 in what follows. First, since u satisfies the Noether equation (1.21), we can use Lemma 4.2 and setting $\gamma = p$ in (4.1) it follows that

$$a_R := \int_{B_R} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}}|\nabla u \cdot x|^2}{|x|^{d-p+2}}\,dx \le CR^{\delta_A} + C\int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\,dx. \tag{6.2}$$

To estimate the integral on the right hand side, we apply Lemma 5.1, in particular we use (5.1) with $c \equiv 0$ to conclude

$$\int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le CR^p + C\int_{B_{4R}} \frac{|u|^p}{R^{d-p}} + \frac{|u|^{\frac{p}{1+\delta_A}}}{R^{\frac{p}{1+\delta_A}+d-p}}\,dx + CI_{2R}^{\alpha}Y_{2R}^{1-\alpha}, \tag{6.3}$$

where the I_R, Y_R and α are defined in (5.3). To estimate the term with $|u|^p$, we use the Poincaré inequality to conclude

$$\int_{B_{4R}} |u|^p\,dx \le C\int_{B_{4R}} |u - u_{4R}|^p + CR^d|u_{4R}|^p \le CR^d \int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}} + CR^d|u_{4R}|^p,$$

where u_{4R} denotes the mean value of u over a ball B_{4R}. Similarly, with the help of the Hölder and the Young inequalities we also deduce that

$$\begin{aligned}\int_{B_{4R}} |u|^{\frac{p}{1+\delta_A}}\,dx &\le C\int_{B_{4R}} |u - u_{4R}|^{\frac{p}{1+\delta_A}} + CR^d|u_{4R}|^{\frac{p}{1+\delta_A}} \\ &\le CR^{\frac{d\delta_A}{1+\delta_A}}\left(\int_{B_{4R}} |u - u_{4R}|^p\right)^{\frac{1}{1+\delta_A}} + CR^d|u_{4R}|^{\frac{p}{1+\delta_A}} \\ &\le CR^d\left(\int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}}\right)^{\frac{1}{1+\delta_A}} + CR^d|u_{4R}|^{\frac{p}{1+\delta_A}}.\end{aligned}$$

Thus, inserting these two estimates into (6.3) and using the definition of Y_R we get

$$
\begin{aligned}
\int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le CR^p + CR^p \int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}} + CR^{\frac{p\delta_A}{1+\delta_A}} \left(\int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}}\right)^{\frac{1}{1+\delta_A}} \\
+ CR^p |u_{4R}|^p + CR^{\frac{p\delta_A}{1+\delta_A}} |u_{4R}|^{\frac{p}{1+\delta_A}} \\
+ CI_{2R}^\alpha \left(\int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}} + |u_{4R}|^p\right)^{1-\alpha}.
\end{aligned} \tag{6.4}
$$

Next, we find the largest $R_* \le \frac{1}{2}$ such that $B_{2R_*} \subset \Omega$ and denoting

$$
K := \int_{B_{R_*}} \frac{|\nabla u|^p}{R_*^{d-p}}\,dx, \tag{6.5}
$$

we can use the estimate (4.2) to substitute $\int_{B_{4R}} \frac{|\nabla u|^p}{R^{d-p}}$ by K in (6.4). In addition, we use (1.23) to substitute the mean values in (6.4) and using also the definition on α (see (5.3)), we finally simplify (6.4) in the following way

$$
\begin{aligned}
\int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\,dx &\le CC_{\ln}^p |\ln R|^p R^{\frac{p\delta_A}{1+\delta_A}} + CR^{\frac{p\delta_A}{1+\delta_A}} K \\
&\quad + CI_{2R}^\alpha (1+K)^{1-\alpha} + CC_{\ln}^p (I_{2R}|\ln R|)^\alpha \\
&\le CR^\beta(1+K) + CI_{2R}^\alpha (1+K)^{1-\alpha} + C (I_{2R}|\ln R|)^\alpha
\end{aligned} \tag{6.6}
$$

with some $C, \beta > 0$ depending only on data, i.e., on $\alpha_0, \alpha_0^*, \delta_0, C_{\ln}$ and $|\Omega|$. Then we combine (6.2) and (6.6) and assuming that β is chosen such that $\beta \le \delta_A$, we get

$$
\begin{aligned}
a_R &\le CR^\beta(1+K) + CI_{2R}^\alpha (1+K)^{1-\alpha} + C (I_{2R}|\ln R|)^\alpha \\
&\le CR^\beta(1+K) + \frac{C(1+K)}{|\ln|\ln R||^{\frac{\alpha}{1-\alpha}}} + I_{2R}|\ln|\ln R||| \ln R|,
\end{aligned} \tag{6.7}
$$

where we used the Young inequality. Finally, using the definition of I_{2R} and the fact that $R \le \frac{1}{2}$ we get

$$
a_R \le \frac{C(1+K)}{|\ln|\ln R||^{\frac{\alpha}{1-\alpha}}} + |\ln R||\ln|\ln R||(a_{4R} - a_R). \tag{6.8}
$$

Consequently, by a simple algebraic manipulation we deduce that

$$
a_R \le \frac{|\ln R||\ln|\ln R||}{1 + |\ln R||\ln|\ln R||} a_{4R} + \frac{C(1+K)}{|\ln R||\ln|\ln R||^{\frac{\alpha}{1-\alpha}+1}}. \tag{6.9}
$$

Before, we continue we show that there exists some $R_{**} > 0$ such that for all $R \in (0, R_{**})$ we have

$$
g_R := \frac{|\ln R||\ln|\ln R||}{1 + |\ln R||\ln|\ln R||} \frac{|\ln|\ln|\ln R|||}{|\ln|\ln|\ln 4R|||} \le 1. \tag{6.10}
$$

To prove it, we first consider $R_{**} < \frac{1}{4}$ so small that $\ln|\ln 4R| > 1$. Then to prove (6.10) it is equivalent to show that

$$
|\ln R| \ln|\ln R| (\ln\ln|\ln R| - \ln\ln|\ln 4R|) \le \ln\ln|\ln 4R|. \tag{6.11}
$$

Since

$$\begin{aligned}\ln\ln|\ln R| - \ln\ln|\ln 4R| &= \ln\left(\frac{\ln|\ln R|}{\ln|\ln 4R|}\right) = \ln\left(1+\frac{\ln|\ln R| - \ln|\ln 4R|}{\ln|\ln 4R|}\right)\\ &= \ln\left(1+\frac{\ln\left(\frac{\ln R}{\ln 4R}\right)}{\ln|\ln 4R|}\right) = \ln\left(1+\frac{\ln\left(1+\frac{\ln R-\ln 4R}{\ln 4R}\right)}{\ln|\ln 4R|}\right)\\ &= \ln\left(1+\frac{\ln\left(1+\frac{\ln 4}{|\ln 4R|}\right)}{\ln|\ln 4R|}\right) \le \frac{C}{|\ln 4R|\ln|\ln 4R|},\end{aligned}$$

we see that to prove (6.11) it is enough to show that

$$\frac{C|\ln R|\ln|\ln R|}{|\ln 4R|\ln|\ln 4R|} \le \ln\ln|\ln 4R|. \tag{6.12}$$

But since the left hand side is bounded and the right hand side tends to infinity as $R \to 0_+$ we get (after a possible redefinition of R_{**}) that (6.12) is valid for all $R \in (0, R_{**})$. Hence, multiplying (6.9) by $\ln\ln|\ln R|$ and using (6.10), we observe

$$\begin{aligned}\ln\ln|\ln R|a_R &\le g_R \ln\ln|\ln 4R|a_{4R} + \frac{C(1+K)\ln\ln|\ln R|}{|\ln R||\ln|\ln R||^{\frac{\alpha}{1-\alpha}+1}}\\ &\le \ln\ln|\ln 4R|a_{4R} + \frac{C(1+K)\ln\ln|\ln R|}{|\ln R||\ln|\ln R||^{\frac{\alpha}{1-\alpha}+1}}\\ &\le \ln\ln|\ln 4R|a_{4R} + \frac{C(1+K)}{|\ln R||\ln|\ln R||^{1+\varepsilon}}\end{aligned} \tag{6.13}$$

for some $\varepsilon > 0$ depending only on data. Finally, since $\sum_k \frac{1}{k(\ln k)^{1+\varepsilon}} < \infty$ we can iterate the inequality (6.13) and show that

$$a_R \le \frac{\ln\ln|\ln R_{**}|}{\ln\ln|\ln R|} a_{R_{**}} + \frac{C(1+K)}{\ln\ln|\ln R|}. \tag{6.14}$$

Hence using the definition of a_R and (4.1) to bound the term on the right hand side we conclude

$$\int_{B_R} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}}|\nabla u \cdot x|^2}{|x|^{d-p+2}}\,dx \le \frac{C\ln\ln|\ln R_{**}|(1+K)}{\ln\ln|\ln R|}. \tag{6.15}$$

Finally, using the same argument as before, namely

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le CR^p + C\sup_{x_0\in B_R}\int_{B_{2R}(x_0)} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x|^2}{|x|^{d-p+2}}\,dx,$$

we conclude (6.1) from (6.15).

□

The next result is of a stronger character than the previous one. In fact we do not need a priori knowledge about the possible explosion rate of mean values of u as in (1.23) but we replace it by a more restrictive assumption on F_u (either (1.24) or (1.25)) and it can be applied only for minimizers.

Lemma 6.2. *Let Ω be an open set, b satisfy* (1.20)*, F satisfy* (1.5)–(1.7)*,* (1.13)–(1.15)*,* (1.19) *and let one of* (1.24) *and* (1.25) *hold. Assume that $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$*

is a weak solution to (1.1) *that in addition satisfies* (1.22). *Then for all* $x_0 \in \Omega$ *there holds*

$$\limsup_{R\to 0_+} \int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}}\, dx = 0. \tag{6.16}$$

Proof. For simplicity we consider only the case $x_0 = 0$ and assume for a contradiction that

$$\limsup_{R\to 0_+} \int_{\Omega} \frac{|\nabla u(x)|^p \tau_R^p}{R^{d-p}}\, dx = L > 0. \tag{6.17}$$

First note that $L < \infty$ which follows from Lemma 4.2. Next, we choose a not relabeled subsequence of R's for which the L is attained and in what follows we consider just this sequence (that may be again changed by taking a subsequence). First, we find $\bar{u}$

$$\bar{u} := \lim_{R\to 0} u_{2R} := \lim_{R\to 0} \frac{1}{|B_{2R}|} \int_{B_{2R}} u\, dx,$$

where we allow the values $\pm\infty$. Next, we define a vectors $c_R \in \mathbb{R}^N$ as follows

$$c_R^\alpha := \begin{cases} 0 & \text{if } \bar{u}^\alpha \in \mathbb{R}, \\ u_{2R} & \text{if } \bar{u}^\alpha = \pm\infty. \end{cases} \tag{6.18}$$

Note that it directly follows from this definition that there exists $C > 0$ (depending of course on x_0 and u) such that

$$|u_{2R} - c_R| \le C. \tag{6.19}$$

Consequently, with the help of the Poincaré inequality and Lemma 4.2 we deduce that for all $B_{2R} \subset \Omega$

$$\begin{aligned} \int_{B_{2R}} \frac{|u - c_R|^p}{R^d}\, dx &\le C \int_{B_{2R}} \frac{|u - u_{2R}|^p + |u_{2R} - c_R|^p}{R^d}\, dx \\ &\le C + C \int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}}\, dx \le C. \end{aligned} \tag{6.20}$$

Next, we apply Lemma 5.1. First, it follows from the uniform bound (1.21) that

$$I_R \to 0 \qquad \text{as } R \to 0_+,$$

where I_R is defined in (5.3). Next, we set $c := c_R$ in (5.1)–(5.2), and using the fact that $Y_R \le C$ (which follows from (6.20)) we can let $R \to 0_+$ in (5.1)–(5.2) to arrive to the following inequalities

$$L \le C \limsup_{R\to 0_+} \int_{B_{2R}} \frac{F_u(u, \nabla u) \cdot c_R \tau_R^p}{R^{d-p}}\, dx, \tag{6.21}$$

$$L \le \limsup_{R\to 0_+} - \int_{B_{2R}} \frac{2F_u(u, \nabla u) \cdot (u - c_R)\tau_R^p}{\alpha_0 R^{d-p}}\, dx. \tag{6.22}$$

Next, using the definition of c_R, we see that (6.21) reduces to

$$\begin{aligned} L \le{} & C \limsup_{R\to 0_+} \sum_{\alpha;\, \bar{u}^\alpha = \pm\infty} \int_{B_{2R}} \frac{F_{u^\alpha}(u, \nabla u) u^\alpha \tau_R^p}{R^{d-p}}\, dx \\ & + C \limsup_{R\to 0_+} \sum_{\alpha;\, \bar{u}^\alpha = \pm\infty} \int_{B_{2R}} \frac{|F_{u^\alpha}(u, \nabla u)||u^\alpha - u_{2R}^\alpha|}{R^{d-p}}\, dx, \end{aligned} \tag{6.23}$$

while from (6.22) it follows that

$$
\begin{aligned}
L \le \limsup_{R\to 0_+} & \sum_{\alpha;\ \bar u^\alpha \ne \pm\infty} -\!\!\!\!\!\!\int_{B_{2R}} \frac{2F_{u^\alpha}(u,\nabla u)u^\alpha \tau_R^p}{\alpha_0 R^{d-p}}\, dx \\
&+ C \limsup_{R\to 0_+} \sum_{\alpha;\bar u^\alpha = \pm\infty} \int_{B_{2R}} \frac{|F_{u^\alpha}(u,\nabla u)||u^\alpha - u^\alpha_{2R}|}{R^{d-p}}\, dx.
\end{aligned} \tag{6.24}
$$

Hence, if F satisfies (1.25) we derive from (6.23) that

$$
L \le C \limsup_{R\to 0_+} \sum_{\alpha;\bar u^\alpha = \pm\infty} \int_{B_{2R}} \frac{a(u^\alpha)|\nabla u|^p (1+|u^\alpha - u^\alpha_{2R}|)}{R^{d-p}}\, dx. \tag{6.25}
$$

In case F satisfies (1.24), we can absorb the first term on the right hand side of (6.24) by the left hand side and we again deduce the inequality (6.25).

Thus, it remains to discuss the behavior of the term on the right hand side of (6.25). For this purpose, we use the fact that u is a minimizer. First, we define

$$
v_R(x) := u(Rx) - c_R \qquad \text{for } x \in B_2 \tag{6.26}
$$

and using the substitution theorem we have the identity

$$
\begin{aligned}
L &\le \limsup_{R\to 0_+} \sum_{\alpha;\ \bar u^\alpha = \pm\infty} \int_{B_{2R}} \frac{a(u^\alpha)|\nabla u|^p (1+|u^\alpha - u^\alpha_{2R}|)}{R^{d-p}}\, dx \\
&= \limsup_{R\to 0_+} \sum_{\alpha;\ \bar u^\alpha = \pm\infty} \int_{B_2} a(v^\alpha_R + c^\alpha_R)|\nabla v_R|^p (1+|v_R|)\, dx.
\end{aligned} \tag{6.27}
$$

For the last integral, we first deduce a priori bound for v_R. Hence, using (6.20), (4.2) and the substitution theorem we see that

$$
\int_{B_2} |v_R|^p \, dx \le C.
$$

Moreover, using the substitution theorem again and the fact that u is a minimizer and therefore satisfies the reverse Hölder inequality with some $0<\varepsilon<d-p$ from (3.4), we find that for all $B_r(x_0) \subset B_2$

$$
\begin{aligned}
\int_{B_r(x_0)} |\nabla v_R|^{p+\varepsilon}\, dx &= R^{p+\varepsilon} \int_{B_r(x_0)} |\nabla u(Rx)|^{p+\varepsilon}\, dx \\
&= r^d R^{p+\varepsilon} \int_{B_{Rr}(Rx_0)} \frac{|\nabla u|^{p+\varepsilon}}{R^d r^d}\, dx \\
&\le C r^d R^{p+\varepsilon} \left(1 + \left(\int_{B_{2Rr}(Rx_0)} \frac{|\nabla u|^p}{R^d r^d}\, dx \right)^{\frac{p+\varepsilon}{p}} \right) \\
&\le C r^{d-p-\varepsilon} \left(1 + \left(\int_{B_{2Rr}(Rx_0)} \frac{|\nabla u|^p}{(Rr)^{d-p}}\, dx \right)^{\frac{p+\varepsilon}{p}} \right) \\
&\le C r^{d-p-\varepsilon},
\end{aligned} \tag{6.28}
$$

where for the last inequality we used (4.2). Consequently, in addition to (6.27) we have the following uniform estimate

$$
\|v_R\|_{BMO} + \|v_R\|_{1,p+\varepsilon} \le C. \tag{6.29}
$$

Therefore, using the embedding $BMO(B_2) \hookrightarrow L^q(B_2)$ valid for all $q < \infty$, the uniform bound (6.29) and the Hölder inequality, we see that (6.27) implies that

$$L \le C \limsup_{R\to 0_+} \sum_{\alpha; \bar{u}^\alpha = \pm\infty} \|a(v_R^\alpha + c_R^\alpha)\|_{\frac{2p}{\varepsilon}}. \tag{6.30}$$

However, from the compact embedding we have that there exists $v \in W^{1,p}(B_2; \mathbb{R}^N)$ such that

$$v_R \to v \qquad \text{almost everywhere}$$

and it also follows from the definition of c_R that

$$c_R^\alpha \to \pm\infty \qquad \text{if } \bar{u}^\alpha = \pm\infty.$$

Hence, using the assumption on a we observe that

$$a(v_R^\alpha + c_R^\alpha) \to a(v \pm \infty) = 0 \quad \text{almost everywhere in } B_2$$

and consequently since a is a bounded function, we can use the Lebesgue dominated convergence theorem and we see that the right hand side of (6.30) is zero, which however contradicts (6.17) and therefore the proof is complete. $\square$

7. Hölder continuity of solution

This section is devoted to the proof of the Hölder continuity of the solution u, i.e., to the proof of Theorems 1.1–1.2. We start this section with the proof for minimizers.

7.1. **Proof of Theorems 1.1–1.2 for the case 1).** The proof is based on the method developed in [5] and a proper VMO estimates stated in the previous section which will finally imply the Hölder continuity of any minimizer, i.e, when 1) is valid. In addition we skip the proof of the boundary regularity here and we refer to the next subsection, where it is proved for u being even non-minimizer. We prove the result only for $x_0 = 0$, i.e., we show that there is $\beta > 0$ (depending only $\alpha_0, \alpha_0^*, \delta_A$ such that for some $R_0 > 0$ any minimizer belongs to $\mathcal{C}^\beta(B_{R_0})$.

First, since we consider the case 1), we can apply Lemma 6.2 and we see that for $\delta > 0$, that will be specified later, we can find $R_1 > 0$ such that $B_{4R_1} \subset \Omega$ and

$$\int_{B_{R_1}} \frac{|\nabla u|^p}{R_1^{d-p}} \, dx \le \delta. \tag{7.1}$$

Consequently, using (4.2) with $\Omega := B_{4R_1}$, we find that for all $x_0 \in B_{R_1}$ and all $R \in (0, R_1)$, we have

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}} \, dx \le C(R^{\delta_A} + \delta). \tag{7.2}$$

Hence, we can fix R_0 (still depending on δ) such that for any $x_0 \in B_{R_0}$ and all $R \in (0, 2R_0)$, we have

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}} \, dx \le C\delta. \tag{7.3}$$

Next, in order to shorten the formula we denote

$$\begin{aligned} Z^1_{R,x_0} &:= \int_{B_R(x_0)} \frac{(\delta_0+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u\cdot x|^2}{|x|^{d-p+2}}\,dx,\\ Z^2_{R,x_0} &:= \int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}}\,dx. \end{aligned}$$

The starting point of the proof is the Caccioppoli inequality (5.2) with

$$c := u_{B_{2R}(x_0)}.$$

Note that by using the Poincaré inequality and in view of the previous definition we have that

$$I_{R,x_0} = Z^1_{2R,x_0} - Z^1_{R,x_0}, \qquad Y_{R,x_0} \le CZ^2_{R,x_0}.$$

Moreover, by using the Poincaré inequality, the first integral on the right hand side of (5.2) is a lower order term simply estimated by R^{δ_A} and therefore using also the growth assumptions on F_u (1.19) the inequality (5.2) reduces to (with $\alpha \in (0,1)$ defined in (5.3))

$$\begin{aligned} Z^2_{R,x_0} \le CR^{\delta_A} &+ C\left(Z^1_{2R,x_0}-Z^1_{R,x_0}\right)^{\alpha}\left(Z^2_{2R,x_0}\right)^{1-\alpha}\\ &+ C\int_{B_{2R}(x_0)} \frac{|\nabla u|^p|u-\bar u_{B_{2R}(x_0)}|}{R^{d-p}}\,dx. \end{aligned} \tag{7.4}$$

First, we focus on the last term. Using the Hölder inequality and (3.4) (with some ε depending only on data) we have

$$\begin{aligned} &\int_{B_{2R}(x_0)} \frac{|\nabla u|^p|u-\bar u_{B_{2R}(x_0)}|}{R^{d-p}}\,dx\\ &\le R^p\left(\int_{B_{2R}(x_0)}\frac{|\nabla u|^{p+\varepsilon}}{R^d}\,dx\right)^{\frac{p}{p+\varepsilon}}\left(\int_{B_{2R}(x_0)}\frac{|u-u_{B_{2R}(x_0)}|^{\frac{p+\varepsilon}{\varepsilon}}}{R^d}\,dx\right)^{\frac{\varepsilon}{p+\varepsilon}}\\ &\le C(R^p+Z^2_{4R,x_0})(Z^2_{2R,x_0})^{\gamma}\|u\|^{1-\gamma}_{BMO} \le C\delta^{\gamma}(R^p+Z^2_{4R,x_0}), \end{aligned} \tag{7.5}$$

where the last inequality follows from (4.3) and (7.3) and $\gamma > 0$ depends on ε. Hence, substituting this into (7.4) and using the Young inequality we obtain (after possible extension of integration domain in Z^1)

$$Z^2_{R,x_0} \le CR^{\delta_A} + (\frac{1}{8}+C\delta)Z^2_{4R,x_0} + C(Z^1_{4R,x_0}-Z^1_{R,x_0}).$$

At this point, we finally fix δ to be sufficiently small (depending on data and consequently also on ε) such that we get

$$Z^2_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{4R,x_0}}{4} + C(Z^1_{4R,x_0}-Z^1_{R,x_0}). \tag{7.6}$$

Next, using (4.1) with $\gamma = p$ and after a possible division by some constant we get

$$C^{-1}Z^1_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{4R,x_0}}{4}. \tag{7.7}$$

Hence, summing (7.6) and (7.7) we obtain

$$Z^2_{R,x_0} + (C+C^{-1})Z^1_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{4R,x_0}}{2} + CZ^1_{4R,x_0} \tag{7.8}$$

and consequently, defining

$$\theta := \max\left(\frac{1}{2}, \frac{C}{C+C^{-1}}\right) < 1,$$
$$W_{R,x_0} := Z^2_{R,x_0} + (C+C^{-1})Z^1_{R,x_0},$$

we get

$$W_{R,x_0} \le CR^{\delta_A} + \theta W_{4R,x_0}. \tag{7.9}$$

Thus, we can find $\beta > 0$ such that

$$p\beta < \delta_A, \qquad 4^{p\beta}\theta \le 1$$

and we can rewrite (7.9) as (with some $\omega > 0$)

$$\frac{W_{R,x_0}}{R^{p\beta}} \le CR^{\omega} + \frac{W_{4R,x_0}}{(4R)^{p\beta}},$$

which after a simple iteration leads to the estimate

$$\frac{W_{R,x_0}}{R^{p\beta}} \le C\left(1 + \frac{W_{4R_0,x_0}}{R_0^{p\beta}}\right) \le C(R_0). \tag{7.10}$$

and the Morrey embedding finishes the proof of $u \in \mathcal{C}^{\beta}(B_{R_0})$.

7.2. **Proof of Theorems 1.1–1.2 - the case 2).** Here we derive uniform estimates in case 2) is valid. In this section we prove everything up to the boundary and it will be evident that one can mimic such a procedure also for the proof of Theorem 1.2 - the case 1), which was missing in the previous subsection. The proof is again based on the paper [5] and on the VMO property of the solution u. However, since we want to use a supremum argument, we need to assume the continuity of the solution a priori, which however do not affect the final uniform estimate which will depend only on $C_{\ln}$ and $\|\nabla u\|_p$. In addition, it will be clear from the proof that near the boundary the estimates are independent of $C_{\ln}$, which is caused by the fact that u is fixed (and smooth as it is assumed to be equal to zero) on the boundary.

The proof is split onto two parts. First, we show that any continuous solution is in fact Hölder continuous, but with all estimates dependent on the modulus of continuity of u. Next, having such Hölder continuity, we almost repeat step by step the same procedure but finally we use a supremum argument - for this we however need to know Hölder continuity a priori - and get desired estimates. Before we start, we recall the notation used in the previous section, namely

$$Z^1_{R,x_0} := \int_{B_R(x_0)} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u \cdot x|^2}{|x|^{d-p+2}}\, dx,$$
$$Z^2_{R,x_0} := \int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p}}\, dx,$$
$$W_{R,x_0} := Z^2_{R,x_0} + (C+C^{-1})Z^1_{R,x_0}.$$

First, we derive the global VMO-like information. Therefore for any $x_0 \in \partial\Omega$, we use (4.4) with $\gamma = p$ and find $C > 0$ such that

$$C^{-1}Z^1_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{2R,x_0}}{4}. \tag{7.11}$$

Similarly, using (5.1) with $c \equiv 0$ (note that it is a correct setting since $x_0 \in \partial\Omega$), we deduce that (using also the Young inequality)

$$
\begin{aligned}
Z^2_{R,x_0} &\le CR^{\delta_A}\left(1+\int_{B_{2R}(x_0)}\frac{|u|^p}{R^d}\,dx\right)\\
&\quad + C\left(\int_{B_{2R}(x_0)}\frac{|u|^p}{R^d}\,dx\right)^{1-\alpha}\left(Z^1_{2R,x_0}-Z^1_{R,x_0}\right)^{\alpha}.
\end{aligned}
\tag{7.12}
$$

But since $x_0 \in \partial\Omega$, $\Omega \in \mathcal{C}^{0,1}$ and $u \equiv 0$ outside Ω we can use the Poincaré inequality to conclude that

$$\int_{B_{2R}(x_0)}\frac{|u|^p}{R^d}\,dx \le CZ^2_{2R,x_0} \le C$$

and substituting this into (7.12) we get after applying the Young inequality

$$Z^2_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{R,x_0}}{4} + C(Z^1_{2R,x_0}-Z^1_{R,x_0}). \tag{7.13}$$

Thus, summing up (7.11) and (7.13) and using the notation from the previous section, we obtain

$$W_{R,x_0} \le CR^{\delta_A} + \theta W_{2R,x_0}, \tag{7.14}$$

which finally gives that there is $\beta_0 > 0$ (depending only on $\alpha_0, \alpha_0^*, \delta_A, \Omega$ and R_0) such that for all $R \in (0, R_0)$ and all $x_0 \in \partial\Omega$ we have

$$\frac{W_{R,x_0}}{R^{p\beta_0}} \le C(1 + R_0^{-d+p-p\beta_0}\|\nabla u\|_p^p). \tag{7.15}$$

Note here, that at this estimate there is no dependence on $C_{\ln}$ and therefore for the boundary regularity problem stated in Theorem 1.2, we restrict ourselves to so small neighborhood of $\partial\Omega$ such that (7.15) implies the smallness of the Dirichlet integral near the boundary.

Next, we focus on the estimates in the interior of Ω. Let $x_0 \in \Omega$ and $R \in (0, R_0)$ be arbitrary. Our aim is to show a uniform variant of Lemma 6.1, i.e., that

$$Z^2_{R,x_0} \le \frac{C(\alpha_0, \alpha_0^*, \delta_A, C_{\ln}, R_0)\|\nabla u\|_p^p}{\ln\ln|\ln R|}. \tag{7.16}$$

To prove it, we first consider the case when $B_{2R}(x_0) \not\subseteq \Omega$. In this case, we can surely find $x_1 \in \partial\Omega$ such that $B_R(x_0) \subset B_{8R}(x_1)$ and consequently

$$Z^2_{R,x_0} \le CZ^2_{8R,x_1} \le CR^{p\beta_0}(1 + R_0^{-d+p-p\beta_0}\|\nabla u\|_p^p), \tag{7.17}$$

where for the second inequality we used (7.15) and we see that (7.16) follows. In case $B_{2R}(x_0) \subset \Omega$ we use (6.1) to conclude (however, starting from here the constant C also depends on $C_{\ln}$)

$$Z^2_{R,x_0} \le CR^{\delta_A} + \frac{C\ln\ln|\ln R_{**}|Z^2_{R_{**},x_0}}{\ln\ln|\ln R|},$$

where R_{**} is the maximal radius such that $B_{R_{**}}(x_0) \subset \Omega$. Then since $B_{2R_{**}}(x_0) \not\subseteq \Omega$ we can use (7.17) to iterate once again and to get

$$Z^2_{R,x_0} \le CR^{\delta_A} + \frac{C\ln\ln|\ln R_{**}|}{\ln\ln|\ln R|}R_{**}^{p\beta_0}(1 + R_0^{-d+p-p\beta_0}\|\nabla u\|_p^p).$$

Finally, using a simple inequality $\ln\ln|\ln R_{**}|R_{**}^{p\beta_0} \le C$ we obtain (7.16).

Then for any $x_0 \in \Omega$ and any $R \in (0, R_0)$ such that $B_{2R}(x_0) \subset \Omega$ we can apply (4.1) with $\gamma = p$ to get

$$C^{-1} Z^1_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{2R,x_0}}{4}. \tag{7.18}$$

Similarly using (5.1) with $c \equiv \bar{u}_{2R}(x_0)$, the Poincaré inequality, the uniform bound (7.16) and the assumption (1.19) we get that

$$\begin{aligned} Z^2_{R,x_0} \le & CR^{\delta_A} + CZ^2_{2R,x_0} \| u - \bar{u}_{2R}(x_0) \|_{L^\infty(B_{2R}(x_0))} \\ & + C(Z^1_{2R,x_0} - Z^1_{R,x_0})^\alpha (Z^2_{2R,x_0})^{1-\alpha}. \end{aligned} \tag{7.19}$$

Thus, using the Young inequality and summing (7.18) and (7.19) we get

$$\begin{aligned} Z^2_{R,x_0} + (C + C^{-1}) Z^1_{R,x_0} \le CR^{\delta_A} + \frac{Z^2_{2R,x_0}}{2} + CZ^1_{2R,x_0} \\ + CZ^2_{2R,x_0} \| u - \bar{u}_{2R}(x_0) \|_{L^\infty(B_{2R}(x_0))}, \end{aligned} \tag{7.20}$$

which finally leads to

$$W_{R,x_0} \le CR^{\delta_A} + \theta W_{2R,x_0} + CZ^2_{2R,x_0} \| u - \bar{u}_{2R}(x_0) \|_{L^\infty(B_{2R}(x_0))}, \tag{7.21}$$

where

$$\theta := \max \left(\frac{1}{2}, \frac{C}{C + C^{-1}} \right) < 1.$$

This is the starting inequality for further investigation. First, since $u \in \mathcal{C}(\overline{\Omega})$, there surely exists $R_1 > 0$ (depending however strongly on the modulus of continuity of u) such that for all $R \in (0, R_1)$ we have

$$C \| u - \bar{u}_{2R}(x_0) \|_{L^\infty(B_{2R}(x_0))} \le \frac{1 - \theta}{2}. \tag{7.22}$$

Consequently, it follows from (7.21) and (7.22) that for all $R \in (0, R_1)$ and all $x_0 \in \Omega$ such that $B_{2R}(x_0) \subset \Omega$ we have

$$W_{R,x_0} \le CR^{\delta_A} + \theta_1 W_{2R,x_0}, \qquad \theta_1 := \frac{1 + \theta}{2} < 1$$

and repeating the iterative procedure we find that there is some $\beta_1 \in (0, \beta_0)$ depending on θ such that

$$\frac{W_{R,x_0}}{R^{p\beta_1}} \le C + \frac{CW_{R_{***},x_0}}{R^{p\beta_1}_{***}}, \tag{7.23}$$

where R_{***} is the maximal satisfying $R_{***} \le R_1$ and $B_{4R_{***}(x_0)} \subset \Omega$. Hence, in case $R_{***} = R_1$ it leads to

$$\frac{W_{R,x_0}}{R^{p\beta_1}} \le C(R_1),$$

while in case $B_{4R_{***}(x_0)} \nsubseteq \Omega$ we use (4.1) to obtain

$$\frac{W_{R_{***},x_0}}{R^{p\beta_1}_{***}} \le C \frac{W_{4R_{***},x_0}}{R^{p\beta_1}_{***}} \le C \frac{W_{16R_{***},x_1}}{R^{p\beta_1}_{***}} \le C,$$

where $x_1 \in \partial\Omega$ and the second inequality follows from (7.17) and the fact that we choose $\beta_1 < \beta_0$. Hence, combining these estimates with the boundary estimates (7.15) we can conclude that for any $x_0 \in \Omega$ and any $R \in (0, R_1)$ we have

$$\int_{B_R(x_0)} \frac{|\nabla u|^p}{R^{d-p+p\beta_1}} \, dx \le C(R_1, \|\nabla u\|_p), \tag{7.24}$$

which by the use of the Morrey embedding leads to

$$u \in \mathcal{C}^{\beta_1}(\overline{\Omega}). \tag{7.25}$$

Note here that the estimate (7.24) heavily relies on R_1 and consequently on the modulus of continuity of u and therefore in what follows we avoid this dependence.

Next, we proceed slightly differently. We fix some $R_2 > 0$ and $\beta \in (0, \beta_1)$ that will be specified later and for any $x_0 \in \overline{\Omega}$ we define

$$w_{R_2,x_0} := \sup_{R\in(0,2R_2)} \frac{W_{R,x_0}}{R^{p\beta}} < \infty, \qquad w_{R_2} := \sup_{x_0\in\overline{\Omega}} w_{R_2,x_0}. \tag{7.26}$$

The fact that w_{R_2,x_0} is finite follows from (7.24) and the simple inequality

$$\begin{aligned}
&\int_{B_R} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u \cdot x|^2}{|x|^{d-p+2} R^{\beta p}}\, dx \le \frac{C}{R^{\beta p}} \sum_{k=1}^{\infty} \int_{A_{R2^{-k}}} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2}{(R2^{-k})^{d-p}}\, dx \\
&\le \frac{CR^{\beta_1 p}}{R^{\beta p}} \sum_{k=1}^{\infty} 2^{-kp\beta_1} \int_{B_{R2^{-k+1}}} \frac{(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2}{(R2^{-k})^{d-p+p\beta_1}} \le C(R_1) R^{p(\beta_1-\beta)} \\
&\le C(R_1).
\end{aligned}$$

Finally, we derive uniform bound on w_{R_2,x_0}. Thus, if the supremum is attained form some $R \in (R_2, 2R_2)$ then using (4.1) we get

$$w_{R_2,x_0} \le C\left(1 + \frac{\|\nabla u\|_p^p}{R_0^{d-p+\beta p}}\right) \le C(R_2). \tag{7.27}$$

If the opposite is true, i.e., if for some $R \in (0, R_0]$ we have

$$w_{R_0,x_0} = \frac{W_{R,x_0}}{R^{p\beta}},$$

we use (7.20), and assume that β is so small that

$$2^{p\beta}\theta < 1, \qquad \beta < \delta_A$$

to get

$$\begin{aligned}
w_{R_2,x_0} &= \frac{W_{R,x_0}}{R^{p\beta}} \\
&\le R^{\delta_A - p\beta} + \theta 2^{p\beta} \frac{W_{2R,x_0}}{(2R)^{p\beta}} + \frac{CZ^2_{2R,x_0} \|u - \bar{u}_{2R}(x_0)\|_{L^\infty(B_{2R}(x_0))}}{R^{p\beta}} \\
&\le C + \theta 2^{p\beta} w_{R_2,x_0} + C(Z^2_{2R,x_0})^{\frac{1}{p}} \left(\frac{W_{2R,x_0}}{R^{p\beta}}\right)^{\frac{p-1}{p}} |u|_{\beta,R_2} \\
&\le C + \frac{\theta 2^{p\beta}+1}{2} w_{R_2,x_0} + C(\theta,\beta) Z^2_{2R,x_0} |u|^p_{\beta,2R_2}.
\end{aligned} \tag{7.28}$$

where

$$|u|_{\beta,R_2} := \sup_{x,y\in\overline{\Omega};\ 0<|x-y|<2R_2} \frac{|u(x)-u(y)|}{|x-y|^\beta}. \tag{7.29}$$

Since $\frac{\theta 2^{p\beta}+1}{2} < 1$, we can absorb the second term to the left hand side to get

$$w_{R_0,x_0} \le C + C(\theta,\beta)|u|^p_{\beta,2R_2} \sup_{R\in(0,2R_2)} Z^2_{2R,x_2}. \tag{7.30}$$

From now, we assume that β is fixed and we only choose R_0 in a proper way. First, using the Morrey embedding, we see that

$$|u|^p_{\beta,2R_2} \le Cw_{R_2}$$

with some uniform constant C depending only on β and Ω and substituting this into (7.30), we get

$$w_{R_2,x_0} \le C + Cw_{R_2} \sup_{R\in(0,2R_0)} Z^2_{R,x_0}. \tag{7.31}$$

Finally, using (7.16), it reduces to

$$w_{R_2,x_0} \le C + \frac{Cw_{R_2}}{\ln\ln|\ln R_2|}. \tag{7.32}$$

Hence, choosing R_2 so small that

$$\frac{C}{\ln\ln|\ln R_2|} \le \frac{1}{2}$$

we get

$$2w_{R_2,x_0} \le C + w_{R_2}. \tag{7.33}$$

Consequently taking the supremum with respect to $x_0 \in \Omega$, we get ide to conclude

$$w_{R_2} \le C \tag{7.34}$$

and by the Morrey embedding we end the proof of the theorem.

8. Proof of the Liouville theorem

In this section we provide the detail proof of Theorem 1.3. First, it is easy to observe that due to the p-homogeneity of F with respect to the second variable, the assumptions (1.6), (1.7), (1.13) and (1.14)–(1.18) reduce to

$$\alpha_0|\eta|^p \le F(u,\eta) \le \alpha_0^*|\eta|^p, \tag{8.1}$$

$$\alpha_0|\eta|^p \le F_\eta(u,\eta)\cdot\eta + F_u(u,\eta)\cdot u, \tag{8.2}$$

$$F_{\eta_l^\alpha} = \sum_{\beta=1}^N \sum_{m=1}^d A^{\alpha\beta}(u,\eta)h_{lm}\eta_l^\beta, \tag{8.3}$$

$$\alpha_0|\eta|^{p-2}|\mu|^2 \le \sum_{\alpha,\beta=1}^N A^{\alpha\beta}(u,\eta)\mu^\alpha\mu^\beta, \quad |A(u,\eta)| \le \alpha_0^*|\eta|^{p-2}, \tag{8.4}$$

$$\alpha_0|\eta|^p \le F_\eta(u,\eta)\cdot\eta \le pF(u,\eta). \tag{8.5}$$

We start the proof with observing that there exists $C>0$ such that for all $R>1$ we have

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le C. \tag{8.6}$$

Indeed, in case we assume that u is a minimizer we can use (3.1) (without absolute constant which disappear due to the homogeneity of F, or precisely due to (8.1)) to get

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le C\int_{B_{2R}} \frac{|u-\bar{u}_{2R}|^p}{R^d}\,dx \le C, \tag{8.7}$$

where the second inequality follows from (1.28) and we see that (8.6) holds. In case u is a bounded solution, we repeat the procedure as in the proof of Lemma 5.1. Thus, multiplying (1.1) by $u\tau_R^p$ and using (8.3)–(8.5) we deduce that

$$\begin{aligned}\alpha_0 \int_{\mathbb{R}^d} |\nabla u|^p \tau_R^p \, dx &\le C \int_{\mathbb{R}^d} |\nabla u|^{p-1} \tau_R^{p-1} |u| |\nabla \tau_R| \, dx \\ &\le \frac{\alpha_0}{2} \int_{\mathbb{R}^d} |\nabla u|^p \tau_R^p \, dx + C \int_{B_{2R}} \frac{|u|^p}{R^p} \, dx \\ &\le \frac{\alpha_0}{2} \int_{\mathbb{R}^d} |\nabla u|^p \tau_R^p \, dx + C R^{d-p},\end{aligned}$$

where the last inequality follows from the fact the u is bounded. Hence, absorbing the first integral on the right hand side to the left hand side and dividing by R^{d-p} we get (8.6).

The next step is to show that

$$\int_{\mathbb{R}^d} \frac{|\nabla u|^{p-2} |\nabla u \cdot x|^2}{|x|^{d-p+2}} \, dx \le C \limsup_{R\to\infty} \int_{B_R} \frac{|\nabla u|^p}{R^{d-p}} \, dx \le C. \tag{8.8}$$

To prove it we use the fact that the solution satisfies (1.21), which is either assumed a priori or follows from the fact that u is a minimizer. Therefore, we can use Lemma 4.2. Moreover, going back to the proof of Lemma 4.2 and using the homogeneity of F, we see that (4.1) reduces to

$$\int_{B_R} \frac{|\nabla u|^{p-2} |\nabla u \cdot x|^2}{|x|^{d-p+2}} \, dx \le C \int_{B_{2R}} \frac{|\nabla u|^p}{R^{d-p}} \, dx \le C,$$

where the second inequality follows from (8.6). Thus letting $R \to \infty$ we find (8.8).

We continue the proof by showing that

$$L := \limsup_{R\to\infty} \int_{B_R} \frac{|\nabla u|^p}{R^{d-p}} \, dx = 0. \tag{8.9}$$

For this purpose we mimic the procedure developed in Section 6. Thus, first for a bounded solution, we use Lemma 5.1 with $c \equiv 0$ and using again the homogeneity of F the relation (5.1) reduces to

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}} \, dx \le C I_R^{\alpha} Y_R^{1-\alpha}.$$

Then using the fact that u is bounded and the definition of I_R and Y_R we get that

$$L \le C \limsup_{R\to\infty} \left(\int_{\mathbb{R}^d \setminus B_R} \frac{|\nabla u|^{p-2} |\nabla u \cdot x|^2}{|x|^{d-p+2}} \, dx \right)^{\alpha} = 0, \tag{8.10}$$

where the second inequality follows from (8.8) and the basic properties of integrable functions.

In case u is not bounded but a minimizer, we mimic the procedure as in the proof of Lemma 6.2. Hence, if F satisfies (1.25) we derive from (6.23) that

$$L \le C \limsup_{R\to\infty} \sum_{\alpha; \bar{u}^\alpha = \pm\infty} \int_{B_{2R}} \frac{a(u^\alpha) |\nabla u|^p (1 + |u^\alpha - u^\alpha_{2R}|)}{R^{d-p}} \, dx. \tag{8.11}$$

where $\bar{u} := \lim_{R\to\infty} u_{B_{2R}}$. Hence, keeping the same notation as in Lemma 6.2, we define

$$v_R(x) := u(Rx) - c_R \qquad \text{for } x \in B_2$$

and using the substitution theorem we have the identity

$$\begin{aligned}
&\limsup_{R\to\infty} \sum_{\alpha;\bar{u}^\alpha=\pm\infty} \int_{B_{2R}} \frac{a(u^\alpha)|\nabla u|^p(1+|u^\alpha-u^\alpha_{2R}|)}{R^{d-p}}\,dx \\
&\quad= \limsup_{R\to\infty} \sum_{\alpha;\bar{u}^\alpha=\pm\infty} \int_{B_2} a(v_R^\alpha+c_R^\alpha)|\nabla v_R|^p(1+|v_R|)\,dx.
\end{aligned} \tag{8.12}$$

Then repeating step by step the proof of Lemma 6.2, we get the uniform bound

$$\|v_R\|_{BMO(B_2)} + \|v_R\|_{W^{1,p+\varepsilon}(B_2)} \le C \tag{8.13}$$

and consequently

$$L \le \limsup_{R\to\infty} \sum_{\alpha;\bar{u}^\alpha=\pm\infty} \|a(v_R^\alpha+c_R^\alpha)\|_{\frac{2p}{\varepsilon}}. \tag{8.14}$$

But since

$$a(v_R^\alpha+c_R^\alpha)\to 0 \quad \text{almost everywhere in } B_2$$

we can use the Lebsegue dominated convergence theorem and to show the validity of (8.9).

Consequently, substituting (8.9) into (8.8) we get

$$\int_{\mathbb{R}^d} \frac{|\nabla u|^{p-2}|\nabla u\cdot x|^2}{|x|^{d-p+2}}\,dx = 0. \tag{8.15}$$

Finally, using (5.1) once again with $c\equiv 0$ we get

$$\int_{B_R} \frac{|\nabla u|^p}{R^{d-p}}\,dx \le CY_R^{1-\alpha} I_R^\alpha = 0,$$

where we used the definition of I_R and (8.15). Since R is arbitrary, we see that $\nabla u\equiv 0$ in $\mathbb{R}^d$ and consequently u is a constant vector. Thus, the proof is complete.

Appendix A. A Logarithmic estimate for mean values

In this section we derive an estimate for the mean value of a nonnegative function v in terms of the p - Dirichlet integral for its derivatives. So, the main result of this section is following.

Lemma A.1. *Let $p\in(1,\infty)$ and $\Omega\subset\mathbb{R}^d$ be an open set. Then there exists $C>0$ such that for any $x_0\in\Omega$ and any $0<R_1\le R_2$ such that $B_{R_2}(x_0)\subset\Omega$ the following estimate holds true for all nonnegative $v\in W^{1,p}(\Omega)$:*

$$\int_{B_{R_1}(x_0)} \frac{v(x)}{R_1^d}\,dx \le \int_{B_{R_2}} \frac{v(x)}{R_2^d}\,dx + C\, I_1^{\frac{\alpha}{p}}\, I_2^{\frac{1-\alpha}{p}} + Y_1^{\frac{\alpha}{p}}\, Y_2^{\frac{1-\alpha}{p}}\, (\ln(R_2/R_1))^{\frac{1}{p'}}, \tag{A.1}$$

where

$$\begin{aligned}
I_1 &:= \int_{B_{R_1}(x_0)} \frac{|\nabla v(x)|^{p-2}|\nabla v(x)\cdot(x-x_0)|^2}{R_1^{d-p+2}}\, dx,\\
I_2 &:= \int_{B_{R_1}(x_0)} \frac{|\nabla v(x)|^{p}}{R_1^{d-p}}\, dx,\\
Y_1 &:= \int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{|\nabla v(x)|^{p-2}|\nabla v(x)\cdot(x-x_0)|^2}{|x-x_0|^{d-p+2}}\, dx,\\
Y_2 &:= \int_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} \frac{|\nabla v(x)|^{p}}{|x-x_0|^{d-p}}\, dx,
\end{aligned}$$

and α is defined as

$$\alpha := \min(1, \frac{p}{2}).$$

Proof. To simplify the proof, we consider only the point $x_0 = 0$. For other x_0 the proof is the same. In addition, to shorten all formulae appearing in the following, we denote

$$B_R := B_R(x_0), \qquad A_R^{1,2} := B_{R_2} \setminus B_{R_1}.$$

We start the proof with the following identity that is a consequence of an integration by parts formula (here n denotes the unit outward normal vector)

$$\begin{aligned}
R\int_{\partial B_R} v\, dS &= \int_{\partial B_R} vx\cdot n\, dS\\
&= \int_{B_R} \operatorname{div}(vx)\, dx = \int_{B_R} \nabla v\cdot x\, dx + d\int_{B_R} v\, dx
\end{aligned}$$

Thus dividing the result by R^{d+1} we see that

$$\frac{d}{dR}\left(R^{-d}\int_{B_R} v\, dx\right) = R^{-d-1}\int_{B_R}\nabla v\cdot x\, dx$$

and therefore integration over $R \in (R_1, R_2)$ gives

$$\int_{B_{R_1}} \frac{v}{R_1^d}\, dx \le \int_{B_{R_2}} \frac{v}{R_2^d}\, dx + \int_{R_1}^{R_2}\int_{B_R} \frac{|\nabla v\cdot x|}{R^{d+1}}\, dx\, dR. \tag{A.2}$$

Thus, we see that to prove (A.1) it is enough to estimate the second integral on the right hand side of (A.2). First, we use integration by parts (now w.r.t. R) to deduce that

$$\begin{aligned}
\int_{R_1}^{R_2}\int_{B_R} \frac{|\nabla v\cdot x|}{R^{d+1}}\, dx\, dR &= -\frac{1}{d}\left[\int_{B_R}\frac{|\nabla v\cdot x|}{R^d}\, dx\right]_{R_1}^{R_2}\\
&\quad + \frac{1}{d}\int_{R_1}^{R_2}\int_{\partial B_R}\frac{|\nabla v\cdot x|}{R^d}\, dS\, dR\\
&\le \frac{1}{d}\left(\int_{B_{R_1}}\frac{|\nabla v\cdot x|}{R_1^d}\, dx + \int_{A_R^{1,2}}\frac{|\nabla v\cdot x|}{|x-x_0|^d}\, dx\right).
\end{aligned} \tag{A.3}$$

Hence, it only remains to estimate the last two integrals in (A.3). First, we focus on the case $p \ge 2$. For such p's we use the Hölder inequality and the fact that

$|x| \le R_1$ in B_{R_1} to obtain

$$\begin{aligned}\int_{B_{R_1}} \frac{|\nabla v \cdot x|}{R_1^d}\,dx &\le C\left(\int_{B_{R_1}} \frac{|\nabla v \cdot x|^p}{R_1^d}\,dx\right)^{\frac{1}{p}}\\ &\le C\left(\int_{B_{R_1}} \frac{|\nabla v|^{p-2}|\nabla v \cdot x|^2}{R_1^{d-p+2}}\,dx\right)^{\frac{1}{p}}\end{aligned}$$

and also

$$\begin{aligned}\int_{A_R^{1,2}} \frac{|\nabla v \cdot x|}{|x|^d}\,dx &\le \int_{A_R^{1,2}} \frac{|\nabla v|^{\frac{p-2}{p}}|\nabla v \cdot x|^{\frac{2}{p}}}{|x|^{\frac{d-p+2}{p}}} \cdot \frac{1}{|x|^{\frac{d}{p'}}}\,dx\\ &\le C\left(\int_{A_R^{1,2}} \frac{|\nabla v|^{p-2}|\nabla v \cdot x|^2}{|x|^{d-p+2}}\,dx\right)^{\frac{1}{p}} (\ln(R_2/R_1))^{\frac{p-1}{p}}\end{aligned}$$

and combining these estimates with (A.2) and (A.3) we find (A.1) for $p \ge 2$.

Similarly, for $p \in (1,2)$ we can deduce by using the Hölder inequality that

$$\begin{aligned}\int_{B_{R_1}} \frac{|\nabla v \cdot x|}{R_1^d}\,dx &= \int_{B_{R_1}} \frac{|\nabla v|^{\frac{p-2}{2}}|\nabla v \cdot x|}{R_1^{\frac{d-p+2}{2}}} \cdot \frac{|\nabla v|^{\frac{2-p}{2}}}{R_1^{\frac{(d-p)(2-p)}{2p}}} \cdot \frac{1}{R_1^{\frac{d}{p'}}}\,dx\\ &\le C\left(\int_{B_{R_1}} \frac{|\nabla v|^{p-2}|\nabla v \cdot x|^2}{R_1^{d-p+2}}\,dx\right)^{\frac{1}{2}} \left(\int_{B_{R_1}} \frac{|\nabla v|^p}{R_1^{d-p}}\,dx\right)^{\frac{2-p}{2p}}\end{aligned}$$

and that

$$\begin{aligned}&\int_{A_R^{1,2}} \frac{|\nabla v \cdot x|}{|x|^d}\,dx = \int_{A_R^{1,2}} \frac{|\nabla v|^{\frac{p-2}{2}}|\nabla v \cdot x|}{|x|^{\frac{d-p+2}{2}}} \cdot \frac{|\nabla v|^{\frac{2-p}{2}}}{|x|^{\frac{(d-p)(2-p)}{2p}}} \cdot \frac{1}{|x|^{\frac{d}{p'}}}\,dx\\ &\le C\left(\int_{A_R^{1,2}} \frac{|\nabla v|^{p-2}|\nabla v \cdot x|^2}{|x|^{d-p+2}}\,dx\right)^{\frac{1}{2}} \left(\int_{A_R^{1,2}} \frac{|\nabla v|^p}{|x|^{d-p}}\,dx\right)^{\frac{2-p}{2p}} (\ln(R_2/R_1))^{\frac{p-1}{p}}\end{aligned}$$

Substituting these estimates into (A.2) and (A.3) we again easily deduce (A.1). Thus, the proof is complete. □

Appendix B. L^∞ a priori bounds for solutions

This section is devoted to deriving L^∞ and exponential bounds for solutions of (1.1). First, we discuss the simple case described by the Uhlenbeck structure.

Lemma B.1. *Let $F(u,\nabla u) = a(u)[|\nabla u|^2 + \delta_0]^{\frac{p}{2}}$ and a satisfy* (1.12). *Assume that $u \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ is a weak solution to* (1.1) *with b satisfying* (1.20). *Then there exists C depending only on α_0, K, p and δ_A such that*

$$\|u\|_\infty \le C. \tag{B.1}$$

Proof. The proof is based on the Moser iteration technique. We test[3] the system (1.1) by $u|u|^m$ with arbitrary $m \ge 0$. Thus, after integration by parts and by using

[3] It is not a possible test function but me can properly truncate such a function to make the proof rigorously.

(1.12) and (1.20) we find that

$$\begin{aligned}&\alpha_0 \int_\Omega (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 |u|^m \, dx \\ &\qquad + \frac{1}{2} \int_\Omega a(u)(\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla |u|^2 \cdot \nabla |u|^m \, dx \\ &\le K \int_\Omega |u|^{m+1}(|\nabla u|^{p-1-\delta_A} + 1) \, dx\end{aligned} \tag{B.2}$$

Next, we use the Young inequality to bound the term on the right hand side. Hence assuming $\delta_A \ll 1$ we get

$$\begin{aligned}&K \int_\Omega |u|^{m+1}(|\nabla u|^{p-1-\delta_A} + 1) \, dx \\ &\le C \int_\Omega |u|^{m+1} |\nabla u|^{p-1-\delta_A} \chi_{|\nabla u| \ge \delta_0} + |u|^{m+1} \, dx \\ &= C \int_\Omega ((\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 |u|^m)^{\frac{p-1-\delta_A}{p}} |u|^{1+\frac{m(1+\delta_A)}{p}} + |u|^{m+1} \, dx \\ &\le \frac{\alpha_0}{2} \int_\Omega (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 |u|^m + C \int_\Omega |u|^{\frac{p}{1+\delta_A}+m} \, dx.\end{aligned} \tag{B.3}$$

Similarly, one can observe that

$$\int_\Omega (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 |u|^m \, dx \ge \int_\Omega |\nabla u|^p |u|^m - |u|^m \, dx$$

and combining it with (B.2) and (B.3), we deduce the final estimate

$$\left(\frac{p}{m+p}\right)^p \int_\Omega |\nabla |u|^{\frac{m+p}{p}}|^p \, dx \le \int_\Omega |\nabla u|^p |u|^m \, dx \le C \int_\Omega |u|^{\frac{p}{1+\delta_A}+m} \, dx. \tag{B.4}$$

Thus, uign the Sobolev embedding theorem, we see that (assuming for simplicity that $p < d$)

$$\|u\|^{m+p}_{\frac{d(m+p)}{d-p}} = \||u|^{\frac{m+p}{p}}\|^p_{\frac{dp}{d-p}} \le C \left(\frac{m+p}{p}\right)^p \|u\|^{\frac{p}{1+\delta_A}+m}_{\frac{p}{1+\delta_A}+m}. \tag{B.5}$$

Note here, that $\delta_A > 0$ is needed just to get the first a priori estimate, i.e., setting $m = 0$ we can get that

$$\|u\|_{1,p} \le C.$$

However, in what follows we consider the worst case, i.e., $\delta_A = 0$ and then (B.5) reduces to

$$\|u\|_{\frac{d(m+p)}{d-p}} \le C^{\frac{1}{m+p}} \left(\frac{m+p}{p}\right)^{\frac{p}{m+p}} \|u\|_{p+m}. \tag{B.6}$$

Therefore, defining

$$p_0 := p, \qquad p_{k+1} = \frac{dp_k}{d-p} \Leftrightarrow p_{k+1} = \left(\frac{d}{d-p}\right)^k p,$$

we get from (B.6)

$$\|u\|_{p_{k+1}} \le C^{\frac{1}{p_k}} \left(\frac{p_k}{p}\right)^{\frac{p}{p_k}} \|u\|_{p_k}, \tag{B.7}$$

which after an iteration procedure leads to

$$\begin{aligned}\|u\|_{p_k} &\le C^{\sum_{i=0}^{k}\frac{1}{p_k}} e^{\sum_{i=0}^{k}\frac{p}{p_k}\ln\frac{p_k}{p}}\|u\|_p \le C^{\sum_{i=0}^{\infty}\frac{1}{p_k}} e^{\sum_{i=0}^{\infty}\frac{p}{p_k}\ln\frac{p_k}{p}}\|u\|_p \\ &= C^{\frac{1}{p}\sum_{i=0}^{\infty}(\frac{d-p}{d})^k} e^{\sum_{i=0}^{\infty}\left(\frac{d-p}{d}\right)^k k\ln\frac{d}{d-p}}\|u\|_p \le K\|u\|_p.\end{aligned} \tag{B.8}$$

Hence letting $k \to \infty$ we get (B.1). $\square$

Appendix C. Exponential a priori estimates for solution

The second estimate of this section is only of the exponential type however works also for more general structure of F.

Lemma C.1. *Let $u \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ be a weak solution to* (1.1)*. Assume that F satisfies* (1.7), (1.17) *and* (1.13)*. Let $m \ge 1$ be given and assume that there exists a smooth bounded mapping $a : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ such that for all $u, \xi, \mu \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^{d\times N}$*

$$\sum_{\alpha,\beta=1}^{N} a^{\alpha\beta}(u)u^\alpha u^\beta \ge \delta_B |u|^2, \tag{C.1}$$

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{N} A^{\alpha\beta}(u,\eta)a^{\gamma\delta}(u)\xi^\alpha\xi^\delta\mu^\beta\mu^\delta \ge 0, \tag{C.2}$$

$$|a_u(u)||u|^{2m+1} \le K. \tag{C.3}$$

Then there exist constants $C, \lambda_1 > 0$ depending only on $K, \delta_A, \delta_B, \lambda_0, \alpha_0$ such that

$$\int_\Omega |u|^2 e^{\lambda_1 |u|^{2m}}\, dx \le C. \tag{C.4}$$

Proof. First we define a quadratic form $B(u)$ as

$$B(u) := \sum_{\gamma,\delta=1}^{N} a^{\gamma\delta}(u)u^\gamma u^\delta.$$

Next, we test (1.1) by $ue^{\lambda(B(u))^m}$ with some $\lambda := \frac{\alpha_0}{2Km} > 0$. Therefore after integration by parts and using (1.7), (1.13) and (1.20) we find that

$$\begin{aligned}&\alpha_0\int_\Omega (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|^2 e^{\lambda(B(u))^m} \\ &\quad+ \int_\Omega \sum_{\alpha,\beta} A^{\alpha\beta}(u,\nabla u)u^\alpha \nabla u^\beta \cdot \nabla e^{\lambda(B(u))^m}\, dx \le \int_\Omega |b||u|e^{\lambda(B(u))^m}\, dx.\end{aligned} \tag{C.5}$$

First we focus on the last term on the left hand side of (C.5). This term can be estimated as

$$
\begin{aligned}
&2\int_\Omega \sum_{\alpha,\beta=1}^N A^{\alpha\beta}(u,\nabla u)u^\alpha \nabla u^\beta \cdot \nabla e^{\lambda(B(u))^m}\, dx \\
&= \lambda m \int_\Omega (B(u))^{m-1} e^{\lambda(B(u))^m} \sum_{\alpha,\beta,\gamma,\delta=1}^N A^{\alpha\beta}(u,\nabla u)\nabla(u^\alpha u^\beta)\cdot \nabla(a^{\gamma\delta}(u)u^\gamma u^\delta)\, dx \\
&= \lambda m \int_\Omega (B(u))^{m-1} e^{\lambda(B(u))^m} \sum_{\alpha,\beta,\gamma,\delta=1}^N A^{\alpha\beta}(u,\nabla u)a^{\gamma\delta}(u)\nabla(u^\alpha u^\beta)\cdot \nabla(u^\gamma u^\delta)\, dx \\
&\quad + \lambda m \int_\Omega (B(u))^{m-1} e^{\lambda(B(u))^m} \sum_{\alpha,\beta,\gamma,\delta=1}^N A^{\alpha\beta}(u,\nabla u)u^\gamma u^\delta \nabla(u^\alpha u^\beta)\cdot \nabla(a^{\gamma\delta}(u))\, dx.
\end{aligned}
$$

Next using the assumption (C.2) we see that the first term is nonnegative and using the assumption (C.3) we can estimate the second term as

$$
\begin{aligned}
&\lambda m \left| \int_\Omega (B(u))^{m-1} e^{\lambda(B(u))^m} \sum_{\alpha,\beta,\gamma,\delta=1}^N A^{\alpha\beta}(u,\nabla u)u^\gamma u^\delta \nabla(u^\alpha u^\beta)\cdot \nabla(a^{\gamma\delta}(u))\, dx \right| \\
&\qquad \le \lambda m \int_\Omega |u|^{2m+1} e^{\lambda(B(u))^m} (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 |a_u(u)|\, dx \\
&\qquad \le \frac{\alpha_0}{2} \int_\Omega (\delta_0 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 e^{\lambda(B(u))^m},
\end{aligned}
$$

where, for the last inequality we used the choice of λ. Then we can estimate the right hand side of (C.5) via the similar procedure as in the preceding Lemma and we can finally conclude that

$$\int_Q |u|^2 e^{\lambda(B(u))^m}\, dx \le C.$$

Thus, using (C.1) we find (C.4). $\square$

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