Nečas Center for Mathematical Modeling

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Preprint no. 2013-028

http://ncmm.karlin.mff.cuni.cz/

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August 19, 2013

Abstract

In this paper we show existence of a global classical solution to a quasilinear hyperbolic integrodifferential equation of non-convolutionary type for small data. We apply the result to show global existence for a one-dimensional model of a chemically reacting viscoelastic body.

Key words: nonconvolution integral equation, quasilinear hyperbolic integral equations, chemically reacting viscoelastic materials

AMS subject classification: 45K05, 45G10, 74D10.

1 Introduction

Recently, flows in materials with a stress-strain relation dependent on an external factor (e.g. concentration of a chemical) started to have been studied intensively, see Bulíček, Málek and Rajagopal [3] and references therein. An integral model for a viscoelastic material was introduced by Rajagopal and Wineman in [6] and applied by Bárta in [1], [2] to parabolic models of viscoelastic fluids.

In this paper, we consider the following system

$$\begin{aligned} u_{tt} &= \chi(c, u_x)u_{xx} + \int_0^t k(c(t,x), t-s)\psi(u_x(s))_x ds + g, \\ c_t &= c_{xx}. \end{aligned} \tag{1}$$

A one-dimensional viscoelastic body is represented by the interval $\Omega = [0,1]$, $x \in [0,1]$. The displacement of a particle x at time t is denoted by $u(t,x)$ and $c(t,x)$ is a concentration of a chemical in (t,x). Function g represents an external force. Since we assume that difusivity is independent of u, the two equations are not coupled.

Since we can get c from the second equation and insert it to the first equation, we will be interested in equations of the form

$$u_{tt} = \chi(t,x,u_x)u_{xx} + \int_0^t a_s(t,x,t-s)\psi(u_x(s))_x ds + g \tag{IDE}$$

(here a_s is the derivative of a with respect to the third variable). Equations similar to this one were studied by Dafermos and Nohel [4], Hrusa and Nohel [5], Renardy, Hrusa and Nohel [7] and others.

Unlike the works mentioned in the previous paragraph, in our case the functions χ and a depend explicitly on t and x and a is not a convolution kernel any more. However, we will assume that the dependence on t and x is not very strong and a is almost convolution kernel. In this case, we show existence and uniqueness of classical solutions by the same methods as in the above mentioned papers.

Let us mention that Rajagopal and Wineman introduced a model with

$$k(c(t,x), t-s) = e^{-\lambda(c(t,x))(t-s)},$$

see [6], where λ is a positive function. If λ is smooth and $\lambda \geq \varepsilon > 0$, then for small changes of concentration the kernel k is almost convolutionary, as was shown in [1].

We consider the following initial and boundary conditions.

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad u(\cdot,0) = u(\cdot,1) = 0. \tag{Cu}$$

$$c(0,\cdot) = c_0, \quad c_x(\cdot,0) = c_x(\cdot,1) = 0. \tag{Cc}$$

2 Notation and the main results

Let us start with introducing another form of (IDE). Let us apply integration by parts to the integral in (IDE). We obtain

$$u_{tt} = \varphi(t,x,u_x)u_{xx} + \int_0^t a(t,x,t-s)\psi(u_x(s,x))_{xt}ds + f(t,x), \tag{IDE1}$$

where

$$f(t,x) = g(t,x) + a(t,x,t)\psi'(u_0'(x))u_0''(x)$$

and

$$\varphi(t,x,u_x) = \chi(t,x,u_x) - a(t,x,0)\psi'(u_x(t,x)).$$

For the function $\chi = \chi(t,x,u_x(t,x))$ we will denote χ_t, resp. χ_x the derivatives with respect to the first, resp. second variable, the derivative with respect to the third variable is denoted by $\chi'(t,x,u_x(t,x))$. Similarly for the function φ. For $a : J \times \Omega \times J$ we will denote the partial derivatives with respect to first, second and third variable respectively by a_t, a_x, a_s.

In the following, $\Omega = [0,1]$ and $J := [0, T_{max}]$ or $J := [0, +\infty) = R_+$. For a function $u : J \times \Omega \to R$ we will often use notation $u(t) := u(t,\cdot)$ and $\|u(t)\|_2 := \|u(t,\cdot)\|_{L^2(\Omega)}$. On the other hand, by $\|u\|_2$ we mean $\|u\|_{L^2(J\times\Omega)}$.

We say that a kernel $a : J \times \Omega \times J \to R$ is of strong positive type (or strongly positive definite), if there exists $c > 0$ such that

$$Q(a,T,v) \geq cQ(e,T,v)$$

for all $v \in C(J, L^2(\Omega))$ and all $T \in J$, where

$$Q(a, T, v) := \int_0^T \int_\Omega \int_0^t a(t, x, t-s) v(s, x)\, \mathrm{d}s\ v(t, x)\, \mathrm{d}x\, \mathrm{d}t$$

and $Q(e, T, v)$ similarly with a replaced by $e(t, x, t-s) := e^{t-s}$.

Throughout the paper, $C > 0$ will be a generic constant and $Z : R_+ \to R_+$ will be a generic function which is continuous nondecreasing and $Z(0) = 0$.

Our aim is to show global existence for small data and small values of u_x. So, let us fix a small neighborhood of 0 and denote it by B.

Let us introduce our assumptions.

(A1) χ, $\varphi \in C_b^2(J \times \Omega \times B)$, $\psi \in C_b^3(B)$ and all derivatives of χ, φ, ψ (up to second resp. third order) are pointwise bounded by C_ψ and χ, φ, $\psi' \geq c_\psi > 0$ on $J \times \Omega \times B$, resp. B

(A2) $a \in C_b^2(J \times \Omega \times J)$ with a_{tt}, $a_{ts} \in L^1(J^2(L^\infty(\Omega)))$, $a_s(0, 0, \cdot) \in L^2(J)$, a_t, a_s, $a_{xs}(T, \cdot, T - \cdot) \in L^1(J, L^\infty(\Omega))$.

(A3) $u_0 \in H^3(\Omega)$, $u_1 \in H^2(\Omega)$, $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$.

(A4) $g, g_x, g_t \in C_b(J, L^2(\Omega))$, $g, g_x, g_t, g_{tt} \in L^2(J, L^2(\Omega))$.

(A5) $\chi'(u_0'(0))u_0''(0) + g(0, 0) = \chi'(u_0'(1))u_0''(1) + g(1, 0) = 0$.

(A6) a is of strong positive type.

Let us introduce several quantities measuring the data

$$U_0(u_0, u_1) := \int_\Omega u_0^2 + (u_0')^2 + (u_0'')^2 + (u_0''')^2 + u_1^2 + (u_1')^2 + (u_1'')^2\, \mathrm{d}x \tag{2}$$

$$F(f) := \sup_{t \in J} \int_\Omega (f^2 + f_x^2 + f_t^2)(t, x)\, \mathrm{d}x + \int_J \int_\Omega f^2 + f_x^2 + f_t^2 + f_{tt}^2)(t, x)\, \mathrm{d}x\, \mathrm{d}t \tag{3}$$

and similarly $F(g)$. We will often write F instead of $F(f) + F(g)$ and U_0 instead of $U_0(u_0, u_1)$. Let us define

$$\varepsilon_\varphi := \max_{J \times \Omega \times B} \{\varphi_t, \varphi_{tx}, \varphi_t', \varphi_{ttx}, \varphi_{tx}', \varphi_x', \varphi_{tt}'\}, \tag{4}$$

$$\varepsilon_\chi := \max_{J \times \Omega \times B} \{\chi_t, \chi_{tx}, \chi_t', \chi_{ttx}, \chi_{tx}', \chi_x', \chi_{tt}', \chi_x\}, \tag{5}$$

$$\varepsilon_a := \sup_t \left(\tilde{a}_t(t, 0) + \int_J |\tilde{a}_t(t, t-s)|\, \mathrm{d}s + \int_J \tilde{a}_{ts}(t, s)^2 + \tilde{a}_{xs}(t, s)^2\, \mathrm{d}s \right) + \int_J \int_J |\tilde{a}_{tt}(t, s)| + |\tilde{a}_{ts}(t, s)|\, \mathrm{d}s\, \mathrm{d}t, \tag{6}$$

where $\tilde{a}_t$ means $\sup_x a_t$. Further we introduce two quantities measuring the solution

$$\nu(t) := \max_{x \in \Omega, s \in [0, t]} (u_x^2 + u_{xx}^2 + u_{tx}^2)^{1/2}(s, x) \tag{7}$$

and

$$E(t) := \max_{s \in [0, t]} \int_\Omega (u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{tx}^2 + u_{tt}^2 + u_{xxx}^2 + u_{txx}^2 + u_{ttx}^2 + u_{ttt}^2)(s, x)\, \mathrm{d}x$$
$$+ \int_0^t \int_\Omega (u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{tx}^2 + u_{tt}^2 + u_{xxx}^2 + u_{txx}^2 + u_{ttx}^2 + u_{ttt}^2)(s, x)\, \mathrm{d}x\, \mathrm{d}s,$$

Theorem 2.1 *Assume (A1) – (A6) hold. There exists $\mu > 0$ such that for every u_0, u_1, g, χ and φ satisfying*

$$U_0(u_o, u_1) + F(g) + \varepsilon_\varphi + \varepsilon_\chi \leq \mu, \tag{8}$$

the initial-boundary value problem (IDE) has a unique solution $u : \Omega \times J \to R$ with

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, u_{xxx}, u_{txx}, u_{ttx}, u_{ttt} \in C_b(J, L^2(\Omega)) \cap L^2(J, L^2(\Omega)).$$

If $J = R_+$ then

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt} \to 0$$

uniformly on Ω as $t \to +\infty$.

Denote

$$C_0(c_0) := \int_\Omega c_0^2 + (c_0')^2 + (c_0'')^2 + (c_0''')^2 + (c_0'''')^2 \, dx. \tag{9}$$

Let $U \subset R$ be a neigborhood of zero. Consider the following assumptions

(A1') $\chi \in C_b^2(U \times J)$, $\psi \in C_b^3(B)$ and all derivatives of χ, ψ (up to second resp. third order) are pointwise bounded by C_ψ and $\chi(t, x, 0) > 0$, $\psi'(0) > 0$ and $\chi(t, x, 0) - a(t, x, 0)\psi'(0) > 0$.

(A2') $k \in C_b^2(U \times J)$ with k, k', k'', k_s, $k_s' \in L^1(J, L^\infty(U))$.

(A3') $u_0 \in H^3(\Omega)$, $u_1 \in H^2(\Omega)$, $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$, $c_0 \in H^4(\Omega)$.

(A4') = (A4)

(A5') = (A5)

(A6') $(t, s) \mapsto k(c(t, x), s)$ is of positive type for every $c \in C_b^2$ with $\|c_t\|_\infty$ small enough.

The assumption (A6') is not easy to verify, but Theorem 2.4 and Example 2.8 in [1] give sufficient conditions under which (A6') holds. In fact, the assumption (A6') is satisfied if $k(z, \cdot)$ is of η-strong c-positive type for all $z \in U$, $k'(z, 0) = \lim_{s\to+\infty} k'(z, s) = \lim_{s\to+\infty} k_s'(z, s) = 0$, and $\|k_s'(z, 0)\|$, $\|k'(z, \cdot)\|_1$, and $\|k_{ss}'(z, \cdot)\|_1$ are bounded by a constant independent of z.

In particular, if $k(c(t, x), t - s) = e^{-\lambda(c(t,x))(t-s)}$, where λ is a smooth function with values in $[\alpha, \beta]$, $\alpha > 0$, then (A6') holds (see Example 2.8 and the third section of [1]).

Theorem 2.2 *Assume (A1') – (A6') hold. There exists $\mu > 0$ such that for every u_0, u_1, c_0, g and χ satisfying*

$$U(u_o, u_1) + F(g) + C_0(c_0) + \varepsilon_\chi \leq \mu,$$

the system (1) has a unique solution $(u, c) : J \times \Omega \to R^2$ with

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, u_{xxx}, u_{txx}, u_{ttx}, u_{ttt} \in C_b(R_+, L^2(I)) \cap L^2(R_+, L^2(I))$$

If $J = R_+$ then

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt} \to 0$$

uniformly on Ω as $t \to +\infty$.

Remark 2.3 *(1) If χ does not depend on c (i.e., the instant response does not depend on concentration) then $\varepsilon_\chi = 0$ and Theorem 2.2 yields global existence provided the initial values and the external force g are small enough.*

(2) If moreover $k(c(t,x),0)$ is independent of c (e.g. in the case $k(c(t,x),t-s) = e^{-\lambda(c(t,x))(t-s)}$), then also $\varepsilon_\varphi = 0$. In this case, the proof would be shorter since many terms in the estimates disappear.

3 Local existence

We will generalize Theorem III.5 from [7]. Consider the following equation

$$u_{tt} = A(t,x,u_x)u_{xx} + \int_0^t K(t,t-s,x,u_x(s))u_{xx}(s)ds + F(t) \tag{10}$$

with initial and boundary conditions

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad u(t,0) = u(t,1) = 0. \tag{11}$$

Assume

(S1) $A \in C^2(J \times \Omega \times U)$.

(S2) $u_0 \in H^3([0,1])$, $u_1 \in H^2([0,1])$, $\nabla u_0(x) \in U$ for all $x \in [0,1]$.

(S3) $F \in \bigcap_{k=0}^1 C^{1-k}(J, H^k(\Omega))$, $F_{tt} \in L^1(J, L^2(\Omega))$.

(S4) $K \in C^2(J^2 \times \Omega \times U)$.

(S5) $A \geq \varepsilon_A > 0$ on $J \times \Omega \times U$.

(S6) $\partial_t^k u(\cdot,0) = \partial_t^k u(\cdot,1) = 0$ for $k = 0,1,2$.

Theorem 3.1 *Let (S1) – (S6) hold. Then there exists $T \in (0, T_{max}]$ such that (10) has a solution $u \in \bigcap_{k=0}^3 C^{3-k}([0,T), H^k(\Omega))$. Moreover, if*

$$\sup_{t\in[0,T)} \int_\Omega u^2(t) + u_x^2(t) + u_{xx}^2(t) + u_{xxx}^2(t) + u_t^2(t) + u_{tx}^2(t) + u_{txx}^2(t)\, dx < +\infty, \tag{12}$$

then $T = T_{max}$.

Proof. The proof is very similar to the proof of Theorem III.5 in [7] where the same statement is proved with A, K independent of (t,x). Therefore, we will just give the main idea and point out differences.

We consider the linearized equation

$$u_{tt} = A(x,t,w_x)u_{xx} + \int_0^t K(t,t-s,x,w_x(s))w_{xx}(s)ds + F(t).$$

It has a solution $u \in \bigcup_{k=0}^m C^{m-k}([0,T_{max}], H^k(\Omega))$ for each w by Lemma III.3 in [7]. We show that the mapping $S : w \mapsto u$ is a contraction on $X(T', M)$ for T' small and M large enough, where

$$X(T,M) = \{w \in \bigcap_{k=0}^3 W^{3-k,\infty}([0,T'], H^k(\Omega)),$$
$$\partial_t^k w(\cdot,0) = \partial_t^k w(\cdot,0), k = 0,1,2, \sum_{k=0}^3 \|w\|_{k,3-k} \leq M\},$$

where $\|\cdot\|_{k,l}$ is the norm of $W^{k,\infty}([0,T'],H^l(\Omega))$.

To show that $S : X(T,M) \to X(T,M)$ for appropriate T, M one can use the same procedure as in Lemma III.8 in [7], there only appear several new terms that can be easily estimated. In fact,

$$\int_0^t \|\partial_1 A(s,\cdot,w_x(s,\cdot))\|_{H^1}\,\mathrm{d}s \le C,$$

$$\left\|\int_0^t K(t,t-s,\cdot,w_x(s,\cdot))w_{xx}(s\cdot)\,\mathrm{d}s\right\|_{H^1} \le C + T\cdot P(M),$$

and

$$\left\|\frac{d}{dt}\int_0^t K(t,t-s,\cdot,w_x(s,\cdot))w_{xx}(s\cdot)\,\mathrm{d}s\right\|_2 \le C + T\cdot P(M)$$

hold by the same argument as in [7] ($P(M)$ is a generic continuous function of M).

To show that S is contractive in the metric

$$d(w,\bar{w}) := (\|w-\bar{w}\|_{0,2}^2 + \|w-\bar{w}\|_{1,1}^2 + \|w-\bar{w}\|_{2,0}^2)^{1/2}$$

we estimate as in [7]. Taking the difference of equations for w, $u := Sw$ and $\bar{w}$, $\bar{u} := S\bar{w}$, differentiate w.r.t. t, multiply by $U_{tt} := u_{tt} - \bar{u}_{tt}$ and integrate over space and time we obtain some extra terms that will be of lower order than the terms already present. So, they can be estimated via Hölder and Young inequalities as in [7], Lemma III.9.

The moreover-part follows by the standard continuation argument. If $T < T_{max}$, then $\lim_{t\to T-}(u(t),u_t(t))$ exists and it belongs to (H^3,H^2), so it can be considered as a new initial condition and the solution can be extended to $[T,T+\delta)$. $\square$

Corollary 3.2 *Let (A1)-(A5) hold. Then there exist $T \in (0,T_{max}]$ such that (IDE), (Cu) has a unique solution $u \in \bigcap_{k=0}^3 C^{3-k}([0,T),H^k(\Omega))$. If (12) holds, then $T = T_{max}$.*

Proof. It is easy to see that (A1)-(A5) imply (S1)-(S6). $\square$

4 Global existence for small a

To show global existence on the interval J, it is sufficient to show that $E(T)$ will not escape to infinity. This will follow from the key estimate

$$E(T) \le CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon+\varepsilon_\varphi+\varepsilon_\chi+\varepsilon_a) + CE(T)^{3/2} + CE(T)^2. \tag{13}$$

More precise formulation is contained in the following lemma.

Lemma 4.1 *Let (A1)-(A6) hold. Then there exists a continuous function Z satisfying $Z(0) = 0$ and a constant $C > 0$ such that (13) holds for all $T \in J$.*

The proof of this lemma is contained in Lemmas 4.3 - 4.8. Now we show global existence theorem with an additive assumption that derivatives of a are not very large.

Theorem 4.2 *Let (13) hold with $\varepsilon_a < 1/C$. Then Theorem 2.1 holds.*

Proof. Since we have local existence by Corollary 3.2, it is sufficient to show that condition (12) is satisfied. It will follow from (13)

Take ε_φ, ε_χ, ε so small that $C(\varepsilon + \varepsilon_\varphi + \varepsilon_X + \varepsilon_a) = 1 - \delta < 1$. Then we have

$$E(T) \le \frac{C}{\delta} Z(U_0) + \frac{C}{\delta} Z(F) + \frac{C(\varepsilon)}{\delta} \varepsilon_\varphi^2 + \frac{C}{\delta} E(T)^{3/2} + \frac{C}{\delta} E(T)^2$$

for $t \in J$. Take $\gamma > 0$ so small, that $\frac{C}{\delta}\gamma^{3/2} < \gamma/4$ and $\frac{C}{\delta}\gamma^2 < \gamma/4$ and then U_0, F and ε_φ so small that $\frac{C}{\delta} Z(U_0) + \frac{C}{\delta} Z(F) + \frac{C(\varepsilon)}{\delta}\varepsilon_\varphi^2 < \gamma/4$ and $E(0) < \gamma$. Let $T_\gamma := \sup\{T \in J : \; E(t) \le \gamma \; \forall t \in [0,T]\}$. Then $E(t) \le \frac{3}{4}\gamma$ on $[0, T_\gamma)$. Hence, $T_\gamma = T_{max}$ by Corollary 3.2. □

Now we will prove six lemmas that together with Remark 4.9 give a proof of Lemma 4.1.

Lemma 4.3 *Let (A1) – (A6) hold and $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T_1), H^k(\Omega))$ is a solution to (IDE), (Cu). Then*

$$\|u_{tx}(T)\|_2^2 + \|u_{xx}(T)\|_2^2 + Q(a, T, \psi(u_x)_{xt}) \le$$
$$CZ(U_0) + CZ(F) + CE(T)^{3/2} + C(\varepsilon_\varphi + \varepsilon)E(T). \quad (14)$$

Proof. Multiply (IDE1) by $\psi(u_x)_{xt} = \psi''(u_x)u_{xt}u_{xx} + \psi'(u_x)u_{xxt}$ and integrate over $[0,1] \times [0,T]$. We obtain

$$\int_0^T [u_{tt}\psi(u_x)_t]_{x=0}^1 - \int_0^1 \frac{d}{dt}[\frac{1}{2}\psi'(u_x)u_{tx}^2] + \psi''(u_x)u_{xt}^3 \,\mathrm{d}x\,\mathrm{d}t =$$
$$\int_0^T \int_0^1 \frac{d}{dt}\left[\frac{1}{2}\varphi(t,x,u_x)\psi'(u_x)u_{xx}^2\right] - \frac{1}{2}\varphi_t\psi' u_{xx}^2 - \frac{1}{2}\varphi'\psi' u_{xt}u_{xx}^2 + \frac{1}{2}\varphi\psi'' u_{xx}^2 u_{xt} \,\mathrm{d}x\,\mathrm{d}t$$
$$+ Q(a, T, \psi(u_x)) + \int\int f\varphi(u_x)_{xt}.$$

Hence,

$$\frac{1}{2}\int_0^1 \psi'(u_x)u_{tx}^2(T) + \varphi(t,x,u_x)\psi'(u_x)u_{xx}^2(T)\,\mathrm{d}x + Q(a,T,\psi(u_x)) =$$
$$\frac{1}{2}\int_0^1 \psi'(u_0')(u_1')^2 + \varphi(0,x,u_0')\psi'(u_0')(u_0'')^2 -$$
$$\int_0^T \int_0^1 \psi''(u_x)u_{xt}^3 - \frac{1}{2}\varphi_t\psi' u_{xx}^2 - \frac{1}{2}\varphi'\psi' u_{xt}u_{xx}^2 + \frac{1}{2}\varphi\psi'' u_{xx}^2 u_{xt} + f\psi(u_x)_{xt}\,\mathrm{d}x\,\mathrm{d}t.$$

Estimating ψ' and φ on the left-hand side from bellow and the derivatives of φ, ψ on the right-hand side from above and using Hölder and Young inequality we obtain (14). □

Lemma 4.4 *Let (A1) – (A6) hold and* $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T_1), H^k(\Omega))$ *is a solution to (IDE), (Cu). Then*

$$\|u_{ttx}(T)\|_2^2 + \|u_{txx}(T)\|_2^2 + \lim_{h\to 0}\frac{1}{h^2}Q(a,T,\Delta_h\psi(u_x)_{xt}) \leq$$
$$CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon+\varepsilon_\varphi+\varepsilon_a) + CE(T)^{3/2} + CE(T)^2 \quad (15)$$

Proof. Applying Δ_h to (IDE1) we obtain (using Lemma 6.4)

$$\Delta_h u_{tt}(t) = \Delta_h\varphi(t,x,u_x(t))_x + \int_0^t a(t,x,t-\tau)\Delta_h\psi(u_x)_{xt}(\tau)d\tau + \Delta_h f(t)$$
$$+\int_0^h a(t+h,x,t+h-s)\psi(u_x)_{xt}(s)\,\mathrm{d}s + \int_0^t [a(t+h,x,t-s)-a(t,x,t-s)]\psi(u_x)_{xt}(s+h)\,\mathrm{d}s \quad (16)$$

We multiply both sides by $\Delta_h(\psi(u_x)_{xt})$ and integrate over $[0,1]\times[0,T]$. Denote the six terms we obtain by $I_1, \ldots, I_6$. We will compute the limits $\lim_{h\to 0}\frac{1}{h^2}I_j =: L_j$.

Let us start with the first term.

$$I_1 = -\int_0^T\int_0^1 \Delta_h[\psi'(u_x)u_{tx}]\Delta_h u_{ttx}\,\mathrm{d}x\,\mathrm{d}t =$$
$$-\int_0^T\int_0^1 \frac{1}{2}[\psi'(u_x)(\Delta_h u_{tx})^2]_t - \frac{1}{2}\psi''(u_x)u_{xt}[\Delta_h u_{tx}]^2 + \Delta_h\psi'(u_x)u_{tx}(t+h)\Delta_h u_{tx}\,\mathrm{d}x\,\mathrm{d}t =$$
$$-\int_0^1 \frac{1}{2}[\psi'(u_x)(\Delta_h u_{tx})^2](T)\,\mathrm{d}x - \int_0^1 \frac{1}{2}[\psi'(u_x)(\Delta_h u_{tx})^2](0)\,\mathrm{d}x -$$
$$\int_0^T\int_0^1 \frac{1}{2}\psi''(u_x)u_{xt}[\Delta_h u_{tx}]^2 + \Delta_h\psi'(u_x)u_{tx}(t+h)\Delta_h u_{tx}\,\mathrm{d}x\,\mathrm{d}t$$

After dividing by h^2 and taking the limit for $h\to 0$ we obtain L_1 equal to

$$-\int_0^1 \frac{1}{2}\psi'(u_x(T))u_{ttx}^2(T) + \int_0^1 \frac{1}{2}\psi'(u_0')u_{ttx}^2(0) - \int_0^T\int_0^1 \frac{1}{2}\psi''(u_x)u_{xt}u_{ttx}^2 + [\psi'(u_x)]_t u_{tx}u_{ttx}\,\mathrm{d}x\,\mathrm{d}t. \quad (17)$$

The second term in (16) gives

$$I_2 = \int_0^T\int_0^1 \Delta_h[\varphi_x(t,x,u_x) + \varphi'(t,x,u_x)u_{xx}]\Delta_h[\psi'(u_x)u_{xx}]_t\,\mathrm{d}x\,\mathrm{d}t =$$
$$\int_0^T\int_0^1 [\Delta_h\varphi_x(t,x,u_x) + \varphi'(t,x,u_x)\Delta_h u_{xx} + \Delta_h\varphi'(t,x,u_x)u_{xx}(t+h)]\Delta_h[\psi'(u_x)u_{xx}]_t\,\mathrm{d}x\,\mathrm{d}t \quad (18)$$

The most problematic term in (18) gives

$$\int_0^T\int_0^1 \varphi'(t,x,u_x)\Delta_h u_{xx}\Delta_h[\psi'(u_x)u_{xx}]_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_0^1 \varphi'(t,x,u_x)\Delta_h u_{xx}\psi'(u_x)\Delta_h u_{xxt} +$$
$$\varphi'(t,x,u_x)\Delta_h u_{xx}\Delta_h[\psi''(u_x)u_{xx}^2] + \varphi'(t,x,u_x)\Delta_h u_{xx}\Delta_h\psi'(u_x)u_{xxt}(t+h)\,\mathrm{d}x\,\mathrm{d}t \quad (19)$$

Here the first term on the right-hand side is

$$\int_0^T \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)(\Delta_h u_{xx})^2]_t - \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)]_t(\Delta_h u_{xx})^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)(\Delta_h u_{xx})^2](T)\,\mathrm{d}x - \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)(\Delta_h u_{xx})^2](0)\,\mathrm{d}x - \int_0^T \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)]_t(\Delta_h u_{xx})^2\,\mathrm{d}x\,\mathrm{d}t$$

After taking the limit

$$\int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)u_{txx}^2](T)\,\mathrm{d}x - \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)u_{txx}^2](0)\,\mathrm{d}x - \int_0^T \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)]_t u_{txx})^2\,\mathrm{d}x\,\mathrm{d}t \quad (20)$$

The remaining terms on the right-hand side in (19) give

$$\int_0^T \int_0^1 \varphi'(t,x,u_x)\Delta_h u_{xx}\Delta_h[\psi''(u_x)u_{xx}^2] + \varphi'(t,x,u_x)\Delta_h u_{xx}\Delta_h\psi'(u_x)u_{xxt}(t+h)\,\mathrm{d}x\,\mathrm{d}t$$

and the limit is

$$\int_0^T \int_0^1 \varphi'(t,x,u_x)u_{txx}[\psi''(u_x)u_{xx}^2]_t + \varphi'(t,x,u_x)u_{txx}\psi''(u_x)u_{tx}u_{txx}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_0^1 \varphi'(t,x,u_x)u_{txx}[\psi'''(u_x)u_{tx}u_{xx}^2 + \psi''(u_x)2u_{xx}u_{txx}] + \varphi'(t,x,u_x)u_{txx}\psi''(u_x)u_{tx}u_{txx}\,\mathrm{d}x\,\mathrm{d}t \quad (21)$$

Taking the limit in the remaining terms of (18) we obtain

$$\int_0^T \int_0^1 [\Delta_h\varphi_x(t,x,u_x) + \Delta_h\varphi'(t,x,u_x)u_{xx}(t+h)] \cdot \Delta_h[\psi'(u_x)u_{xxt}]\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_0^1 \Delta_h[\Delta_h\varphi_x(t,x,u_x) + \Delta_h\varphi'(t,x,u_x)u_{xx}(t+h)]\psi'(u_x)u_{xxt}\,\mathrm{d}x\,\mathrm{d}t + \left(-\int_0^h + \int_T^{T+h}\right)\int_0^1 [\Delta_h\varphi_x(t,x,u_x) + \Delta_h\varphi'(t,x,u_x)u_{xx}(t+h)]\psi'(u_x)u_{xxt}\,\mathrm{d}x\,\mathrm{d}t.$$

Taking the limit we have

$$-\int_0^T \int_0^1 [[\varphi_x(t,x,u_x)]_t + [\varphi'(t,x,u_x)]_t u_{xx}]_t\psi'(u_x)u_{xxt}\,\mathrm{d}x\,\mathrm{d}t - \int_0^1 [(\varphi_x(t,x,u_x))_t + (\varphi'(t,x,u_x))_t u_{xx}]\psi'(u_x)u_{xxt}(0)\,\mathrm{d}x + \int_0^1 [(\varphi_x(t,x,u_x))_t + (\varphi'(t,x,u_x))_t u_{xx}]\psi'(u_x)u_{xxt}(T)\,\mathrm{d}x \quad (22)$$

Hence,

$$L_2 = (20) + (21) + (22).$$

The third term in (16) is $I_3 = Q(a, T, \Delta_h \psi(u_x)_{xt})$, taking limsup we have

$$\limsup_{h\to 0} \frac{1}{h^2} Q(a, T, \Delta_h \psi(u_x)_{xt}). \tag{23}$$

The fourth term in (16) yields

$$I_4 = \int_0^T \int_0^1 \Delta_h f \Delta_h[\psi'(u_x) u_{xxt}] \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_0^1 \Delta_h[\Delta_h f] \psi'(u_x) u_{xxt} \,\mathrm{d}x\,\mathrm{d}t +$$
$$\left(-\int_0^h + \int_T^{T+h}\right) \int_0^1 \Delta_h f \psi'(u_x) u_{xxt} \,\mathrm{d}x\,\mathrm{d}t$$

and the limit is

$$L_4 = \int_0^T \int_0^1 f_{tt} \psi'(u_x) u_{xxt} \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 f_t \psi'(u_x) u_{xxt}(T) \,\mathrm{d}x\,\mathrm{d}t - \int_0^1 f_t \psi'(u_x) u_{xxt}(0) \,\mathrm{d}x\,\mathrm{d}t. \tag{24}$$

In the fifth and sixth term in (16) we need to move Δ_h from $\psi'(u_x)_{xt}$ to the integral term

$$\int_0^h a(t+h, x, t+h-s) \psi(u_x)_{xt}(s) \,\mathrm{d}s$$

resp.

$$\int_0^t [a(t+h, x, t-s) - a(t, x, t-s)] \psi(u_x)_{xt}(s+h) \,\mathrm{d}s$$

In the fifth term we obtain (using Lemma 6.3)

$$I_5 = \int_0^T \int_0^1 \int_0^h \Delta_h a(t+h, x, t+h-s) \psi(u_x)_{xt}(s) \,\mathrm{d}s (\psi'(u_x) u_{xx})_t \,\mathrm{d}x\,\mathrm{d}t +$$
$$\left(-\int_0^h + \int_T^{T+h}\right) \int_0^1 \int_0^h a(t+h, x, t+h-s) \psi(u_x)_{xt}(s) \,\mathrm{d}s (\psi'(u_x) u_{xx})_t \,\mathrm{d}x\,\mathrm{d}t$$

Taking the limit we have

$$L_5 = \int_0^T \int_0^1 (a_t + a_s)(t, x, t) \psi(u_x)_{xt}(0) (\psi'(u_x) u_{xx})_t \,\mathrm{d}x\,\mathrm{d}t +$$
$$\int_0^1 a(T, x, T) \psi(u_x)_{xt}(0) (\psi'(u_x) u_{xx})_t(T) \,\mathrm{d}x - \int_0^1 a(0, x, 0) \psi(u_x)_{xt}(0) (\psi'(u_x) u_{xx})_t(0) \,\mathrm{d}x. \tag{25}$$

The sixth term gives (with help of Lemma 6.3)

$$I_6 = \int_0^T \int_0^1 \Delta_h \left[\int_0^t [a(t+h, x, t-s) - a(t, x, t-s)] \psi(u_x)_{xt}(s+h) \,\mathrm{d}s\right] (\psi'(u_x) u_{xx})_t \,\mathrm{d}x\,\mathrm{d}t +$$
$$\left(-\int_0^h + \int_T^{T+h}\right) \int_0^t [a(t+h, x, t-s) - a(t, x, t-s)] \psi(u_x)_{xt}(s+h) \,\mathrm{d}s (\psi'(u_x) u_{xx})_t \,\mathrm{d}x\,\mathrm{d}t$$

and taking the limit we have

$$L_6 = \int_0^T \int_0^1 \left[\int_0^t a_t(t,x,t-s)\psi(u_x)_{xt}(s)\,\mathrm{d}s\right]_t (\psi'(u_x)u_{xx})_t\,\mathrm{d}x\,\mathrm{d}t + \\ \int_0^1 \int_0^T a_t(T,x,T-s)\psi(u_x)_{xt}(s)\,\mathrm{d}s(\psi'(u_x)u_{xx})_t(T)\,\mathrm{d}x \quad (26)$$

Since the limit exists in all terms except the one with Q, the limsup in (23) must be in fact a limit and putting (17), (20), (21), (22), (23), (24), (25), (26) together, we obtain

$$\int_0^1 \frac{1}{2}\psi'(u_x(T))u_{ttx}^2(T) + \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)u_{txx}^2](T)\,\mathrm{d}x + \lim_{h\to 0}\frac{1}{h^2}Q(a,T,\Delta_h\psi(u_x)_{xt})$$
$$= \int_0^1 \frac{1}{2}\psi'(u_0')u_{ttx}^2(0) + \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)u_{txx}^2](0)\,\mathrm{d}x$$
$$+ \int_0^1 [(\varphi_x(t,x,u_x))_t + (\varphi'(t,x,u_x))_t u_{xx}](\psi'(u_x)u_{xx})_t(0)\,\mathrm{d}x$$
$$+ \int_0^1 f_t(\psi'(u_x)u_{xx})_t(0)\,\mathrm{d}x$$
$$- \int_0^1 [(\varphi_x(t,x,u_x))_t + (\varphi'(t,x,u_x))_t u_{xx}](\psi'(u_x)u_{xx})_t(T)\,\mathrm{d}x$$
$$- \int_0^1 f_t(\psi'(u_x)u_{xx})_t(T)\,\mathrm{d}x + \int_0^T \int_0^1 \frac{1}{2}[\varphi'(t,x,u_x)\psi'(u_x)]_t u_{txx}^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^T \int_0^1 \varphi'(t,x,u_x)u_{txx}[\psi'''(u_x)u_{tx}u_{xx}^2 + \psi''(u_x)2u_{xx}u_{txx}] + \varphi'(t,x,u_x)u_{txx}\psi''(u_x)u_{tx}u_{txx}\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^T \int_0^1 \frac{1}{2}\psi''(u_x)u_{xt}u_{ttx}^2 + [\psi'(u_x)]_t u_{tx}u_{ttx}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_0^1 [[\varphi_x(t,x,u_x)]_t + [\varphi'(t,x,u_x)]_t u_{xx}]_t(\psi'(u_x)u_{xx})_t\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^T \int_0^1 f_{tt}(\psi'(u_x)u_{xx})_t\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_0^1 (a_t + a_s)(t,x,t)\psi(u_x)_{xt}(0)(\psi'(u_x)u_{xx})_t\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^1 a(T,x,T)\psi(u_x)_{xt}(0)(\psi'(u_x)u_{xx})_t(T)\,\mathrm{d}x + \int_0^1 a(0,x,0)\psi(u_x)_{xt}(0)(\psi'(u_x)u_{xx})_t(0)\,\mathrm{d}x$$
$$- \int_0^T \int_0^1 \left[\int_0^t a_t(t,x,t-s)\psi(u_x)_{xt}(s)\,\mathrm{d}s\right]_t (\psi'(u_x)u_{xx})_t\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^1 \int_0^T a_t(T,x,T-s)\psi(u_x)_{xt}(s)\,\mathrm{d}s(\psi'(u_x)u_{xx})_t(T)\,\mathrm{d}x$$

Here the left-hand side larger or equal to

$$\frac{1}{2}c_\psi\|u_{ttx}(T)\|_2^2 + \frac{1}{2}c_\psi^2\|u_{txx}(T)\|_2^2 + \lim_{h\to 0}\frac{1}{h^2}Q(a,T,\Delta_h\psi(u_x)_{xt})$$

and the terms on the right-hand side are by Hölder inequality less or equal to

$$\frac{1}{2}Z(U_0) + \frac{1}{2}C_\psi Z(U_0) + Z(C_\psi)Z(U_0) + F_0 Z(U_0)$$
$$+ (\varepsilon_\varphi + \varepsilon_\varphi \nu(T) + \varepsilon_\varphi \nu(T) + C_\psi \nu(T)^2)\|\psi(u_x)_{xt}(T)\|_2$$
$$+ F\|\psi(u_x)_{xt}(T)\|_2 + \frac{1}{2}(\varepsilon_\varphi C_\psi + C_\psi^2 \nu(T) + C_\psi^2 \nu(T))\|u_{txx}\|_2^2$$
$$+ C_\psi^2(\nu(T)^2 + 3\nu(T))(\|u_{txx}\|_2^2 + \|u_{xx}\|_2^2)$$
$$+ \frac{1}{2}C_\psi \nu(T)\|u_{ttx}\|_2^2 + C_\psi \nu(T)\|u_{tx}\|_2\|u_{ttx}\|_2$$
$$+\Big(\varepsilon_\varphi + \sqrt{E(T)}(2\varepsilon_\varphi + \varepsilon_\varphi + \varepsilon_\varphi + \varepsilon_\varphi + \nu(T)(\varepsilon_\varphi + 2\varepsilon_\varphi + 2C_\psi) + C_\psi \nu^2(T)\Big) \|\psi(u_x)_{xt}\|_2$$
$$+ F\|\psi(u_x)_{xt}\|_2 + \|(a_t + a_s)(t,x,t)\|_2 Z(U_0)\|\psi(u_x)_{xt}\|_2$$
$$+ \|a(T,\cdot,T)\|_2 C_\psi U_0 \|\psi(u_x)_{xt}(T)\|_2$$
$$+ \|a(0,\cdot,0)\|_2 Z(U_0)$$
$$+ \|\psi(u_x)_{xt}\|_2^2(\|a_t(\cdot,\cdot,0)\|_\infty + \|a_{tt} + a_{st}\|_{L^1_{t,s}L^\infty_x})$$
$$+ \|\psi(u_x)_{xt}\|_2\|\psi(u_x)_{xt}(T)\|_2\|a_t(T,\cdot,T-\cdot)\|_{L^1_s L^\infty_x}$$

Since $|\psi(u_x)_{xt}|_2^2$, $|\psi(u_x)_{xt}(T)|_2^2 \le E(T)$ and $\nu(T) \le \sqrt{E(T)}$ and by Young inequality we have

$$\frac{1}{2}c_\psi\|u_{ttx}(T)\|_2^2 + \frac{1}{2}c_\psi^2\|u_{txx}(T)\|_2^2 + \lim_{h\to 0}\frac{1}{h^2}Q(a,T,\Delta_h\psi(u_x)_{xt}) \le$$
$$CZ(U_0) + CZ(F) + \sqrt{E(T)}(\varepsilon_\varphi + C\varepsilon_\varphi + \varepsilon_\varphi) +$$
$$E(T)(5\varepsilon + 2\varepsilon_\varphi + 2\varepsilon_\varphi + \varepsilon_\varphi + 2\varepsilon_\varphi + \|a_t(\cdot,\cdot,0)\|_\infty + \|a_{tt} + a_{st}\|_{L^1_{t,s}L^\infty_x} + \|a_t(T,\cdot,T-\cdot)\|_{L^1_s L^\infty_x}) +$$
$$CE(T)^{3/2} + CE(T)^2$$

Applying Young inequality to the third term on the right-hand side and using the definition of ε_a and ε_φ we obtain (15). □

Lemma 4.5 *Let (A1) – (A6) hold and* $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T_1), H^k(\Omega))$ *is a solution to (IDE), (Cu). Then*

$$\|u_{ttx}(T)\|_2^2 + \|u_{txx}(T)\|_2^2 + \|u_{tx}(T)\|_2^2 + \|u_{xx}(T)\|_2^2 + \|u_{txx}\|_2^2 \le$$
$$CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon + \varepsilon_\varphi + \varepsilon_a) + CE(T)^{3/2} + CE(T)^2 \quad (27)$$

Proof. This estimate follows immediately summing the estimates (14) and (15) and applying Lemma 6.1. □

Lemma 4.6 *Let (A1) – (A6) hold and* $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T_1), H^k(\Omega))$ *is a solution to (IDE), (Cu). Then*

$$\|u_{tt}(T)\|_2^2 + \|u_{ttt}(T)\|_2^2 + \|u_{ttt}\|_2^2 \le$$
$$CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon + \varepsilon_\varphi + \varepsilon_a) + CE(T)^{3/2} + CE(T)^2$$

Proof. Taking L^2-norms in (IDE) we have

$$\|u_{tt}(t)\|_2^2 \le C\|u_{xx}(t)\|_2^2 + \int_0^t \|a_s(t,\cdot,t-s)\|_2 \|\psi'(u_x(s))u_{xx}\|_{2,x} ds + \|g(t)\|_2^2,$$

hence,

$$\|u_{tt}(t)\|_2^2 \le C\|u_{xx}(t)\|_2^2 + \|a_s(t,\cdot,t-\cdot)\|_{L_s^1 L_x^2} \max_s \|u_{xx}(s)\|_2 + Z(F) + Z(U_0), \quad (28)$$

Differentiating (IDE) with respect to t we obtain

$$u_{ttt} = (\chi(t,x,u_x)u_{xx})_t + a_s(t,x,0)\psi(u_x)_x + \int_0^t (a_{tt} + a_{ts})(t,x,t-s)\psi(u_x(s))_x ds + g_t. \quad (29)$$

Taking L^2-norm we have

$$\|u_{ttt}(t)\|_2^2 \le \|\chi_t(t,\cdot,u_x)\|_\infty \|u_{xx}(t)\|_2^2 + \|\chi'(t)\|_\infty \nu(t)^2 \|u_{xx}(t)\|_2^2 + \|\chi\|_\infty \|u_{xxt}(t)\|_2^2 + \|a_s(t,\cdot,0)\|_\infty \|u_{xx}(t)\|_2^2 + \|(a_{tt}+a_{st})(t,x,t-s)\|_{L_x^\infty L_s^1} \max_s \|u_{xx}(s)\|_2^2 + \|g_t(t)\|_2^2. \quad (30)$$

If we integrate the squared equation (29) over $[0,T]$ we obtain

$$\|u_{ttt}\|_2^2 \le \|\chi_t\|_{L_t^1 L_x^\infty} \max_s \|u_{xx}(s)\|_2^2 + \|\chi'\|_{L_t^1 L_x^\infty} \max_s \nu(s)^2 \|u_{xx}(s)\|_2^2 + \|\chi\|_\infty \|u_{xxt}\|_2^2 + \|a_s(t,\cdot,0)\|_{\infty,1} \max_t \|u_{xx}(t)\|_2^2 + \|(a_{tt}+a_{ts})a(t,x,t-s)\|_{L_x^\infty L_{st}^1} \max_s \|u_{xx}(s)\|_2^2 + Z(U_0) + Z(F). \quad (31)$$

Since all the terms on the right-hand sides in (28), (30), (31) are estimated in previous lemmas, the assertion follows. □

Lemma 4.7 *Let (A1) – (A6) hold and* $u \in \bigcap_{k=0}^3 C^{3-k}([0,T_1), H^k(\Omega))$ *is a solution to (IDE), (Cu). Then*

$$\|u_{ttx}(T)\|_2^2 \le CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon + \varepsilon_\varphi + \varepsilon_a) + CE(T)^{3/2} + CE(T)^2$$

Proof. Using difference operators one can derive the following "integration by parts formula"

$$\int_0^T \int_0^1 u_{ttx}^2 = \int_0^T \int_0^1 u_{ttt}u_{txx} + \int_0^1 u_{txx}u_{tt}(0) - \int_0^1 u_{txx}u_{tt}(T).$$

The assertion then easily follows using Young inequality and previous estimates. □

Lemma 4.8 *Let (A1) – (A6) hold and* $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T_1), H^k(\Omega))$ *is a solution to (IDE), (Cu). Then*

$$\|u_{xxx}(T)\|_2^2 + \|u_{xxx}\|_2^2 + \|u_{xx}\|_2^2 \leq$$
$$CZ(U_0) + CZ(F) + C(\varepsilon)\varepsilon_\varphi^2 + CE(T)(\varepsilon + \varepsilon_\varphi + \varepsilon_\chi + \varepsilon_a) + CE(T)^{3/2} + CE(T)^2 \tag{32}$$

Proof. Rewrite (IDE) in the form

$$\chi(0,0,0)u_{xx} + \int_0^t a_s(0,0,t-s)\psi'(0)u_{xx}(s)\,\mathrm{d}s = G(t,x), \tag{33}$$

where

$$G = u_{tt} - g - (\chi(t,x,u_x) - \chi(0,0,0))u_{xx} - \int_0^t [a_s(t,x,t-s)\psi'(u_x) - a_s(0,0,t-s)\psi'(0)]u_{xx}(s)\,\mathrm{d}s.$$

By Lemma 3.2 in [4] there exists a resolvent kernel $k \in L^1(0,+\infty)$ for (33) and

$$\chi(0,0,0)u_{xx}(t,x) = G(t,x) + \int_0^t k(t-s)G(s,x)\,\mathrm{d}s. \tag{34}$$

Differentiating with respect to x yields

$$\chi(0,0,0)u_{xxx}(t) = u_{ttx} - g_x - (\chi(t,x,u_x) - \chi(0,0,0))u_{xxx} -$$
$$\chi_x(t,x,u_x)u_{xx} - \chi'(t,x,u_x)u_{xx}^2 - \int_0^t [a_s(t,x,t-s)\psi'(u_x) - a_s(0,0,t-s)\psi'(0)]u_{xxx}(s)\,\mathrm{d}s -$$
$$\int_0^t [a_{xs}(t,x,t-s)\psi'(u_x) + a_s(t,x,t-s)\psi''(u_x)u_{xx}]u_{xx}(s)\,\mathrm{d}s \tag{35}$$

Squaring this equation and integrating over Ω yields

$$\chi^2(0,0,0)\|u_{xxx}(t)\|_2^2 \leq C(\|u_{ttx}(t)\|_2^2 + \|g_x(t)\|_2^2 + |(\chi(t,x,u_x) - \chi(0,0,0))|_\infty^2 \|u_{xxx}(t)\|_2^2 +$$
$$\varepsilon_\chi E(t) + \nu(t)^2 E(t) + \int_0^t \|a_s(t,\cdot,t-s)\psi'(u_x(s,\cdot)) - a_s(0,0,t-s)\psi'(0)\|_\infty^2\,\mathrm{d}s \max_s \|u_{xxx}(s)\|_2^2 +$$
$$\int_0^t \|a_{sx}(t,x,t-s)\psi'(u_x) + a_s(t,x,t-s)\psi''(u_x)u_{xx}\|_\infty^2\,\mathrm{d}s \max_s \|u_{xx}(s)\|_2^2.$$

Hence, since

$$|(\chi(t,x,u_x) - \chi(0,0,0))|_\infty^2 \text{ and } \int_0^t \|[a_s(t,\cdot,t-s)\psi'(u_x(s,\cdot)) - a_s(0,0,t-s)\psi'(0)]\|_\infty^2\,\mathrm{d}s$$

are bounded by ε_χ and ε_a, we have

$$\chi^2(0,0,0)\|u_{xxx}(t)\|_2^2 \leq$$
$$C(\|u_{ttx}(t)\|_2^2 + \|g_x(t)\|_2^2 + (\varepsilon_\chi + \varepsilon_a)E(t) + \nu(t)^2 E(t) + \max_s \|u_{xx}(s)\|_2^2).$$

Integration of (35) over $[0,T]$ yields

$$\chi^2(0,0,0)\|u_{xxx}\|_2^2 \le C(\|u_{ttx}\|_2^2 + \|g_x\|_2^2 + \\ \|(\chi(\cdot,\cdot,u_x)-\chi(0,0,0))\|^2_{L^2_t L^\infty_x} \max_s \|u_{xxx}(s)\|_2^2 + \varepsilon_\chi E(t) + \nu(t)^2 E(t) + \\ \int_0^T \int_0^t \|[a_s(t,\cdot,t-s)\psi'(u_x(s,\cdot)) - a_s(0,0,t-s)\psi'(0)]\|_\infty^2 \,\mathrm{d}s\,\mathrm{d}t \max_s \|u_{xxx}(s)\|_2^2 + \\ \int_0^T \int_0^t \|a_{sx}(t,x,t-s)\psi'(u_x) + a_s(t,x,t-s)\psi''(u_x)u_{xx}\|_\infty^2 \,\mathrm{d}s\,\mathrm{d}t \max_s \|u_{xx}(s)\|_2^2,$$

hence,

$$\chi^2(0,0,0)\|u_{xxx}\|_2^2 \le C(\|u_{ttx}\|_2^2 + \|g_x\|_2^2 + \varepsilon_\chi E(t) + \nu(t)^2 E(t) + \sup_s \|u_{xx}(s)\|_2^2).$$

Finally, squaring and integrating (34) we obtain

$$\chi^2(0,0,0)\|u_{xx}\|_2^2 \le C(\|u_{tt}\|_2^2 + \|g\|_2^2 + \|(\chi(\cdot,\cdot,u_x)-\chi(0,0,0))\|^2_{L^2_t L^\infty_x} \max_s \|u_{xx}(s)\|_2^2 + \\ \int_0^T \int_0^t \|[a_s(t,\cdot,t-s)\psi'(u_x(s,\cdot)) - a_s(0,0,t-s)\psi'(0)]\|_\infty^2 \,\mathrm{d}s\,\mathrm{d}t \max_s \|u_{xx}(s)\|_2^2,$$

The assertion follows from this estimate and the estimates proved in the previous lemmas. □

Remark 4.9 *The derivatives of u of lower order can be easily estimated by the derivatives of higher order and the Poincaré inequality.*

5 Proofs of the main results

Let us mention that if χ, a, g and u_0 satisfy (A1) – (A6), then also φ satisfies the regularity condition in (A1) and f satisfies (A4). Further, it is sufficient to assume that $\chi(t,x,0) > 0$ and $\chi(t,x,0) - a(t,x,0)\psi'(0) > 0$ and we obtain the lower bound for χ, φ, ψ in (A1) if the set B is small enough. Moreover, if $U_0(u_0,u_1)$ is small and $F(f)$ is small, then $F(g)$ is also small, i.e. $F(g) \le Z(F(f), U_0)$, Z continuous and $Z(0,0) = 0$. We can also estimate

$$\varepsilon_\varphi \le \varepsilon_\chi + C_\psi \max_{t\in J, x\in\Omega} \{a_t, a_x, a_{tx}, a_{tt}, a_{ttx}\}(t,x,0).$$

If a is of the form $a(t,x,t-s) = k(c(t,x), t-s)$, then we have

$$\varepsilon_a \le Z(\varepsilon_c),$$

where Z is continuous with $Z(0) = 0$ (depending only on $\|k\|_{C^2(U\times J)}$) and

$$\varepsilon_c := \max_{t,x}\{c_t, c_x, c_{tx}, c_{tt}, c_{ttx}\}(t,x) + \int_J |\tilde{c}_{tt}(t)| + |\tilde{c}_t(t)| \,\mathrm{d}t$$

with $\tilde{c}_t(t) := \max_x c(t,x)$. Further, if ε_c is small or k', k'', k''' are small, then ε_φ is small, provided ε_χ is small and (A2') holds.

Proof.[OF THEOREM 2.2] The second equation of (1) has a solution $c \in C_b^4(R_+ \times \Omega)$ with c_t, $c_{tt} \in L^1(J, L^\infty(\Omega))$. Then (A2) is satisfied and by the considerations above. The assumption (A1) holds if B is sufficiently small and assumptions (A3), (A6) follow from (A3'), (A6') respectively.

Moreover, if $C_0(c_0)$ is small, then ε_c is small and therefore ε_a is small and also ε_φ is small (since ε_χ is small). Therefore, the assumptions of Theorem 4.2 are satisfied and the assertion of Theorem 2.2 follows from Theorem 4.2. □

Proof.[of Theorem 2.1] To show that Theorem 2.1 holds without smallness assumptions on the kernel a, we first observe (from the definition of ε_a) that J can be covered by finitely many half-open subintervals $J_1, \ldots, J_r$ that overlap a little and such that $\varepsilon_a < 1/C$ on each of them.

We would like to solve the equation on each of the intervals J_i separately and glue the solutions together. Let u^i be a solution on $J^i := \bigcup_{j=1}^{i-1} J_j$ and $J_i = [T_i, S_i)$ with $T_i \in J^i$. Since $u^i(T_i)$ satisfies (A3) and (A5), it can be taken as a new initial value. Let us reformulate the equation (IDE) for $\tilde{u}(t,x) := u(t+T_i, x)$. We obtain

$$\tilde{u}_{tt} = \tilde{\chi}(t,x,\tilde{u}_x)\tilde{u}_{xx} + \int_0^t \tilde{a}_s(t,x,t-s)\psi(\tilde{u}_x(s))_x ds + \tilde{g}, \qquad \text{(IDEi)}$$

where $\tilde{\chi}(t,x,z) := \chi(t+T_i,x,z)$, $\tilde{a}(t,x,s) := a(t+T_i,x,s)$ and

$$\tilde{g}(t,x) := g(t+T_i) + \int_0^{T_i} a(t,x,t-s)\psi(u_x(s,x))_x \,\mathrm{d}s.$$

Since the new data $\tilde{\chi}$, $\tilde{a}$, $\tilde{g}$ satisfy (A1) – (A6) and $\varepsilon_{\tilde{a}} < 1/C$, there exists a solution to (IDEi) on J^i provided

$$U(\tilde{u}_0, \tilde{u}_1) + F(\tilde{g}) + \varepsilon_{\tilde{\varphi}} + \varepsilon_{\tilde{\chi}} \qquad (36)$$

is sufficiently small. Continuing the solution u^i by $\tilde{u}$ we obtain a solution on $\bigcup_{j=1}^{i} J_j$ and by induction we obtain a solution on J. It remains to show that we can guarantee that (36) is small enough in each step.

There is no problem with $\varepsilon_{\tilde{\varphi}}$, $\varepsilon_{\tilde{\chi}}$. We can simply assume these quantities to be small on the whole J. However, $\tilde{u}_0$, $\tilde{u}_1$ and $\tilde{g}$ depend on the solution on the previous interval, so we have to be more careful. We will start from the last interval J_r and go through the same scheme as in the proof of Theorem 4.2 and proceed to J_{r-1}, J_{r-2}, $\ldots$, J_1.

We will denote the data on J_r by g^r, u_0^r, u_1^r. We remind that $u_0^r = u^{r-1}(T_r)$, $u_1^r = u_t^{r-1}(T_r)$ and

$$g^r(t,x) := g(t+T_r) + \int_0^{T_r} a(t,x,t-s)\psi(u_x^{r-1}(s,x))_x \,\mathrm{d}s.$$

To obtain a solution on J^r we want g^r, u_0^r, u_1^r to be small enough. We will show that

$$U_0(u_0^r, u_1^r) + F(g^r) + \varepsilon_\varphi + \varepsilon_\chi \le \mu^r, \quad E(T_r) \le \gamma^r \qquad (37)$$

follows from

$$U_0(u_0^{r-1}, u_1^{r-1}) + F(g^{r-1}) + \varepsilon_\varphi + \varepsilon_\chi \le \mu^{r-1}, \qquad E(T_{r-1}) \le \gamma^{r-1} \tag{38}$$

for appropriate μ^{r-1}, γ^{r-1} (provided ε_φ, ε_χ are small enough).

Let us go to J_{r-1}. Take γ such that $\gamma < \gamma^r$, $\gamma < \mu^r/4$ and $\overline{F^r} < \mu^r/4$, where

$$\overline{F^r} := \sup\{F(g^r) : \ u \text{ is such that } E(T_r) < \gamma\}.$$

Further, we take γ so small that $\frac{C}{\delta}\gamma^{3/2} < \gamma/4$ and $\frac{C}{\delta}\gamma^2 < \gamma/4$. Now, take μ^{r-1} such that

$$\frac{C}{\delta} Z(\mu^{r-1}) + \frac{C}{\delta} Z(\mu^{r-1})) + \frac{C(\varepsilon)}{\delta}\varepsilon_\varphi^2 < \gamma/4 \tag{39}$$

and set $\gamma^{r-1} := \gamma/4$.

Let u be a solution on $[0, T_{r-1}]$ with $E(T_{r-1}) \le \gamma^{r-1}$. Like in the proof of Theorem 4.2, (38), (39) and the other conditions on γ yield $E(t) < \frac{3}{4}\gamma$ for the solution u_r on $[0, T_r - T_{r-1}]$. If u^r is the continuation of u by u_r, then we have $E(T_r) \le E(T_{r-1}) + \frac{3}{4}\gamma \le \gamma \le \gamma^r$ and we have the second estimate in (37). Further we have

$$U_0(u_0^r, u_1^r) + F(g^r) \le E(T_r) + \overline{F^r} \le \frac{\mu^r}{4} + \frac{\mu^r}{4}$$

and we have the first estimate in (37).

Applying the implication (38) $\Rightarrow$ (37) inductively, we obtain μ^1, γ^1. Since $E(T_1) = E(0) \le Z(F, U_0)$, we obtain that existence of a solution on J is gueranteed by (8) and Theorem 2.1 is proved. □

6 Appendix

Lemma 6.1 *Let $k : J \times \Omega \times J \to R$ be of strong positive type. Then there exists $c > 0$ such that for all $T > 0$ and $w \in C([0,T], L^2(I))$ the following inequality holds*

$$\int_0^t \|w(s)\|_2^2 ds \le c\left(\|w(0)\|_2^2 + Q(k,t,w) + \liminf_{h\searrow 0} \frac{1}{h^2} Q(k,t,\Delta_h w)\right) \tag{40}$$

for all $t \in [0, T)$.

Proof. Lemma 2.5 in [5] gives the result for convolution kernels independent of x. In particular (40) holds if we replace k by $e(t,x,t-s) := e^{t-s}$. Since k is of strong positive type, we have $Q(k,t,w) \ge c_1 Q(e,t,w)$ and $Q(k,t,\Delta_h w) \ge c_1 Q(e,t,\Delta_h w)$ and the proof is complete. □

We formulate three lemmas for working with difference operators. Their proofs are easy.

Lemma 6.2 *Let f, $w \in C(J)$. Then for every $T \in (0, T_{max})$ and h small enough it holds that*

$$\int_0^T f(t)\Delta_h w(t)\, dt = -\int_0^T \Delta_h f(t) w(t+h)\, dt + \left(\int_T^{T+h} - \int_0^h\right) f(t)w(t)\, dt.$$

Lemma 6.3 *Let $w \in C(J)$, $a \in C(J \times J)$. Then for every $T \in (0, T_{max})$ and h small enough it holds that*

$$\int_0^T \int_0^h a(t+h, t+h-s)w(s)\, ds \Delta_h w(t)\, dt = -\int_0^T \int_0^h \Delta_h a(t+h, t+h-s)w(s)\, ds w(t+h)\, dt + \left(\int_T^{T+h} - \int_0^h\right) \int_0^h a(t+h, t+h-s)w(s)\, ds w(t)\, dt.$$

Lemma 6.4 *Let $a \in C^1(J \times J)$, $w \in C(J)$. Then*

$$\Delta_h \int_0^t a(t, t-s)w(s)\, ds = \int_0^h a(t+h, t+h-s)w(s) + \int_0^t [a(t+h, t-s) - a(t, t-s)]w(s+h)\, ds + \int_0^t a(t, t-s)\Delta_h w(s) ds.$$

7 Acknowledgements

This work is part of the project MSM 0021620839 and is partially supported by GACR 201/09/0917.

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