JOURNAL **?** (**????**), 1–3
DOI 10.1515/JOURNAL-**????**-**???**

Optimal spatial error estimates for DG time discretizations

Miloslav Vlasák

Abstract. In this paper a general parabolic problem is considered and discretized by discontinuous Galerkin (DG) method in time and generally in space. Optimal a priori error estimates in space as well as in time are derived and applied to the heat equation and to the nonlinear convection–diffusion equation.

Keywords. parabolic problems, time discontinuous Galerkin, a priori error estimates.

2010 Mathematics Subject Classification. 65M15, 65M60.

We are ready to present the main result.

Theorem 0.1. *Let $u \in W^{q+1,\infty}(0,T,X)$ be the exact solution of (**??**) and $U \in X_h^\tau$ be its discrete approximation given by (**??**). Then*

$$\max_{m=1,\dots,r} \sup_{I_m} \|U-u\| \leq C\Big(\sup_{(0,T)} \|R_h u - u\| + \sup_{(0,T)} \|R_h u' - u'\| + \tau^{q+1} + \|U_-^0 - u^0\|\Big), \tag{1}$$

where the constant C depends on u and T, but is independent of h and τ.

Bibliography

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[2] K. Chrysafinos and N. J. Walkington. Error estimates for discontinuous Galerkin approximations of implicit parabolic equations. *SIAM J. Numer. Anal.*, 43(6):2478–2499, 2006.

[3] K. Chrysafinos and N. J. Walkington. Error estimates for the discontinuous Galerkin methods for parabolic equations. *SIAM J. Numer. Anal.*, 44(1):349–366, 2006.

The author is a junior researcher of the University centre for mathematical modelling, applied analysis and computational mathematics (Math MAC)

[4] P. G. Ciarlet. *The finite element methods for elliptic problems. Repr., unabridged republ. of the orig. 1978.* Classics in Applied Mathematics. 40. Philadelphia, PA: SIAM. xxiv, 530 p. , 2002.

[5] V. Dolejší, M. Feistauer, and V. Sobotíková. Analysis of the discontinuous Galerkin method for nonlinear convection–diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 194:2709–2733, 2005.

[6] V. Dolejší, M. Feistauer, V. Kučera, and V. Sobotíková. An optimal $L^\infty(L^2)$-error estimate for the discontinuous Galerkin approximation of a nonlinear non-stationary convection-diffusion problem. *IMA J. Numer. Anal.*, 28(3):496–521, 2008.

[7] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. *Computational differential equations.* Cambridge: Cambridge Univ. Press. xvi, 538 p., 1996.

[8] M. Feistauer, J. Felcman, and I. Straškraba. *Mathematical and Computational Methods for Compressible Flow*. Oxford University Press, Oxford, 2003.

[9] M. Feistauer, J. Hájek, and K. Švadlenka. Space-time discontinuos Galerkin method for solving nonstationary convection-diffusion-reaction problems. *Appl. Math., Praha*, 52(3):197–233, 2007.

[10] M. Feistauer, V. Kučera, K. Najzar and J. Prokopová. Analysis of space-time discontinuous Galerkin method for nonlinear convection-diffusion problems. *Numer. Math.*, 117(2):251–288, 2011.

[11] J. Jaffre, C. Johnson, and A. Szepessy. Convergence of the discontinuous Galerkin finite element method for hyperbolic conservation laws. *Math. Models Methods Appl. Sci*, 5(3):367–286, 1995.

[12] P. Jamet. Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain. *SIAM J. Numer. Anal.*, 15:912–928, 1978.

[13] L. Kaland, H.-G. Roos. Parabolic singularly perturbed problems with exponential layers: robust discretizations using finite elements in space on Shishkin meshes. *Int. J. Numer. Anal. Model.*, 7(3):593–606, 2010.

[14] C. M. Klaij, J.J.W. van der Vegt, and H. Van der Ven. Pseudo-time stepping for space-time discontinuous Galerkin discretizations of the compressible Navier-Stokes equations. *J. Comput. Phys.*, 219(2):622–643, 2006.

[15] V. Kučera. On diffusion-uniform error estimates for the DG method applied to singularly perturbed problems. The Preprint Series of the School of Mathematics, preprint No. MATHknm- 2011/3 (2011), http://www.karlin.mff.cuni.cz/ms-preprints/prep.php.

[16] N. N. Lebedev. *Special functions and their applications. Rev. engl. ed. Translated and edited by R. A. Silverman.* Dover Publications, New York, 1972.

[17] F. Lörcher, G. Gassner, and C.-D. Munz. A discontinuous Galerkin scheme based on a spacetime expansion. I. Inviscid compressible flow in one space dimension. *J. Sci. Comput.*, 32(2):175–199, 2007.

[18] M. Luskin and R. Rannacher. On the smoothing property of the Galerkin method for parabolic equations. *SIAM J. Numer. Anal.*, 19:93–113, 1982.

[19] B. Rivière. *Discontinuous Galerkin methods for solving elliptic and parabolic equations. Theory and implementation.* Frontiers in Applied Mathematics 35. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). xxii, 190 p., 2008.

[20] D. Schötzau and Ch. Schwab. An hp a priori error analysis of the DG time-stepping for initial value problems. *Calcolo*, 37(4):207–232, 2000.

[21] D. Schötzoau. hp-DGFEM for parabolic evolution problems. application to diffusion and viscous incompressible flow. *Ph.D. thesis, ETH Zürich*, 1999.

[22] J.J. Sudirham, J.J.W. van der Vegt, and R.M.J. van Damme. Space-time discontinuous Galerkin method for advection-diffusion problems on time-dependent domains. *Applied Numerical Mathematics*, (to appear).

[23] V. Thomée. *Galerkin finite element methods for parabolic problems. 2nd revised and expanded ed.* Berlin: Springer. xii, 370 p., 2006.

[24] M. Vlasák, V. Dolejší, and J. Hájek. A priori error estimates of an extrapolated space-time discontinuous Galerkin method for nonlinear convection-diffusion problems. *Numer. Methods Partial Differ. Equations*, 27(6):1456–1482, 2011.

[25] T. Werder, K. Gerdes, D. Schötzau, and Ch. Schwab. hp-discontinuous Galerkin time stepping for parabolic problems. *Comput. Methods Appl. Mech. Eng.*, 190(49-50):6685–6708, 2001.

[26] Q. Zhang and Ch.-W. Shu. Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws. *SIAM J. Numer. Anal.*, 44(4):1703–1720, 2006.

Received **???**.

Author information

Miloslav Vlasák, Charles University in Prague, Faculty of Mathematics and Physics,
Department of Numerical Mathematics, Sokolovská 83, Praha 8, 186 75, Czech Republic.
E-mail: `vlasak@karlin.mff.cuni.cz`