Nečas Center for Mathematical Modeling

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Preprint no. 2011-006

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THEORY OF THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY PARABOLIC PROBLEMS WITH NONLINEAR CONVECTION AND DIFFUSION*

JAN ČESENEK † AND MILOSLAV FEISTAUER‡

Abstract. The paper presents the theory of the space-time discontinuous Galerkin finite element method for the discretization of a nonstationary convection-diffusion initial-boundary value problem with nonlinear convection and nonlinear diffusion. The discontinuous Galerkin method is applied separately in space and time using, in general, different space grids on different time levels and different polynomial degrees p and q in space and time dicretization. In the space discretization the nonsymmetric, symmetric and incomplete interior and boundary penalty (NIPG, SIPG, IIPG) approximations of diffusion terms is used. The paper is concerned with the analysis of error estimates in "$L^2(L^2)$"- and "DG"-norm formed by the "$L^2(H^1)$"-seminorm and penalty terms. An important ingredient used in the derivation of the error estimate is the concept of the discrete characteristic function and its properties. In the "DG"-norm the error estimates are optimal with respect to the size of the space grid. They are optimal with respect to the time step, if the Dirichlet boundary condition behaves in time as a polynomial of degree $\leq q$. In a general case, the derived estimates become suboptimal in time.

Key words. nonstationary nonlinear convection-diffusion equation, space-time discontinuous Galerkin finite element discretization, NIPG, SIPG and IIPG treatment of diffusion terms, error estimates

AMS subject classifications. 65M15, 65M60, 65M12

1. Introduction. During the last decade the discontinuous Galerkin finite element (DGFE) method, using piecewise polynomial discontinuous approximations, appeared as an efficient tool for the space discretization of a number of problems described by partial differential equations. This technique was used first in [43] for the solution of a neutron transport linear equation and analyzed theoretically in [42]. Since this time the DGFE method has been applied to a series of problems and the number of publications concerned with this technique is growing fast. We can mention here only some of them. The DGFE method has been used for the solution of elliptic and parabolic problems ([2], [3], [4], [44], [53]), transport-reaction problems ([13]), nonlinear conservation laws ([14], [39]), convection-diffusion linear or nonlinear problems ([9], [15], [24], [29], [33], [40]), compressible flow ([6], [7], [8], [18], [20], [22], [30], [32] [34], [52])), incompressible viscous flow ([48], [51]), porous media flow ([49]), shallow water flow ([16]), the Hamilton-Jacobi equations ([37]) and the Maxwell equations ([36]). In [17], [38], analysis of hp-version of the DGFE method is analyzed.

The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy both in space and in time. For some applications, the standard Euler schemes or θ-schemes ([21], [44]) are not sufficiently accurate in time. In computational fluid dynamics, Runge-Kutta methods are very popular. Let us mention, for example the well-known Runge-Kutta discontinuous Galerkin methods (see e.g. [15]). They are applicable to the numerical solution of a wide class of problems, including nonlinear conservation laws and nonlinear convection-diffusion problems, but they are conditionally stable. An example of unconditionally stable method is the technique using the backward difference formula (BDF). It was used for the solution of compressible flow, e.g. in [20] and analyzed theoretically in the case of a scalar nonlinear convection-diffusion equation in [25]. In the paper [5], a time discretization of arbitrary order of parabolic problems was proposed and analyzed. Unfortunately, it is applicable to linear problems only.

It appears suitable to use the discontinuous Galerkin discretization with respect to space as well as time for the construction of numerical schemes with high accuracy in space and time for the solution of nonlinear nonstationary problems. The discontinuous Galerkin time discretization was introduced and analyzed, e.g. in [26] for the solution of ordinary differential equations. In [1], [10], [11], [27], [28], [46]

*This work is supported by the research project MSM 0021620839 (M. Feistauer) and by the Nečas Center for Mathematical Modelling, project LC06052 (J. Česenek), both financed by the Ministry of Education of the Czech Republic. The research of M. Feistauer was also partly supported by the project No. 201/08/0012 of the Czech Science Foundation.

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and [47] the solution of parabolic problems is carried out with the aid of conforming finite elements in space combined with the DG time discretization. See also the monograph [50]. In [29], the space-time DGFE method was analyzed for a linear nonstationary convection-diffusion-reaction problem. The paper [31] is devoted to the theory of error estimates for the space-time DGFE method applied to a nonstationary convection-diffusion problem with a nonlinear convection and linear diffusion. An important tool in the derivation of error estimates was the time Gauss-Radau numerical integration and interpolation.

In the present paper we are concerned with the space-time discontinuous Galerkin discretization applied to the numerical solution of a nonstationary convection-diffusion problem with nonlinear convection and nonlinear diffusion. The time interval is split into subintervals and on each time level a different space mesh may be used in general. Moreover, the triangulations used for the space discretization may be nonconforming with hanging nodes. In the discontinuous Galerkin formulation we use the nonsymmetric, symmetric or incomplete version of the discretization of the diffusion terms and interior and boundary penalty (i.e., NIPG, SIPG or IIPG versions). For the space and time discretization, piecewise polynomial approximations of different degrees p and q, respectively, are used. We assume that the diffusion coefficient depends on the sought solution, but we do not allow its degeneration, as happens in some physical models. In comparison to the paper [31], it is necessary to overcome the difficulty in the derivation of the abstract error estimate caused by the fact that the mentioned technique based on the Gauss-Radau numerical quadrature and interpolation is not applicable to the nonlinear diffusion. In this case, the concept of the discrete characteristic function introduced in [10] is used with success. Under the assumption that the triangulations on all time levels are uniformly shape regular, and the exact solution has some regularity properties, error estimates are derived for the space-time DGFE method. These estimates are optimal in time, if the Dirichlet boundary conditions have behaviour in time as a polynomial of degree $\leq q$. In a general case, the derived estimates become suboptimal.

The paper has the following structure: First, the continuous problem is formulated. In the next section, the discontinuous Galerkin space-time discretization is introduced. Further, some important assumptions are formulated and auxiliary estimates are established. In what follows, abstract error estimates are proven. These estimates are used for the derivation of error estimates in terms of the sizes of the space and time meshes. Finally, some unsolved problems are mentioned.

2. Formulation of the problem. Let $\Omega \subset I\!R^d$ ($d = 2$ or 3) be a bounded polyhedral domain and $T > 0$. We consider the following initial-boundary value problem: Find $u : Q_T = \Omega \times (0,T) \rightarrow I\!R$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u)) = g \quad \text{in } Q_T, \tag{2.1}$$

$$u\big|_{\partial\Omega\times(0,T)} = u_D, \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

We assume that g, u_D, u^0, f_s are given functions and $f_s \in C^1(I\!R)$, $\quad |f_s'| \leq C,\ s = 1,2$. Moreover, let

$$\beta : I\!R \rightarrow [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.4}$$

$$|\beta(u_1) - \beta(u_2)| \leq L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in I\!R. \tag{2.5}$$

(If $\beta_0 = \beta_1$, then we get a problem with a linear diffusion, treated in [31].) In the derivation and analysis of the discrete problem we assume that the exact solution is regular in the following sense:

$$u \in L^2(0,T;H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;H^1(\Omega)), \tag{2.6}$$

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \leq C_R \ \text{ for a.e. } \ t \in (0,T). \tag{2.7}$$

Using techniques from [45], it is possible to prove the existence and uniqueness of a weak solution to problem (2.1) – (2.3).

We use the standard notation of function spaces (see, e.g. [41]). If ω is a bounded domain, then we define the Lebesgue spaces

$$L^\infty(\omega) = \{\text{measurable functions } \varphi;\ \|\varphi\|_{L^\infty(\omega)} = \text{essup}_{x\in\omega}|\varphi(x)| < \infty\},$$
$$L^2(\omega) = \{\text{measurable functions } \varphi;\ \|\varphi\|_{L^2(\omega)} = \Big(\int_\omega |\varphi|^2\,dx\Big)^{1/2} < \infty\}$$

and the Sobolev space

$$H^k(\omega) = \{\varphi \in L^2(\omega);\ \|\varphi\|_{H^k(\omega)} = \big(\sum_{|\alpha|\le k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\big)^{1/2} < \infty\},$$

with the seminorm

$$|\varphi|_{H^k(\omega)} = \Big(\sum_{|\alpha|=k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\Big)^{1/2}.$$

We also use the Bochner spaces. Let X be a Banach space with a norm $\|\cdot\|_X$ and a seminorm $|\cdot|_X$ and let s be an integer. Then we define:

$$C([0,T];X) = \{\varphi : [0,T] \to X,\ \text{continuous},\ \|\varphi\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|\varphi\|_X < \infty\},$$
$$L^2(0,T;X) = \Big\{\varphi : (0,T) \to X,\ \text{strongly measurable},\ \|\varphi\|^2_{L^2(0,T;X)} = \int_0^T \|\varphi\|^2_X\,dt < \infty\Big\},$$
$$H^s(0,T;X) = \Big\{\varphi \in L^2(0,T;X);\ \|\varphi\|^2_{H^s(0,T;X)} = \int_0^T \sum_{\alpha=0}^{s} \Big\|\frac{\partial^\alpha \varphi}{\partial t^\alpha}\Big\|^2_X < \infty\Big\}.$$

Moreover, we set

$$|\varphi|_{C([0,T];X)} = \sup_{t\in[0,T]} |\varphi|_X, \quad |\varphi|_{L^2(0,T;X)} = \Big(\int_0^T |\varphi|^2_X\,dt\Big)^{1/2},$$
$$|\varphi|_{H^s(0,T;X)} = \Big(\int_0^T \Big|\frac{\partial^s \varphi}{\partial t^s}\Big|^2_X\Big)^{1/2}.$$

3. Space-time discretization. In the time interval $[0,T]$ we shall construct a partition $0 = t_0 < \cdots < t_M = T$ and denote $I_m = (t_{m-1}, t_m),\ \tau_m = t_m - t_{m-1}$, $\tau = \max_{m=1,\dots,M} \tau_m$. For each I_m we consider a partition $\mathcal{T}_{h,m}$ of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles with mutually disjoint interiors. The partitions $\mathcal{T}_{h,m}$ are in general different for different m.

By $\mathcal{F}_{h,m}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,m}$. Further, we denote the set of all inner faces by $\mathcal{F}^I_{h,m}$ and the set of all boundary faces by $\mathcal{F}^B_{h,m}$. Each $\Gamma \in \mathcal{F}_{h,m}$ will be associated with a unit normal vector $\boldsymbol{n}_\Gamma$, which has the same orientation as the outer normal to $\partial\Omega$ for $\Gamma \in \mathcal{F}^B_{h,m}$. We set $h_K = \text{diam}(K)$ for $K \in \mathcal{T}_{h,m}$, $h_m = \max_{K\in\mathcal{T}_{h,m}} h_K,\ h = \max_{m=1,\dots,M} h_m$. By ρ_K we denote the radius of the largest ball inscribed into K.

For a function φ defined in $\bigcup_{m=1}^M I_m$ we put $\varphi_m^\pm = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t)$ and $\{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$.

Over a triangulation $\mathcal{T}_{h,m}$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_{h,m}) = \{v; v|_K \in H^k(K)\ \forall\, K \in \mathcal{T}_{h,m}\}, \tag{3.1}$$

with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_{h,m})} = \left(\sum_{K\in\mathcal{T}_{h,m}} |v|^2_{H^k(K)}\right)^{1/2}. \tag{3.2}$$

For each face $\Gamma \in \mathcal{F}^I_{h,m}$ there exist two neighbours $K^{(L)}_\Gamma, K^{(R)}_\Gamma \in \mathcal{T}_{h,m}$ such that $\Gamma \subset \partial K^{(L)}_\Gamma \cap \partial K^{(R)}_\Gamma$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K^{(L)}_\Gamma$ and the inner normal to $\partial K^{(R)}_\Gamma$. If $\Gamma \in \mathcal{F}^B_{h,m}$, then $K^{(L)}_\Gamma$ will denote the element adjacent to Γ. For $v \in H^1(\Omega, \mathcal{T}_{h,m})$ and $\Gamma \in \mathcal{F}_{h,m}$ we use the notation $v^{(L)}_\Gamma$ for the trace of $v|_{K^{(L)}_\Gamma}$ on Γ. Moreover, if $\Gamma \in \mathcal{F}^I_{h,m}$, then we set $v^{(R)}_\Gamma =$ the trace of $v|_{K^{(R)}_\Gamma}$ on Γ, $\langle v \rangle_\Gamma = \frac{1}{2}\left(v^{(L)}_\Gamma + v^{(R)}_\Gamma\right)$, $[v]_\Gamma = v^{(L)}_\Gamma - v^{(R)}_\Gamma$.

We use the notation

$$h(\Gamma) = \frac{h_{K^{(L)}_\Gamma} + h_{K^{(R)}_\Gamma}}{2} \quad \text{for } \Gamma \in \mathcal{F}^I_{h,m}, \quad h(\Gamma) = h_{K^{(L)}_\Gamma} \quad \text{for } \Gamma \in \mathcal{F}^B_{h,m}. \tag{3.3}$$

By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the scalar product and the norm in $L^2(\Omega)$.

If $v, w, \varphi \in H^2(\Omega, \mathcal{T}_{h,m})$ and $C_W > 0$ is a fixed constant., we define the forms

$$a_{h,m}(v,w,\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(v)\nabla w\cdot\nabla\varphi\,\mathrm{d}x \tag{3.4}$$

$$- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\langle\beta(v)\nabla w\rangle\cdot\boldsymbol{n}_\Gamma[\varphi] + \theta\langle\beta(v)\nabla\varphi\rangle\cdot\boldsymbol{n}_\Gamma\,[w]\right)\mathrm{d}S$$

$$- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\beta(v)\nabla w\cdot\boldsymbol{n}_\Gamma\,\varphi + \theta\,\beta(v)\nabla\varphi\cdot\boldsymbol{n}_\Gamma\,w - \theta\beta(v)\nabla\varphi\cdot\boldsymbol{n}_\Gamma u_D\right)\,\mathrm{d}S,$$

$$J_{h,m}(w,\varphi) = C_W \sum_{\Gamma\in\mathcal{F}^I_{h,m}} h(\Gamma)^{-1}\int_\Gamma [w]\,[\varphi]\,\mathrm{d}S + C_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1}\int_\Gamma w\,\varphi\,\mathrm{d}S, \tag{3.5}$$

$$A_{h,m}(v,w,\varphi) = a_{h,m}(v,w,\varphi) + \beta_0 J_{h,m}(w,\varphi), \tag{3.6}$$

$$b_{h,m}(w,\varphi) = -\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^2 f_s(w)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \tag{3.7}$$

$$+ \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma H\left(w^{(L)}_\Gamma, w^{(R)}_\Gamma, \boldsymbol{n}_\Gamma\right)[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma H\left(w^{(L)}_\Gamma, w^{(L)}_\Gamma, \boldsymbol{n}_\Gamma\right)\varphi\,\mathrm{d}S.$$

$$\ell_{h,m}(\varphi) = (g,\varphi) + \beta_0 C_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S. \tag{3.8}$$

In (3.7), H is a numerical flux with the following properties.
(H1) $H(u,v,\boldsymbol{n})$ is defined in $I\!R^2\times B_1$, where $B_1 = \{\boldsymbol{n}\in I\!R^2; |\boldsymbol{n}| = 1\}$, and is *Lipschitz-continuous* with respect to $u,\ v$.
(H2) $H(u,v,\boldsymbol{n})$ is *consistent*: $H(u,u,\boldsymbol{n}) = \sum_{s=1}^2 f_s(u)\,n_s, u\in I\!R, \boldsymbol{n} = (n_1,n_2)\in B_1$.
(H3) $H(u,v,\boldsymbol{n})$ is *conservative*: $H(u,v,\boldsymbol{n}) = -H(v,u,-\boldsymbol{n}), \quad u,\ v\in I\!R,\ \boldsymbol{n}\in B_1$.

In the above forms we take $\theta = -1,\ \theta = 0$ and $\theta = 1$ and obtain the nonsymmetric (NIPG), incomplete (IIPG) and symmetric (SIPG) variants of the approximation of the diffusion terms, respectively.

In the space $H^1(\Omega,\mathcal{T}_{h,m})$, the following norm will be used:

$$\|\varphi\|_{DG,m} = \Big(\sum_{K\in\mathcal{T}_{h,m}} |\varphi|^2_{H^1(K)} + J_{h,m}(\varphi,\varphi)\Big)^{1/2}. \tag{3.9}$$

Let $p,\ q\geq 1$ be integers. For each $m = 1,\ldots,M$ we define the finite-dimensional space

$$S^p_{h,m} = \left\{\varphi\in L^2(\Omega); \varphi|_K\in P^p(K)\ \forall\, K\in\mathcal{T}_{h,m}\right\}. \tag{3.10}$$

Here $P^p(K)$ denotes the space of all polynomials on K of degree $\le p$. We denote by Π_m the $L^2(\Omega)$-projection on $S^p_{h,m}$. This means that if $u \in L^2(\Omega)$, then $\Pi_m u \in S^p_{h,m}$ and

$$(\Pi_m u - u, \varphi) = 0 \quad \forall \varphi \in S^p_{h,m} \tag{3.11}$$

The approximate solution will be sought in the space

$$S^{p,q}_{h,\tau} = \Big\{ \varphi \in L^2(Q_T); \varphi\big|_{I_m} = \sum_{i=0}^{q} t^i \varphi_i \quad \text{with } \varphi_i \in S^p_{h,m},\ m = 1, \dots, M \Big\}. \tag{3.12}$$

In what follows we shall use the notation $U' = \partial U/\partial t, u' = \partial u/\partial t$.

DEFINITION 3.1. *We say that a function U is an approximate solution of problem (2.1) – (2.3), if $U \in S^{p,q}_{h,m}$ and*

$$\int_{I_m} \left((U', \varphi) + A_{h,m}(U, U, \varphi) + b_{h,m}(U, \varphi) \right) \mathrm{d}t + \left(\{U\}_{m-1}, \varphi^+_{m-1} \right) \tag{3.13}$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\, \mathrm{d}t, \quad \forall \varphi \in S^{p,q}_{h,\tau}, \quad m = 1, \dots, M,\ U^-_0 := \Pi_1 u^0.$$

The exact regular solution u satisfies the identity

$$\int_{I_m} \left((u', \varphi) + A_{h,m}(u, u, \varphi) + b_{h,m}(u, \varphi) \right) \mathrm{d}t + \left(\{u\}_{m-1}, \varphi^+_{m-1} \right) \tag{3.14}$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\, \mathrm{d}t \quad \forall \varphi \in S^{p,q}_{h,\tau}, \quad \text{with } u(0-) = u(0).$$

It is also possible to consider $q = 0$. In this case, scheme (3.13) represents a version of the backward Euler method. Therefore, we shall be concerned only with $q \ge 1$.

4. Assumptions and some auxiliary estimates. In the derivation of the error estimates we shall use the $S^{p,q}_{h,\tau}$-interpolation π of functions $v \in H^1(0, T; L^2(\Omega))$ defined by

$$\text{a)} \quad \pi\, v \in S^{p,q}_{h,\tau}, \quad \text{b)} \quad (\pi\, v)\,(t_m-) = \Pi_m\, v(t_m-), \tag{4.1}$$
$$\text{c)} \quad \int_{I_m} (\pi v - v,\, \varphi^*)\, \mathrm{d}t = 0 \quad \forall \varphi^* \in S^{p,q-1}_{h,\tau}, \quad \forall\, m = 1, \dots, M.$$

In [29], Lemma 4, it was proven that πu is uniquely determined.

Our main goal will be the derivation of the estimation of the error $e = U - u$, which can be expressed in the form

$$e = \xi + \eta, \tag{4.2}$$

where

$$\xi = U - \pi u \in S^{p,q}_{h,\tau}, \quad \eta \;=\; \pi u - u. \tag{4.3}$$

Then, in virtue of (3.13) and (3.14), for each $\varphi \in S^{p,q}_{h,\tau}$ we have

$$\int_{I_m} \left((\xi', \varphi) + A_{h,m}(U, U, \varphi) - A_{h,m}(u, u, \varphi) \right) \mathrm{d}t + \left(\{\xi\}_{m-1}, \varphi^+_{m-1} \right) \tag{4.4}$$
$$= \int_{I_m} \left(b_{h,m}(u, \varphi) - b_{h,m}(U, \varphi) \right) \mathrm{d}t - \int_{I_m} (\eta', \varphi) \mathrm{d}t - \left(\{\eta\}_{m-1}, \varphi^+_{m-1} \right).$$

In our further considerations, by $C, C_1, C_2, \dots, \tilde{C}, \tilde{C}_1, \tilde{C}_2, \dots$ we shall denote positive generic constants, independent of h, τ, m, M, K, u, U, which can attain different values in different places.

In the sequel, we shall consider a system of partitions $\{0 = t_0 < t_1 < \ldots < t_M = T\}$ with any positive integer M and $\tau \in (0, \tau_0),\ \tau_0 > 0$, and a system of triangulations $\mathcal{T}_{h,m}$, $m = 1, \ldots, M$, $h \in (0, h_0),\ h_0 > 0$, which is shape regular and locally quasiuniform: There exist positive constants C_R and C_Q, independent of K, Γ, m, M and h, such that for all $m = 1, \ldots, M$ and $h \in (0, h_0)$

$$\frac{h_K}{\rho_K} \le C_R, \quad \forall K \in \mathcal{T}_{h,m}, \tag{4.5}$$

$$h_{K_\Gamma^{(L)}} \le C_Q\, h_{K_\Gamma^{(R)}}, \quad h_{K_\Gamma^{(R)}} \le C_Q\, h_{K_\Gamma^{(L)}} \quad \forall \Gamma \in \mathcal{F}_{h,m}^I. \tag{4.6}$$

Important tools in the analysis of the DGFE method are the multiplicative trace inequality and the inverse inequality: There exist constants $C_M, C_I > 0$ independent of $h \in (0, h_0), m, M$, $K \in \mathcal{T}_{h,m}$ and v such that

$$\|v\|^2_{L^2(\partial K)} \le C_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \quad v \in H^1(K), \tag{4.7}$$

and

$$|v|_{H^1(K)} \le C_I h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K). \tag{4.8}$$

(For proofs, see, e.g. [23] and [12].) We shall also use the inequalities

$$J_{h,m}(v, w) \le (J_{h,m}(v, v))^{1/2} (J_{h,m}(w, w))^{1/2} \le \frac{1}{2} (\delta J_{h,m}(v, v) + \delta^{-1} J_{h,m}(w, w)), \tag{4.9}$$

with any $\delta > 0$, which are the consequence of the definition of the form $J_{h,m}$ and the Cauchy and Young's inequalities.

The analysis of the form $b_{h,m}$ ([19] or [24]) implies that for each $k_b > 0$ there exists a constant $C_b = C_b(k_b)$ such that

$$\begin{aligned} &|b_{h,m}(U, \varphi) - b_{h,m}(u, \varphi)| \\ &\le \frac{\beta_0}{k} \|\varphi\|^2_{DG,m} + C_b(\|\xi\|^2 + \|\eta\| + \sum_{K \in \mathcal{T}_{h,m}} h_K^2 |\eta|^2_{H^1(K)}). \end{aligned} \tag{4.10}$$

As for the coercivity of the forms $A_{h,m}$, we can prove the following result.

LEMMA 4.1. *Let*

$$C_W > 0, \quad \text{for } \theta = -1\ (NIPG), \tag{4.11}$$

$$C_W \ge \left(\frac{4\beta_1}{\beta_0} \right)^2 C_{MI} \quad \text{for } \theta = 1\ (SIPG), \tag{4.12}$$

$$C_W \ge 2 \left(\frac{2\beta_1}{\beta_0} \right)^2 C_{MI} \quad \text{for } \theta = 0\ (IIPG), \tag{4.13}$$

where $C_{MI} = C_M(C_I + 1)(C_Q + 1)$. *Then*

$$a_{h,m}(U, U, \xi) - a_{h,m}(U, \pi u, \xi) + \beta_0 J_{h,m}(\xi, \xi) \ge \frac{\beta_0}{2} \|\xi\|^2_{DG,m}. \tag{4.14}$$

Proof. From (3.4) and (3.5) we immediately see that for $\theta = -1$ inequality (4.14) holds. Let us assume that $\theta = 1$. Then, by (3.4) and (2.4),

$$
\begin{aligned}
& a_{h,m}(U,U,\xi) - a_{h,m}(U,\pi u,\xi) + \beta_0 J_{h,m}(\xi,\xi) \qquad (4.15)\\
&= \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(U)\nabla\xi\cdot\nabla\xi \mathrm{d}x + \beta_0 J_{h,m}(\xi,\xi)\\
&-2\sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle\beta(U)\nabla\xi\rangle\cdot\boldsymbol{n}_\Gamma[\xi]\mathrm{d}S - 2\sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \beta(U)\nabla\xi\cdot\boldsymbol{n}_\Gamma\xi\mathrm{d}S\\
&\geq \beta_0 \sum_{K\in\mathcal{T}_{h,m}} \int_K |\nabla\xi|^2\,\mathrm{d}x + \beta_0 J_{h,m}(\xi,\xi) - 2\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{|\nabla\xi_\Gamma^{(L)}| + |\nabla\xi_\Gamma^{(R)}|}{2}|[\xi]|\mathrm{d}S\\
&-2\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla\xi|\,|\xi|\,\mathrm{d}S.
\end{aligned}
$$

If $\delta > 0$, then Young's inequality implies that

$$
\begin{aligned}
& a_{h,m}(U,U,\xi) - a_{h,m}(U,\pi u,\xi) + \beta_0 J_{h,m}(\xi,\xi) \qquad (4.16)\\
&\geq \beta_0 \|\xi\|^2_{DG,m} - \frac{2\beta_1}{\delta C_W}\sum_{i\in I}\int_{\partial K_i} h_{K_i}|\nabla\xi|^2\,\mathrm{d}S - 2\beta_1\delta J_{h,m}(\xi,\xi).
\end{aligned}
$$

If we set $\delta = \frac{\beta_0}{4\beta_1}$ and use inequalities (4.7), (4.8) and assumption (4.12), we get

$$
\begin{aligned}
& a_{h,m}(U,U,\xi) - a_{h,m}(U,\pi u,\xi) + \beta_0 J_{h,m}(\xi,\xi) \qquad (4.17)\\
&\geq \beta_0 \|\xi\|^2_{DG,m} - \frac{8\beta_1^2 C_{MI}}{\beta_0 C_W}\sum_{i\in I}\int_{K_i} |\nabla\xi|^2\,\mathrm{d}S - \frac{\beta_0}{2}J_{h,m}(\xi,\xi) \geq \frac{\beta_0}{2}\|\xi\|^2_{DG,m}.
\end{aligned}
$$

Similarly we prove (4.14) for $\theta = 0$, provided (4.13) is satisfied.

□

LEMMA 4.2. *There exists a constant $C > 0$ independent of U, ξ, φ, h, m, M such that*

$$
a_{h,m}(U,U,\varphi) - a_{h,m}(U,\pi u,\varphi) + \beta_0 J_{h,m}(\xi,\varphi) \leq C(\|\xi\|^2_{DG,m} + \|\varphi\|^2_{DG,m}) \qquad (4.18)
$$

for any $\varphi \in S^p_{h,m}$.

Proof. Using (3.4), (3.5), (2.4), the Cauchy inequality and Young's inequality, we find that

$$
\begin{aligned}
& a_{h,m}(U,U,\varphi) - a_{h,m}(U,\pi u,\varphi) + \beta_0 J_{h,m}(\xi,\varphi) \leq \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla\xi|^2 + |\nabla\varphi|^2\right)\mathrm{d}x\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W}\left(|\nabla\xi_\Gamma^{(L)}|^2 + |\nabla\xi_\Gamma^{(R)}|^2\right) + \frac{C_W}{h(\Gamma)}|[\varphi]|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W}\left(|\nabla\varphi_\Gamma^{(L)}|^2 + |\nabla\varphi_\Gamma^{(R)}|^2\right) + \frac{C_W}{h(\Gamma)}|[\xi]|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W}|\nabla\xi|^2 + \frac{C_W}{h(\Gamma)}|\varphi|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W}|\nabla\varphi|^2 + \frac{C_W}{h(\Gamma)}|\xi|^2\right)\mathrm{d}S\\
&+\beta_0 J_{h,m}(\xi,\varphi)\\
&\leq \beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(|\nabla\xi|^2 + |\nabla\varphi|^2\right)\mathrm{d}x + \frac{\beta_1}{C_W}\sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} h_K\left(|\nabla\xi|^2 + |\nabla\varphi|^2\right)\mathrm{d}S\\
&+\beta_1 J_{h,m}(\xi,\xi) + \beta_1 J_{h,m}(\varphi,\varphi) + \beta_0 J_{h,m}(\xi,\varphi).
\end{aligned}
$$

Now, this, (4.7), (4.8) and (4.9) yield the estimate

$$\begin{aligned}
&a_{h,m}(U,U,\varphi)-a_{h,m}(U,\pi u,\varphi)+\beta_0 J_{h,m}(\xi,\varphi)\\
&\le \beta_1 \sum_{K\in\mathcal{T}_{h,m}}\int_K \left(|\nabla\xi|^2+|\nabla\varphi|^2\right)\mathrm{d}x+\frac{\beta_1 C_{MI}}{C_W}\sum_{K\in\mathcal{T}_{h,m}}\int_K\left(|\nabla\xi|^2+|\nabla\varphi|^2\right)\mathrm{d}x\\
&+\beta_1(J_{h,m}(\xi,\xi)+J_{h,m}(\varphi,\varphi))+\beta_0(J_{h,m}(\xi,\xi)+J_{h,m}(\varphi,\varphi))\\
&\le C(\|\xi\|^2_{DG,m}+\|\varphi\|^2_{DG,m}),
\end{aligned}$$

which we wanted to prove.
□

LEMMA 4.3. *For arbitrary $k_a, k_c > 0$ there exist constants $C_a = C_a(k_a),\ C_c = C_c(k_c) > 0$ independent of U, ξ, φ, h, m, M such that for each $\varphi \in S^p_{h,m}$ the following estimates hold:*

$$a_{h,m}(U,\pi u,\varphi)-a_{h,m}(u,\pi u,\varphi)\le \frac{\beta_0}{k_a}\|\varphi\|^2_{DG,m}+C_a(\|\xi\|^2+R^2_m(\eta)), \tag{4.19}$$

$$a_{h,m}(u,\pi u,\varphi)-a_{h,m}(u,u,\varphi)\le \frac{\beta_0}{k_c}\|\varphi\|^2_{DG,m}+C_c\tilde{R}_m(\eta), \tag{4.20}$$

where

$$R^2_m(\eta)=\|\eta\|^2_{DG,m}+\|\eta\|^2+\sum_{K\in\mathcal{T}_{h,m}}\left(h^2_K|\eta|^2_{H^1(K)}+h^2_K|\eta|^2_{H^2(K)}\right), \tag{4.21}$$

$$\tilde{R}^2_m(\eta)=\|\eta\|^2_{DG,m}+\sum_{K\in\mathcal{T}_{h,m}}h^2_K|\eta|^2_{H^2(K)}. \tag{4.22}$$

Proof. Since $\nabla\pi u=\nabla\eta+\nabla u,\ [u]=0$, $[\pi u]=[\eta]$, we can write

$$\begin{aligned}
&a_{h,m}(U,\pi u,\varphi)-a_{h,m}(u,\pi u,\varphi) \qquad (4.23)\\
&=\sum_{K\in\mathcal{T}_{h,m}}\int_K((\beta(U)-\beta(u))\nabla\eta\cdot\nabla\varphi+(\beta(U)-\beta(u))\nabla u\cdot\nabla\varphi)\,\mathrm{d}x\\
&-\sum_{\Gamma\in\mathcal{F}^I_{h,m}}\int_\Gamma(\langle(\beta(U)-\beta(u))\nabla\eta\rangle\cdot\boldsymbol{n}_\Gamma[\varphi]+\langle(\beta(U)-\beta(u))\nabla u\rangle\cdot\boldsymbol{n}_\Gamma[\varphi])\,\mathrm{d}S\\
&-\theta\sum_{\Gamma\in\mathcal{F}^I_{h,m}}\int_\Gamma\langle(\beta(U)-\beta(u))\nabla\varphi\rangle\cdot\boldsymbol{n}_\Gamma[\eta]\mathrm{d}S\\
&-\sum_{\Gamma\in\mathcal{F}^B_{h,m}}\int_\Gamma((\beta(U)-\beta(u))\nabla\eta\cdot\boldsymbol{n}_\Gamma\varphi+(\beta(U)-\beta(u))\nabla u\cdot\boldsymbol{n}_\Gamma\varphi)\,\mathrm{d}S\\
&-\theta\sum_{\Gamma\in\mathcal{F}^B_{h,m}}\int_\Gamma(\beta(U)-\beta(u))\nabla\varphi\cdot\boldsymbol{n}_\Gamma(\pi u-u_D)\mathrm{d}S.
\end{aligned}$$

This, conditions (2.2), (2.4), (2.5), (2.7) and the Cauchy and Young's inequalities imply that for any

$\delta_1, \delta_2, \delta_3 > 0$ we have

$$
\begin{aligned}
& a_{h,m}(U,\pi u,\varphi) - a_{h,m}(u,\pi u,\varphi) \leq \\
& (\beta_1-\beta_0) \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(\frac{|\nabla\eta|^2}{\delta_1} + \delta_1 |\nabla\varphi|^2 \right) \mathrm{d}x \\
& + L_\beta C_R \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(\frac{|U-u|^2}{\delta_2} + \delta_2 |\nabla\varphi|^2 \right) \mathrm{d}x \\
& + (\beta_1-\beta_0) \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_1} \left(|\nabla\eta_\Gamma^{(L)}|^2 + |\nabla\eta_\Gamma^{(R)}|^2 \right) + \frac{C_W\delta_1}{h(\Gamma)} |[\varphi]|^2 \right) \mathrm{d}S \\
& + L_\beta C_R \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_2} \left(|(U-u)_\Gamma^{(L)}|^2 + |(U-u)_\Gamma^{(R)}|^2 \right) + \frac{C_W\delta_2}{h(\Gamma)} |[\varphi]|^2 \right) \mathrm{d}S \\
& + (\beta_1-\beta_0) \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)\delta_3}{C_W} \left(|\nabla\varphi_\Gamma^{(L)}|^2 + |\nabla\varphi_\Gamma^{(R)}|^2 \right) + \frac{C_W}{h(\Gamma)\delta_3} |[\eta]|^2 \right) \mathrm{d}S \\
& + (\beta_1-\beta_0) \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_1} |\nabla\eta|^2 + \frac{C_W\delta_1}{h(\Gamma)} |\varphi|^2 \right) \mathrm{d}S \\
& + L_\beta C_R \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_2} |U-u|^2 + \frac{C_W\delta_2}{h(\Gamma)} |\varphi|^2 \right) \mathrm{d}S \\
& + (\beta_1-\beta_0) \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)\delta_3}{C_W} |\nabla\varphi|^2 + \frac{C_W}{h(\Gamma)\delta_3} |\eta|^2 \right) \mathrm{d}S.
\end{aligned}
$$

Now, using the inequalities $|U-u|^2 = |\xi+\eta|^2 \leq 2\left(|\xi|^2 + |\eta|^2\right)$, (4.7) and (4.8), we find that

$$
\begin{aligned}
& a_{h,m}(U,\pi u,\varphi) - a_{h,m}(u,\pi u,\varphi) \\
& \leq \left((\beta_1-\beta_0)\delta_1 + L_\beta C_R\delta_2\right) \|\varphi\|^2_{DG,m} + \frac{(\beta_1-\beta_0)C_{MI}\delta_3}{C_W} \sum_{K\in\mathcal{T}_{h,m}} \int_K |\nabla\varphi|^2 \,\mathrm{d}x \\
& \quad + \left(\frac{2L_\beta C_R C_{MI}}{C_W\delta_2} + \frac{2L_\beta C_R}{\delta_2} \right) \sum_{K\in\mathcal{T}_{h,m}} \int_K |\xi|^2 \,\mathrm{d}x + \frac{2L_\beta C_R}{\delta_2} \sum_{K\in\mathcal{T}_{h,m}} \int_K |\eta|^2 \,\mathrm{d}x \\
& \quad + \frac{4L_\beta C_R C_M}{C_W\delta_2} \sum_{K\in\mathcal{T}_{h,m}} \left(\|\eta\|^2_{L^2(K)} + h_K^2 |\eta|^2_{H^1(K)} \right) + \frac{\beta_1-\beta_0}{\delta_3} J_h(\eta,\eta) \\
& \quad + \frac{(\beta_1-\beta_0)2C_M}{C_W\delta_1} \sum_{K\in\mathcal{T}_{h,m}} \left(|\eta|^2_{H^1(K)} + h_K^2 |\eta|^2_{H^2(K)} \right) + \frac{\beta_1-\beta_0}{\delta_1} \sum_{K\in\mathcal{T}_{h,m}} |\eta|^2_{H^1(K)}\,.
\end{aligned}
$$

Finally, choosing

$$
\delta_1 = \frac{\beta_0}{3k_a(\beta_1-\beta_0)}, \quad \delta_2 = \frac{\beta_0}{3k_a L_\beta C_R}, \quad \delta_3 = \frac{\beta_0 C_W}{3k_a(\beta_1-\beta_0)C_{MI}},
$$

we obtain estimate (4.19).

Similarly we proceed in the proof of (4.20), (4.22). From the definition of the form $a_{h,m}$ and the properties of the function β it follows that

$$
\begin{aligned}
&a_{h,m}(u,\pi u,\varphi) - a_{h,m}(u,u,\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(u)\nabla\eta\cdot\nabla\varphi \mathrm{d}x \qquad (4.24)\\
&- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle\beta(u)\nabla\eta\rangle\cdot\boldsymbol{n}[\varphi]\mathrm{d}S - \theta \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle\beta(u)\nabla\varphi\rangle\cdot\boldsymbol{n}[\eta]\mathrm{d}S\\
&- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \beta(u)\nabla\eta\cdot\boldsymbol{n}\varphi\mathrm{d}S - \theta \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \beta(u)\nabla\varphi\cdot\boldsymbol{n}\eta\mathrm{d}S\\
\le\ &\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_K |\nabla\eta|\,|\nabla\varphi|\,\mathrm{d}x\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \frac{|\nabla\eta^{(L)}_\Gamma| + |\nabla\eta^{(R)}_\Gamma|}{2}\,|[\varphi]|\,\mathrm{d}S + \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \frac{|\nabla\varphi^{(L)}_\Gamma| + |\nabla\varphi^{(L)}_\Gamma|}{2}\,|[\eta]|\,\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla\eta|\,|\varphi|\,\mathrm{d}S + \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma |\nabla\varphi|\,|\eta|\,\mathrm{d}S.
\end{aligned}
$$

The use of Young's inequality, the multiplicative trace inequalities (4.7) and the inverse inequality (4.8), for arbitrary $\delta_1, \delta_2 > 0$ imply that

$$
\begin{aligned}
&a_{h,m}(u,\pi u,\varphi) - a_{h,m}(u,u,\varphi)\\
\le\ &\beta_1 \sum_{K\in\mathcal{T}_{h,m}} \int_K \left(\frac{|\nabla\eta|^2}{\delta_1} + \delta_1\,|\nabla\varphi|^2\right)\mathrm{d}x\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_1}\left(|\nabla\varphi^{(L)}_\Gamma|^2 + |\nabla\varphi^{(R)}_\Gamma|^2\right) + \frac{C_W\delta_1}{h(\Gamma)}\,|[\varphi]|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)\delta_2}{C_W}\left(|\nabla\varphi^{(L)}_\Gamma|^2 + |\nabla\varphi^{(R)}_\Gamma|^2\right) + \frac{C_W}{h(\Gamma_{ij})\delta_2}\,|[\eta]|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)}{C_W\delta_1}\,|\nabla\eta|^2 + \frac{C_W\delta_1}{h(\Gamma_{ij})}\,|\varphi|^2\right)\mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma \left(\frac{h(\Gamma)\delta_2}{C_W}\,|\nabla\varphi|^2 + \frac{C_W}{h(\Gamma)\delta_2}\,|\eta|^2\right)\mathrm{d}S\\
\le\ &\beta_1\delta_1\,\|\varphi\|^2_{DG} + \frac{\beta_1 C_{MI}\delta_2}{C_W}\sum_{K\in\mathcal{T}_{h,m}} |\varphi|^2_{H^1(K)} + \frac{\beta_1}{\delta_2}J_h(\eta,\eta)\\
&+\frac{\beta_1}{\delta_1}\sum_{K\in\mathcal{T}_{h,m}} |\eta|^2_{H^1(K)} + \frac{2\beta_1 C_M}{C_W\delta_1}\sum_{K\in\mathcal{T}_{h,m}}\left(|\eta|^2_{H^1(K)} + h_K^2\,|\eta|^2_{H^2(K)}\right).
\end{aligned}
$$

If we put

$$
\delta_1 = \frac{\beta_0}{2k_c\beta_1}, \quad \delta_2 = \frac{C_W\beta_0}{2k_c\beta_1 C_{MI}}, \quad C_c = \max\left\{\frac{\beta_1}{\delta_2}, \frac{\beta_1}{\delta_1} + \frac{2\beta_1 C_M}{C_W\delta_1}\right\},
$$

we get (4.20).
□

5. Abstract error estimates.

5.1. Estimation of ξ. Let us substitute $\varphi := \xi$ in (4.4). It follows from the definition of the form $A_{h,m}$ that

$$\int_{I_m} ((\xi', \xi) + a_h(U,U,\xi) - a_h(U,\pi u,\xi) + \beta_0 J_h(\xi,\xi))\,\mathrm{d}t + \left(\{\xi\}_{m-1}, \xi^+_{m-1}\right) \tag{5.1}$$
$$= \int_{I_m} (-a_h(U,\pi u,\xi) + a_h(u,\pi u,\xi) - a_h(u,\pi u,\xi) + a_h(u,u,\xi) - \beta_0 J_h(\eta,\xi))\,\mathrm{d}t$$
$$+ \int_{I_m} (b_h(u,\xi) - b_h(U,\xi) - (\eta',\xi))\,\mathrm{d}t - \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) \quad \forall \varphi \in S^{p,q}_{h,\tau}.$$

Simple calculation yields

$$\int_{I_m} (\xi', \xi)\mathrm{d}t + \left(\{\xi\}_{m-1}, \xi^+_{m-1}\right) = \frac{1}{2}\left(\left\|\xi^-_m\right\|^2 - \left\|\xi^-_{m-1}\right\|^2 + \left\|\{\xi\}_{m-1}\right\|^2\right). \tag{5.2}$$

Further, the integration by parts, the relations $(\eta^-_m, \xi^-_m) = (\eta^-_{m-1}, \xi^-_{m-1}) = 0$ following from the definitions of the projections π and Π_m yield the relations

$$\int_{I_m} (\eta', \varphi)\mathrm{d}t + \left(\{\eta\}_{m-1}, \varphi^+_{m-1}\right) = -\left(\eta^-_{m-1}, \varphi^+_{m-1}\right) = -\left(\eta^-_{m-1}, \{\varphi\}_{m-1}\right), \; \varphi \in S^{p,q}_{h,m}. \tag{5.3}$$

The use of (5.1) – (5.3), (4.10), (4.9), Young's inequality and Lemmas 4.1 and 4.3 imply that for arbitrary $\delta, k_a, k_b, k_c > 0$ we have

$$\left\|\xi^-_m\right\|^2 - \left\|\xi^-_{m-1}\right\|^2 + \beta_0\left(1 - \frac{2}{k_a} - \frac{2}{k_b} - \frac{2}{k_c} - 2\delta\right)\int_{I_m} \|\xi\|^2_{DG,m}\,\mathrm{d}t$$
$$\le C\left(\int_{I_m} \|\xi\|^2\,\mathrm{d}t + \left\|\eta^-_{m-1}\right\|^2 + \int_{I_m} R_m(\eta)\mathrm{d}t\right).$$

This and the choice $k_a = k_b = k_c = 16$ and $\delta = \frac{1}{16}$ imply that

$$\left\|\xi^-_m\right\|^2 - \left\|\xi^-_{m-1}\right\|^2 + \frac{\beta_0}{2}\int_{I_m} \|\xi\|^2_{\mathrm{DG},m}\,\mathrm{d}t \tag{5.4}$$
$$\le C\left(\int_{I_m} \|\xi\|^2\,\mathrm{d}t + \left\|\eta^-_{m-1}\right\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t\right), \quad m = 1, \dots, M.$$

5.2. Estimate of $\int_{I_m} \|\xi\|^2\,\mathrm{d}t$. An important task is the estimation of the term $\int_{I_m} \|\xi\|^2\,\mathrm{d}t$. The case, when $\beta(u) = \text{ const } > 0$, was analyzed in [31] using the approach from [1] based on the application of the so-called Gauss-Radau quadrature and interpolation. However, in the case of nonlinear diffusion, this technique is not applicable. It appears suitable to apply here the approach from [10] based on the concept of discrete characteristic functions constructed to ξ.

We shall proceed in several steps. Let us set

$$t_{m-1+l/q} = t_{m-1} + \frac{l}{q}(t_m - t_{m-1}) \quad \text{for} \quad l = 0, ..., q,$$

and use the notation $\xi_{m-1+l/q} = \xi(t_{m-1+l/q})$.

LEMMA 5.1. *There exist constants $L_q, M_q > 0$ dependent on q only such that*

$$\sum_{l=0}^{q} \left\|\xi_{m-1+l/q}\right\|^2 \ge \frac{L_q}{\tau_m}\int_{I_m} \|\xi\|^2\,\mathrm{d}t, \tag{5.5}$$

$$\left\|\xi^+_{m-1}\right\|^2 \le \frac{M_q}{\tau_m}\int_{I_m} \|\xi\|^2\,\mathrm{d}t. \tag{5.6}$$

Proof. Let $\hat{p} \in P^q(0,1)$ be an arbitrary polynomial in t. Since the expressions

$$\left(\sum_{l=0}^{q} \hat{p}^2(\frac{l}{q})\right)^{\frac{1}{2}}, \ \left(\int_0^1 \hat{p}^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

are equivalent norms in the finite dimensional space $P^q(0,1)$, there exist constants $L_q, M_q > 0$ dependent on q only such that

$$L_q \int_0^1 \hat{p}^2 \mathrm{d}x \le \sum_{l=0}^{q} \hat{p}^2(\frac{l}{q}) \le M_q \int_0^1 \hat{p}^2 \mathrm{d}x.$$

Using the transformation $\psi(t) = \frac{t - t_{m-1}}{\tau_m}$ and the substitution theorem, we find that

$$\sum_{l=0}^{q} p^2(t_{m-1+l/q}) \ge \frac{L_q}{\tau_m} \int_{I_m} p^2 \mathrm{d}t \tag{5.7}$$

$$p^2(t_{m-1}) \le \frac{M_q}{\tau_m} \int_{I_m} p^2 \mathrm{d}t. \tag{5.8}$$

for each $p \in P^q(I_m)$. The application of these inequalities to the function "$t \in I_m \to \xi(x,\cdot)$" considered for a.e. fixed $x \in \Omega$, integration over Ω and the use of Fubini's theorem immediately yield (5.5) and (5.6). □

Now, for each $y \in I_m$ we define the *discrete characteristic function* to $\xi \in S_{h,\tau}^{p,q}$ as $\zeta_y \in S_{h,\tau}^{p,q}$ such that

$$\int_{I_m} (\zeta_y, \varphi) \mathrm{d}t = \int_{t_{m-1}}^{y} (\xi, \varphi) \mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q-1}, \tag{5.9}$$

$$\zeta_y(t_{m-1}^+) = \xi(t_{m-1}^+). \tag{5.10}$$

Using the technique from [10], it is possible to show that

$$\int_{I_m} \|\zeta_y\|_{DG,m}^2 \mathrm{d}t \le C_q \int_{I_m} \|\xi\|_{DG,m}^2 \mathrm{d}t, \tag{5.11}$$

where the constant C_q depends on q only.

Further, we shall return to identity (5.1), where we use the relations (5.3),

$$(\{\xi\}_{m-1}, \xi_{m-1}^+) = \|\xi_{m-1}^1\|^2 - (\xi_{m-1}^-, \xi_{m-1}^+), \tag{5.12}$$

$$\int_{I_m} (\xi, \xi') \mathrm{d}t + \|\xi_{m-1}^+\|^2 = \frac{1}{2}\left(\left\|\xi_m^-\right\|^2 + \left\|\xi_{m-1}^+\right\|^2\right),$$

and apply Lemmas 4.1, 4.3, inequalities (4.9), (4.10) and Young's inequality. After some manipulation, for any $\delta_1 > 0$, we arrive at the estimate

$$\begin{aligned} &\left\|\xi_m^-\right\|^2 + \left\|\xi_{m-1}^+\right\|^2 + \frac{\beta_0}{2} \int_{I_m} \|\xi\|_{DG,m}^2 \,\mathrm{d}t \\ &\le C_1 \int_{I_m} \|\xi\|^2 \,\mathrm{d}t + C_2 \int_{I_m} R_m(\eta) \mathrm{d}t + 2\frac{\left\|\eta_{m-1}^-\right\|^2}{\delta_1} + 2\frac{\left\|\xi_{m-1}^-\right\|^2}{\delta_1} + 4\delta_1 \left\|\xi_{m-1}^+\right\|^2, \end{aligned} \tag{5.13}$$

with constants C_1, C_2 independent of M, m, h, τ, ξ, η.

Now we prove the following important result.

LEMMA 5.2. *There exist constants* C, $C^* > 0$ *such that*

$$\int_{I_m} \|\xi\|^2 \,\mathrm{d}t \le C\,\tau_m \left(\left\|\xi_{m-1}^-\right\|^2 + \left\|\eta_{m-1}^-\right\|^2 + \int_{I_m} R_m(\eta) \,\mathrm{d}t\right), \tag{5.14}$$

provided

$$0 < \tau_m \le C^* \beta_0. \tag{5.15}$$

Proof. First, let us consider $q = 1$. Then, from (5.13), (5.5) and (5.6) it follows that

$$\begin{aligned} &\frac{L_q}{\tau_m} \int_{I_m} \|\xi\|^2 \,\mathrm{d}t + \frac{\beta_0}{2} \int_{I_m} \|\xi\|^2_{DG,m} \,\mathrm{d}t \\ &\le \left(C_1 + \frac{4M_q \delta_1}{\tau_m} \right) \int_{I_m} \|\xi\|^2 \,\mathrm{d}t + C_2 \int_{I_m} R_m(\eta) \mathrm{d}t + 2 \frac{\left\|\eta^-_{m-1}\right\|^2}{\delta_1} + 2 \frac{\left\|\xi^-_{m-1}\right\|^2}{\delta_1}. \end{aligned}$$

If we set $\delta_1 = \frac{L_q}{8M_q},\ C_3 = \frac{2}{\delta_1}$, then, under the condition

$$0 < \tau_m \le C^* := \frac{L_q}{4C_1}, \tag{5.16}$$

we get

$$\begin{aligned} &\frac{L_q}{4\tau_m} \int_{I_m} \|\xi\|^2 \,\mathrm{d}t + \frac{\beta_0}{2} \int_{I_m} \|\xi\|^2_{DG,m} \,\mathrm{d}t \\ &\le C_2 \int_{I_m} R_m(\eta) \mathrm{d}t + C_3 \left\|\eta^-_{m-1}\right\|^2 + C_3 \left\|\xi^-_{m-1}\right\|^2, \end{aligned} \tag{5.17}$$

which already implies (5.14).

Further, let $q \ge 2$, $l \in \{1, ..., q-1\}$ and $\tilde{\xi}_l = \zeta_{t_{m-1+l/q}}$, where $\zeta_{t_{m-1+l/q}}$ is the discrete characteristic function to the function ξ at the point $t_{m-1+l/q}$ defined by (5.9) - (5.10). Hence, by (5.9) - (5.11),

$$\int_{I_m} (\tilde{\xi}_l, \xi') \mathrm{d}t = \int_{t_{m-1}}^{t_{m-1+l/q}} (\xi, \xi') \mathrm{d}t, \quad \xi(t^+_{m-1}) = \tilde{\xi}_l(t^+_{m-1}), \tag{5.18}$$

$$\int_{I_m} \|\tilde{\xi}_l\|^2_{DG,m} \mathrm{d}t \le C_q \int_{I_m} \|\xi\|^2_{DG,m} \,\mathrm{d}t. \tag{5.19}$$

This yields the relations

$$\begin{aligned} &\int_{I_m} (\xi', \tilde{\xi}_l) \mathrm{d}t + \left(\xi^+_{m-1}, (\tilde{\xi}_l)^+_{m-1} \right) = \int_{t_{m-1}}^{t_{m-1+l/q}} (\xi, \xi') + \left(\xi^+_{m-1}, \xi^+_{m-1} \right) \\ &= \frac{1}{2} \int_{t_{m-1}}^{t_{m-1+l/q}} \frac{d}{dt} \|\xi\|^2 \,\mathrm{d}t + \left\|\xi^+_{m-1}\right\|^2 = \frac{1}{2} \left(\left\|\xi_{m-1+l/q}\right\|^2 + \left\|\xi^+_{m-1}\right\|^2 \right). \end{aligned} \tag{5.20}$$

Now we start from (4.4), where we set $\varphi := \tilde{\xi}_l$. Then, by (3.6) and the definition of $\{\xi\}_{m-1}$ we get

$$\begin{aligned} &\int_{I_m} (\xi', \tilde{\xi}_l) \mathrm{d}t + \left(\xi^+_{m-1}, (\tilde{\xi}_l)^+_{m-1} \right) \\ &= \int_{I_m} \left(-a_h(U, U, \tilde{\xi}_l) + a_h(U, \pi u, \tilde{\xi}_l) - \beta_0 J_h(\xi, \tilde{\xi}_l) - a_h(U, \pi u, \tilde{\xi}_l) + a_h(u, \pi u, \tilde{\xi}_l) \right) \mathrm{d}t \\ &+ \int_{I_m} \left(-a_h(u, \pi u, \tilde{\xi}_l) + a_h(u, u, \tilde{\xi}_l) - \beta_0 J_h(\eta, \tilde{\xi}_l) + b_h(u, \tilde{\xi}_l) - b_h(U, \tilde{\xi}_l) \right) \mathrm{d}t \\ &+ \left(\xi^-_{m-1}, (\tilde{\xi}_l)^+_{m-1} \right) - \int_{I_m} (\eta', \tilde{\xi}_l) \mathrm{d}t - \left(\{\eta\}_{m-1}, (\tilde{\xi}_l)^+_{m-1} \right). \end{aligned}$$

This, (5.3) and (5.20) imply that

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\xi_{m-1+l/q}\right\|^2+\left\|\xi_{m-1}^{+}\right\|^2\right)\\
&\leq \int_{I_m}\left(\left|a_h(U,U,\tilde{\xi}_l)-a_h(U,\pi u,\tilde{\xi}_l)+\beta_0 J_h(\xi,\tilde{\xi}_l)\right|+\left|a_h(U,\pi u,\tilde{\xi}_l)-a_h(u,\pi u,\tilde{\xi}_l)\right|\right)\mathrm{d}t\\
&+\int_{I_m}\left(\left|a_h(u,\pi u,\tilde{\xi}_l)-a_h(u,u,\tilde{\xi}_l)\right|+\left|\beta_0 J_h(\eta,\tilde{\xi}_l)\right|+\left|b_h(U,\tilde{\xi}_l)-b_h(u,\tilde{\xi}_l)\right|\right)\mathrm{d}t\\
&+\left|(\xi_{m-1}^{-},\xi_{m-1}^{+})\right|+\left|(\eta_{m-1}^{-},\xi_{m-1}^{+})\right|.
\end{aligned}$$

Using Lemmas 4.2 and 4.3, where we set $\varphi := \tilde{\xi}_l$, inequalities (4.9), (4.10) and Young's inequality, for an arbitrary $\delta_2 > 0$ we get

$$\begin{aligned}
&\frac{1}{2}\left(\left\|\xi_{m-1+l/q}\right\|^2+\left\|\xi_{m-1}^{+}\right\|^2\right)\\
&\leq \int_{I_m}\left(C\left(\|\xi\|_{DG,m}^2+\|\tilde{\xi}_l\|_{DG,m}^2\right)+\beta_0\|\tilde{\xi}_l\|_{DG,m}^2+C_a(\|\xi\|^2+R_m(\eta))\right)\mathrm{d}t\\
&+\int_{I_m}\left(\beta_0\|\tilde{\xi}_l\|_{DG,m}^2+C_c\tilde{R}_m(\eta)+\beta_0 J_h(\eta,\eta)+\beta_0 J_h(\tilde{\xi}_l,\tilde{\xi}_l)\right)\mathrm{d}t\\
&+\int_{I_m}\left(\beta_0\|\tilde{\xi}_l\|_{DG,m}^2+C_b(\|\xi\|^2+\|\eta\|^2+\sum_{K\in\mathcal{T}_{h,m}}h_K^2|\eta|_{H^1(K)}^2\right)\mathrm{d}t\\
&+\frac{\left\|\xi_{m-1}^{-}\right\|^2}{\delta_2}+\delta_2\left\|\xi_{m-1}^{+}\right\|^2+\frac{\left\|\eta_{m-1}^{-}\right\|^2}{\delta_2}+2\delta_2\left\|\xi_{m-1}^{+}\right\|^2.
\end{aligned}$$

This and (5.11) imply that

$$\begin{aligned}
&\left\|\xi_{m-1+l/q}\right\|^2+\left\|\xi_{m-1}^{+}\right\|^2\\
&\leq \tilde{C}\int_{I_m}\left(\|\xi\|_{DG,m}^2+\|\xi\|^2+R_m(\eta)\right)\mathrm{d}t\\
&+2\frac{\left\|\xi_{m-1}^{-}\right\|^2}{\delta_2}+2\frac{\left\|\eta_{m-1}^{-}\right\|^2}{\delta_2}+4\delta_2\left\|\xi_{m-1}^{+}\right\|^2
\end{aligned}\tag{5.21}$$

with a constant $\tilde{C} > 0$ independent of ξ, η, h, τ. Further, let us multiply (5.21) by $\frac{\beta_0}{4\tilde{C}(q-1)}$, sum over all $l = 1, \ldots, q-1$ and add the inequality (5.13) to the result. Then we obtain the inequality

$$\begin{aligned}
&\tilde{C}_1\left(\left\|\xi_m^{-}\right\|^2+\sum_{l=1}^{q-1}\left\|\xi_{m-1+l/q}\right\|^2+\left\|\xi_{m-1}^{+}\right\|^2\right)+\frac{\beta_0}{2}\int_{I_m}\|\xi\|_{DG}^2\,\mathrm{d}t\\
&\leq \int_{I_m}\left(\frac{\beta_0}{4}\|\xi\|_{DG}^2+\tilde{C}_2\|\xi\|^2+\tilde{C}_3 R(\eta)\right)\mathrm{d}t\\
&+\left(\frac{2}{\delta_1}+\frac{\beta_0}{2\tilde{C}\delta_2}\right)\left(\left\|\xi_{m-1}^{-}\right\|^2+\left\|\eta_{m-1}^{-}\right\|^2\right)+\left(\frac{\beta_0\delta_2}{\tilde{C}}+4\delta_1\right)\left\|\xi_{m-1}^{+}\right\|^2,
\end{aligned}$$

where

$$\tilde{C}_1 = \min\left\{\frac{\beta_0}{4\tilde{C}(q-1)}, 1\right\}, \quad \tilde{C}_2 = \frac{\beta_0}{4}+C_1, \quad \tilde{C}_3 = \frac{\beta_0}{4}+C_2.$$

Hence, by Lemma 5.1,

$$\begin{aligned}
&\frac{\tilde{C}_1 L_q}{\tau_m}\int_{I_m}\|\xi\|^2\,\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,m}\,\mathrm{d}t\\
&\le\left(\frac{\beta_0 M_q\delta_2}{\tilde{C}\tau_m} + \frac{4M_q\delta_1}{\tau_m} + \tilde{C}_2\right)\int_{I_m}\|\xi\|^2\,\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}R_m(\eta)\mathrm{d}t\\
&+\left(\frac{2}{\delta_1} + \frac{\beta_0}{2\tilde{C}\delta_2}\right)\left(\left\|\xi^-_{m-1}\right\|^2 + \left\|\eta^-_{m-1}\right\|^2\right).
\end{aligned}$$

This and the choice

$$\delta_1 = \frac{\tilde{C}_1 L_q}{16M_q},\quad \delta_2 = \frac{\tilde{C}\tilde{C}_1 L_q}{4\beta_0 M_q},\quad \tilde{C}_4 = \frac{2}{\delta_1} + \frac{\beta_0}{2\tilde{C}\delta_2},$$

lead to the inequality

$$\begin{aligned}
&\left(\frac{\tilde{C}_1 L_q}{2\tau_m} - \tilde{C}_2\right)\int_{I_m}\|\xi\|^2\,\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,m}\,\mathrm{d}t\\
&\le\frac{\beta_0}{4}\int_{I_m}R_m(\eta)\mathrm{d}t + \tilde{C}_4\left(\left\|\xi^-_{m-1}\right\|^2 + \left\|\eta^-_{m-1}\right\|^2\right).
\end{aligned}$$

If the condition

$$0 < \tau_m \le C^* := \frac{\tilde{C}_1 L_q}{4\tilde{C}_2} \tag{5.22}$$

is satisfied, then

$$\begin{aligned}
&\frac{\tilde{C}_1 L_q}{4\tau_m}\int_{I_m}\|\xi\|^2\,\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,m}\,\mathrm{d}t\\
&\le\tilde{C}_3\int_{I_m}R_m(\eta)\mathrm{d}t + \tilde{C}_4\left(\left\|\xi^-_{m-1}\right\|^2 + \left\|\eta^-_{m-1}\right\|^2\right).
\end{aligned}\tag{5.23}$$

This already implies that (5.14) holds under condition (5.15) with C^* defined by (5.22).
□

Now we finish the derivation of the abstract error estimates.

THEOREM 5.3. *Let (5.15) hold. Then there exists a constants $C > 0$ such that the error $e = U - u$ satisfies the following estimates:*

$$\begin{aligned}
&\|e^-_m\|^2 + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|^2_{DG,j}\,\mathrm{d}t\\
&\le C\Big(\sum_{j=1}^{m}\|\eta^-_j\|^2 + \sum_{j=1}^{m}\int_{I_j}R_j(\eta)\,\mathrm{d}t\Big) + 2\|\eta^-_m\|^2 + \beta_0\sum_{j=1}^{m}\int_{I_j}\|\eta\|^2_{DG,j}\,\mathrm{d}t,\\
&m = 1,\dots,M,\quad h\in(0,h_0),
\end{aligned}\tag{5.24}$$

and

$$\begin{aligned}
\|e\|^2_{L^2(Q_T)} &\le C\sum_{m=1}^{M}\tau_m\Bigg(\|\eta^-_{m-1}\|^2 + \int_{I_m}R_m(\eta)\,\mathrm{d}t\\
&+\sum_{j=1}^{m-1}\|\eta^-_j\|^2 + \sum_{j=1}^{m-1}\int_{I_j}R_j(\eta)\,\mathrm{d}t\Bigg) + 2\|\eta\|^2_{L^2(Q_T)},\quad h\in(0,h_0).
\end{aligned}\tag{5.25}$$

Proof. 1) Substituting (5.14) in (5.4), we get

$$\begin{aligned}&\left\|\xi_j^-\right\|^2+\frac{\beta_0}{2}\int_{I_j}\|\xi\|^2_{DG,j}\,\mathrm{d}t\\&\le(1+C\tau_j)\left\|\xi_{j-1}^-\right\|^2+C\Big(\left\|\eta_{j-1}^-\right\|^2+\int_{I_j}R_j(\eta)\mathrm{d}t\Big),\quad j=1,\dots,M,\end{aligned}$$

The use of the discrete Gronwall's lemma implies that

$$\left\|\xi_m^-\right\|^2+\frac{\beta_0}{2}\sum_{j=1}^m\int_{I_j}\|\xi\|^2_{\mathrm{DG},j}\,\mathrm{d}t\le C\Big(\left\|\xi_0^-\right\|^2+\sum_{j=1}^m\left\|\eta_{j-1}^-\right\|^2+\sum_{j=1}^m\int_{I_j}R_j(\eta)\,\mathrm{d}t\Big),\tag{5.26}$$

for $m=1,\dots,M$. In view of the definition of U_0^-, we have $\xi_0^-=0$. Now, if we use the relation $e=\xi+\eta$ and the inequalities

$$\|e\|^2\le 2(\|\xi\|^2+\|\eta\|^2),\tag{5.27}$$

$$\|e\|^2_{DG,j}\le 2(\|\xi\|^2_{DG,j}+\|\eta\|^2_{DG,j}),\tag{5.28}$$

from (5.26) we immediately obtain (5.24).

2) It follows from (5.14) and (5.27) that

$$\begin{aligned}&\|e\|^2_{L^2(Q_T)}=\sum_{m=1}^M\int_{I_m}\|e\|^2\,\mathrm{d}t\\&\le C\sum_{m=1}^M\tau_m\Big(\|\xi_{m-1}^-\|^2+\|\eta_{m-1}^-\|^2+\int_{I_m}R_m(\eta)\,\mathrm{d}t\Big)+2\int_0^T\|\eta\|^2\,\mathrm{d}t.\end{aligned}\tag{5.29}$$

Now we use (5.26) with $m:=m-1<M$ for the estimate of $\|\xi_{m-1}^-\|^2$, $\xi_0=0,\ \eta_0^-=\Pi_1u^0-u^0$ and get inequality (5.25). □

REMARK 1. *A detailed analysis shows that the constant C from the abstract error estimates (5.24) and (5.25) behave in dependence on β_0 as $\exp(c/\beta_0)$, which means that this constant blows up for $\beta_0\to0+$ and the obtained error estimates cannot be used in the case of nonlinear singularly perturbed convection-diffusion problems with degenerated diffusion. This is a consequence of the use of Young's inequality and Gronwall's lemma. Uniform error estimates with respect to the diffusion tending to zero were obtained in [29] for the space-time DGFE approximations of linear convection-diffusion-reaction problems.*

6. Error estimate expressed in terms of h and τ. The derivation of error estimates in dependence on h and τ is obtained from the abstract error estimate and estimation of terms containing η. We assume that the exact solution is sufficiently regular, namely

$$u\in H^{q+1}\big(0,T;H^1(\Omega)\big)\cap C([0,T];H^{p+1}(\Omega)),\tag{6.1}$$

and that the meshes satisfy conditions (4.5), (4.6), (5.15) and

$$\tau_m\ge Ch_m^2,\quad m=1,\dots,M,\tag{6.2}$$

with some constant $C>0$. This assumption is not necessary, if the space meshes are independent of time. See Remark 2.

In what follows, we shall use approximation properties of the interpolation operators π and Π_m and estimates of terms containing η appearing in the abstract error estimates formulated in Theorem 5.3. Their proofs are rather technical and can be found in [31], Section 5. Here we summarize the necessary results.

LEMMA 6.1. *For $m = 1, \dots, M,\ K \in \mathcal{T}_{h,m}$ and $h \in (0, h_0)$ we have*

$$\|\eta_m^-\|^2 \le Ch^{p+1}|u(t_m)|_{H^{p+1}(\Omega)}, \tag{6.3}$$

$$\int_{I_m} \|\eta\|^2_{L^2(K)} \mathrm{d}t \le C \left(h_K^{2(p+1)} |u|^2_{L^2(I_m, H^{p+1}(K))} + \tau_m^{2(q+1)} |u|^2_{H^{q+1}(I_m, L^2(K))} \right), \tag{6.4}$$

$$\int_{I_m} |\eta|^2_{H^1(K)} \mathrm{d}t \le C \left(h_K^{2p} |u|^2_{L^2(I_m, H^{p+1}(K))} + \tau_m^{2(q+1)} |u|^2_{H^{q+1}(I_m, H^1(K))} \right), \tag{6.5}$$

$$h_K^2 \int_{I_m} |\eta|^2_{H^2(K)} \mathrm{d}t \le C \left(h_K^{2p} |u|^2_{L^2(I_m, H^{p+1}(K))} + \tau_m^{2(q+1)} |u|^2_{H^{q+1}(I_m, H^1(K))} \right). \tag{6.6}$$

Moreover, we shall need the estimate of the expression $\int_{I_m} J_{h,m}(\eta, \eta)\, \mathrm{d}t$.

LEMMA 6.2. *a) If the exact solution u satisfies (6.1) and moreover, if there exists a constant $C_B > 0$ such that*

$$\tau_m \le C_B h_{K_\Gamma^{(L)}} \tag{6.7}$$

for all $\Gamma \in \mathcal{F}^B_{h,m},\ m = 1, \dots, M,\ h \in (0, h_0)$, then

$$\int_{I_m} J_{h,m}(\eta, \eta)\, \mathrm{d}t \le C(h^{2p}|u|_{L^2(I_m; H^{p+1}(\Omega))} + \tau^{2q}|u|_{H^{q+1}(I_m; L^2(\Omega))}), \tag{6.8}$$
$$m = 1, \dots, M, \quad h \in (0, h_0).$$

b) If u satisfies (6.1) and the Dirichlet data u_D depends on t as a polynomial of degree $\le q$, i.e.

$$u_D(x, t) = \sum_{j=0}^{q} \psi_j(x)\, t^j, \tag{6.9}$$

where $\psi_j \in H^{p+1/2}(\partial\Omega)$ for $j = 0, \dots, q$, then

$$\int_{I_m} J_{h,m}(\eta, \eta)\, \mathrm{d}t \le C(h^{2p}|u|_{L^2(I_m; H^{p+1}(\Omega))} + \tau^{2q+2}|u|_{H^{q+1}(I_m; L^2(\Omega))}), \tag{6.10}$$
$$m = 1, \dots, M, \quad h \in (0, h_0).$$

Now, we can prove the *main result.*

THEOREM 6.3. *Let u be the exact solution of problem (2.1)–(2.3) satisfying the regularity conditions (2.7) and (6.1). Let U be the approximate solution to problem (2.1)–(2.3) obtained by scheme (3.13). Let conditions (4.5), (4.6), (5.15) and (6.2) be satisfied. Then there exists a constant $C > 0$ independent of $h,\ \tau,\ m,\ M,\ u,\ U$ such that*

$$\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|e\|^2_{DG,j}\, \mathrm{d}t \tag{6.11}$$
$$\le C \left(h^{2p} |u|^2_{C([0,T]; H^{p+1}(\Omega))} + \tau^{2q+\gamma} |u|_{H^{q+1}(0,T; H^1(\Omega))} \right),$$
$$m = 1, \dots, M,\ h \in (0, h_0),$$

and

$$\|e\|^2_{L^2(Q_T)} \le C \left(h^{2p} |u|^2_{L^2(0,T; H^{p+1}(\Omega))} + \tau^{2q+\gamma} |u|^2_{H^{q+1}(0,T; H^1(\Omega))} \right)$$
$$m = 1, \dots, M,\ h \in (0, h_0).$$

Here $\gamma = 0$, if (6.7) holds and the function u_D from the boundary condition (2.2) has a general behaviour with respect to t. If u_D is defined by (6.9), then $\gamma = 2$ and condition (6.7) is not required.

Proof. In virtue of (3.9) and Lemmas 6.1 and 6.2,

$$\int_{I_j} \|\eta\|^2_{DG,j}\,\mathrm{d}t \le C \sum_{K\in\mathcal{T}_{h,j}} \Big(h_K^{2p} |u|^2_{L^2(I_j,H^{p+1}(K))} + \tau_j^{2q+\gamma} |u|^2_{H^{q+1}(I_j,H^1(K))} \Big) \tag{6.12}$$

with γ defined in the theorem. This and the inequality $h_K \le h_j$ imply that

$$\int_{I_j} \|\eta\|^2_{DG,j}\,\mathrm{d}t \le C \Big(h_j^{2p} |u|^2_{L^2(I_j,H^{p+1}(\Omega))} + \tau_j^{2q+\gamma} |u|^2_{H^{q+1}(I_j,H^1(\Omega))} \Big). \tag{6.13}$$

Similarly, in view of (4.21), we get

$$\int_{I_j} R_j(\eta)\,\mathrm{d}t \le C \Big(h_j^{2p} |u|^2_{L^2(I_j,H^{p+1}(\Omega))} + \tau_j^{2q+\gamma} |u|^2_{H^{q+1}(I_j,H^1(\Omega))} \Big). \tag{6.14}$$

Further, by (6.3), (6.2) and the relation $\sum_{m=1}^{M} \tau_m = T$,

$$\sum_{j=1}^{m} \|\eta_j^-\|^2 \le C \sum_{j=1}^{M} \tau_j\, h_j^{2p} |u(t_j)|^2_{H^{p+1}(\Omega)} \le C\,T h^{2p} |u|^2_{C([0,T];H^{p+1}(\Omega))}. \tag{6.15}$$

Using (5.26) and (6.13) – (6.15), we arrive at estimate (6.11).

Similarly as above, estimating the individual terms in (5.25) depending on η, with the aid of (6.3), (6.4), (6.13) – (6.15) and the relation $\sum_{m=1}^{M} \tau_m = T$, we obtain (6.12).
□

REMARK 2. *If all meshes $\mathcal{T}_{h,m}$, $m = 1\dots,M$, are identical, which means that $\mathcal{T}_{h,m} = \mathcal{T}_h$ for all $m = 1,\dots,M$, then all spaces $S^p_{h,m}$ and forms $a_{h,m}, b_{h,m},\dots$ are also identical: $S^p_{h,m} = S^p_h$, $a_{h,m} = a_h, b_{h,m} = b_h,\dots$ for all $m = 1,\dots,M$. This implies that $\{\xi\}_{m-1} \in S^p_h$ and by (4.3), (4.1), a), and (3.11), we have $(\eta^-_{m-1}, \{\xi\}_{m-1}) = 0$. Hence, by (5.3),*

$$\int_{I_m} (\eta', \xi)\,\mathrm{d}t + \big(\{\eta\}_{m-1}, \xi^+_{m-1}\big) = 0. \tag{6.16}$$

It follows from (6.16) that the expression $\sum_{j=1}^{m} \left\|\eta_j^-\right\|^2$ does not appear in estimate (5.26) and instead of (5.24) and (5.25) we get the estimates

$$\begin{aligned} &\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|e\|^2_{DG,j}\,\mathrm{d}t \\ &\le C \Big(\sum_{j=1}^{m} \int_{I_j} R_j(\eta)\,\mathrm{d}t \Big) + 2\|\eta_m^-\|^2 + \beta_0 \sum_{j=1}^{m} \int_{I_j} \|\eta\|^2_{DG,j}\,\mathrm{d}t, \ \ m = 1,\dots,M. \end{aligned} \tag{6.17}$$

and

$$\|e\|^2_{L^2(Q_T)} \le C \sum_{m=1}^{M} \tau_m \bigg(\|\eta^-_{m-1}\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t + \sum_{j=1}^{M} \int_{I_j} R_j(\eta)\,\mathrm{d}t \bigg) + 2\|\eta\|^2_{L^2(Q_T)}, \tag{6.18}$$

respectively. From this we deduce that in the case of identical meshes on all time levels Theorem 6.3 is valid without assumption (6.2).

Conclusion. The paper is devoted to the theoretical analysis of error estimates of the space-time discontinuous Galerkin finite element method for the numerical solution of a nonstationary equation with nonlinear convection and nonlinear diffusion, equipped with initial condition and Dirichlet boundary condition. It is assumed that the diffusion coefficient depending on the sought solution is bounded from above and from below by positive constants and, hence, does not degenerate to zero. In the space discretization, the NIPG, IIPG and SIPG polynomial approximations of degree $p \ge 1$ are used. In time the approximations of degree $q \geq 1$, in general $q \neq p$, are used. The space meshes may be different on different time levels. It is proven that the error estimate is optimal in space in $L^2(H^1)$-norm, but suboptimal in $L^2(L^2)$-norm. As for the estimates in time, the optimality was obtained under the assumption that the Dirichlet data u_D behave in time as a polynomial of degree $\leq q$. It is possible to extend the results to the case of a mixed Dirichlet-Neumann boundary condition using estimates from [35]. Another possibility is the derivation of error estimates in the case when the hp space discretization is used.

There are still subjects for further research:

- derivation of optimal error estimates in space and time in the case of the SIPG method,
- analysis of the effect of numerical integration in space and time integrals,
- numerical realization of the discrete problem and the demonstration of results by numerical experiments.

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